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## **DERIVATIONS OF $C^*$ -ALGEBRAS HAVE SEMI-CONTINUOUS GENERATORS**

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## DERIVATIONS OF $C^*$ -ALGEBRAS HAVE SEMI-CONTINUOUS GENERATORS

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**For each derivation  $\delta$  of a  $C^*$ -algebra  $A$  with  $\delta(x^*) = -\delta(x)^*$  there exists a minimal positive element  $h$  in the enveloping von Neumann algebra  $A''$  such that  $\delta(x) = hx - xh$ . It is shown that the generator  $h$  belongs to the class of lower semi-continuous elements in  $A''$ . From this it follows that if the function  $\pi \rightarrow \|\pi \circ \delta\|$  is continuous on the spectrum of  $A$  then  $h$  multiplies  $A$ . This immediately implies that each derivation of a simple  $C^*$ -algebra is given by a multiplier of the algebra. Another application shows that each derivation of a countably generated monotone sequentially closed  $C^*$ -algebra is inner.**

A linear operator  $\delta$  on a  $C^*$ -algebra  $A$  is called a derivation if  $\delta(ab) = \delta(a)b + a\delta(b)$  for all  $a$  and  $b$  in  $A$ . If  $\delta^* = -\delta$  (i.e.,  $\delta(a)^* = -\delta(a^*)$ ) then  $\alpha_t(a) = \exp(it\delta)a$  defines a norm-continuous one-parameter group of  $*$ -automorphisms of  $A$ . Conversely, each such group can be written as  $\exp(it\delta)$  for a suitable derivation  $\delta$  of  $A$ . After a number of partial results, notably by I. Kaplansky and R. V. Kadison, it was proved by S. Sakai that every derivation of a von Neumann algebra  $A$  is inner, i.e.,  $\delta(a) = ha - ah$  for some  $h$  in  $A$  (see [9, III.9.3. Théorème 1]). Recently W. B. Arveson ([3])—see also [4]—gave a new proof of this result, using the theory of spectral subspaces associated with a one-parameter group of automorphisms. The powerful techniques developed in [3] enabled the first author to show that each derivation of an  $AW^*$ -algebra is inner ([12]).

In this paper we use Arveson's technique to show that if  $\delta$  is a derivation of a  $C^*$ -algebra  $A$  with  $\delta^* = -\delta$  then the minimal positive generator for  $\delta$ , or rather for its extension to a derivation of the enveloping von Neumann algebra  $A''$  of  $A$ , is the limit of an increasing net of self-adjoint operators from  $\tilde{A}$ . This shows that the function  $\pi \rightarrow \|\pi \circ \delta\|$  on the spectrum  $\hat{A}$  of  $A$  is lower semi-continuous and that it is continuous if and only if the minimal positive generators for  $\delta$  and  $-\delta$  both multiply  $A$ . This last result was first proved in [2] and has as an immediate consequence that every derivation of a simple  $C^*$ -algebra is given by a multiplier ([17]). We finally show that every derivation of a countably generated monotone sequentially closed  $C^*$ -algebra is inner.

The possibility of using [12] to show that derivations of  $C^*$ -algebras have measurable generators was pointed out to us by E. B. Davies.

1. Spectral subspaces and duality. Let  $\alpha_t$  be a norm-continuous one-parameter group of isometries of a Banach space  $X$ . For each  $f$  in  $L^1(\mathbf{R})$  let  $\pi_\alpha(f)$  denote the bounded operator on  $X$  given by the Bochner integral

$$\pi_\alpha(f)x = \int \alpha_t(x)f(t)dt .$$

With  $\hat{f}(s) = \int f(t)e^{ist}dt$  and  $-\infty \leqq t \leqq w \leqq \infty$  let  $R_\alpha(t, w)$  denote the closed subspace of  $X$  generated by vectors  $\pi_\alpha(f)x, x \in X$  such that  $\hat{f}$  has compact support in  $(t, w)$ . The spectral subspace associated with  $[t, w]$  is

$$M_\alpha[t, w] = \bigcap R_\alpha\left(t - \frac{1}{n}, w + \frac{1}{n}\right) .$$

As shown in [3, Proposition 2.2]—see also [12]—we have

$$M_\alpha[t, w] = \{x \in X \mid \pi_\alpha(f)x = 0 \ \forall f \in I_0[t, w]\}$$

where  $I_0[t, w]$  denotes the set of functions  $f$  in  $L^1(\mathbf{R})$  such that  $\hat{f}$  has compact support disjoint from  $[t, w]$ .

The transposed  $\alpha_t^*$  and bi-transposed  $\alpha_t^{**}$  of  $\alpha_t$  gives rise to norm-continuous (and weak \*-continuous) groups of isometries of  $X^*$  and  $X^{**}$ , respectively. We shall relate the spectral subspaces of the three groups using polar sets (denoted by  $M^\circ$ ).

LEMMA 1.1. *If  $s < t$  then*

$$M_{\alpha^*}[-\infty, s] \subset M_\alpha[t, \infty]^\circ \subset M_{\alpha^{**}}[-\infty, t] .$$

*Proof.* For each  $f$  in  $I_0[-\infty, t]$  and  $x$  in  $X$  we have  $\pi_\alpha(f)x \in R_\alpha(t, \infty)$ . If therefore  $\rho \in R_\alpha(t, \infty)^\circ$  then

$$0 = \langle \pi_\alpha(f)x, \rho \rangle = \langle x, \pi_{\alpha^*}(f)\rho \rangle ,$$

since  $\pi_{\alpha^*}(f)$  is the transposed of  $\pi_\alpha(f)$ . Thus  $\pi_{\alpha^*}(f)\rho = 0$  so that  $\rho \in M_{\alpha^*}[-\infty, t]$ . It follows that

$$R_\alpha(t, \infty)^\circ \subset M_{\alpha^*}[-\infty, t] ,$$

and

$$R_\alpha(t, \infty) \subset M_\alpha[t, \infty]$$

implies that

$$M_\alpha[t, \infty]^\circ \subset R_\alpha(t, \infty)^\circ .$$

Consequently

$$M_\alpha[t, \infty]^0 \subset M_\alpha^*[-\infty, t].$$

If  $s < t$  then  $s < t - (1/n)$  for sufficiently large  $n$ . For each  $f$  in  $L^1(\mathbf{R})$  where  $\hat{f}$  has compact support in  $(t - (1/n), \infty)$  and  $x$  in  $X$  we have

$$\langle \pi_\alpha(f)x, \rho \rangle = \langle x, \pi_\alpha^*(f)\rho \rangle = 0$$

for each  $\rho$  in  $M_\alpha^*[-\infty, s]$ , since  $f \in I_0[-\infty, s]$ . Thus  $\rho \in R_\alpha(t - (1/n), \infty)^0$  and a fortiori  $\rho \in M_\alpha[t, \infty]^0$ . It follows that

$$M_\alpha^*[-\infty, s] \subset M_\alpha[t, \infty]^0$$

and the proof is complete.

REMARK 1.2. The reader may verify that for each  $x$  in  $M_\alpha[t, \infty]$  and  $\rho$  in  $M_\alpha^*[-\infty, t]$  one has  $\langle \alpha_s(x), \rho \rangle = e^{ist} \langle x, \rho \rangle$  for all  $s$ . Despite this extraordinary behavior it is not in general true that  $M_\alpha[t, \infty]^0 = M_\alpha^*[-\infty, t]$ . To see this take any Banach space  $X$  and define  $\alpha_t(x) = e^{it}x$  for all  $x$  in  $X$ . Then  $\alpha_t$  is a norm-continuous one-parameter group of isometries of  $X$ . Since  $\pi_\alpha(f)x = \hat{f}(1)x$  it is easily verified that  $M_\alpha[t, \infty] = X$  for  $t \leq 1$  and zero otherwise. Analogously  $M_\alpha^*[-\infty, t] = X^*$  for  $t \geq 1$  and zero otherwise. Consequently,

$$0 = M_\alpha[1, \infty]^0 \neq M_\alpha^*[-\infty, 1] = X^*.$$

PROPOSITION 1.3. *If  $s < t$  then*

$$M_\alpha^{**}[t, \infty] \subset M_\alpha[s, \infty]^{00} \subset M_\alpha^{**}[s, \infty].$$

*Proof.* Taking polar sets in Lemma 1.1 we get

$$(*) \quad M_\alpha^*[-\infty, s]^0 \subset M_\alpha[s, \infty]^{00} \subset M_\alpha^*[-\infty, w]^0$$

for  $w < s$ . Using Lemma 1.1 with  $\alpha_t^*$  and  $X^*$  instead of  $\alpha_t$  and  $X$  we obtain

$$M_\alpha^{**}[t, \infty] \subset M_\alpha^*[-\infty, s]^0 \quad \text{and} \quad M_\alpha^*[-\infty, w]^0 \subset M_\alpha^{**}[w, \infty],$$

for  $s < t$ . Inserting these inclusions in (\*) yield

$$M_\alpha^{**}[t, \infty] \subset M_\alpha[s, \infty]^{00} \subset M_\alpha^{**}[w, \infty]$$

for  $w < s < t$ . However, by the definition of spectral subspaces

$$M_\alpha^{**}[s, \infty] = \bigcap_{w < s} M_\alpha^{**}[w, \infty]$$

and the proposition follows.

2. Derivations of  $C^*$ -algebras. Let  $A$  be a  $C^*$ -algebra and denote by  $A''$  the enveloping von Neumann algebra of  $A$ , isomorphic with the second dual of  $A$  (see [7, § 12]). For any set  $B$  in  $A''_{s.a.}$  let  $B^-$  denote the norm-closure of  $B$  and let  $B^m$  denote the set of operators in  $A''_{s.a.}$  which can be obtained as strong limits of increasing nets from  $B$ . The class  $((A_{s.a.})^m)^-$  consists of the so called lower semi-continuous elements of  $A''_{s.a.}$ . If  $A_{s.a.}$  is represented as the continuous real affine functions vanishing at 0 on the convex compact set

$$Q = \{\rho \in A^* \mid \|\rho\| \leq 1, \rho \geq 0\}$$

then  $((A_{s.a.})^m)^-$  is precisely the set of lower semi-continuous bounded real affine functions on  $Q$  vanishing at zero. Let  $\tilde{A}$  denote the  $C^*$ -algebra obtained by adjoining the unit 1 of  $A''$  to  $A$ . Then

$$((A_{s.a.})^m)^- + \mathbf{R}1 = (\tilde{A}_{s.a.})^m.$$

If  $M(A)$  denotes the  $C^*$ -algebra in  $A''$  of elements  $x$  such that  $xA \subset A$  and  $Ax \subset A$  then

$$M(A)_{s.a.} = (\tilde{A}_{s.a.})^m \cap (\tilde{A}_{s.a.})_m.$$

([15, Theorem 2.5] see also [1]). It is shown in [8, Theorem 5] (see also [15, Corollary 4.7]) that the center of  $M(A)$ —the ideal center of  $A$ —can be identified with the set of bounded continuous functions on the spectrum  $\hat{A}$  of  $A$ .

Let  $\delta$  be a derivation of  $A$  such that  $\delta^* = -\delta$ . Then  $\alpha_t = \exp it\delta$  defines a norm-continuous one-parameter group of  $*$ -automorphisms of  $A$  so that the results from § 1 are applicable.

**THEOREM 2.1.** *Let  $\delta$  be a derivation of a  $C^*$ -algebra  $A$  such that  $\delta^* = -\delta$ . Then the minimal positive operator  $h$  in  $A''$  for which  $\delta = adh$  is a lower semi-continuous element.*

*Proof.* The bi-transposed  $\bar{\delta}$  of  $\delta$  is an extension of  $\delta$  to a derivation of  $A''$ . With  $\bar{p}(t)$  as the left annihilator projection of  $M_\alpha^{**}[t, \infty]$  we know from [12, Proposition 3] that the operator-valued Riemann-Stieltjes integral

$$\int_0^{\|\delta\|} t d\bar{p}(t)$$

with respect to the increasing projection-valued map  $t \rightarrow \bar{p}(t)$  defines a positive operator  $h$  in  $A''$  and that  $h$  is the minimal positive operator in  $A''$  such that  $\bar{\delta} = adh$ .

Let  $p(t)$  denote the left annihilator projection in  $A''$  of  $M_\alpha[t, \infty]$ . Since the annihilators of a subspace and its weak closure (= bi-polar)

coincide we see from Proposition 1.3 that

$$\bar{p}(s) \leq p(s) \leq \bar{p}(t)$$

for  $s < t$ . For each positive functional  $\rho$  on  $A''$  define  $g$  and  $\bar{g}$  on  $[0, \|\delta\|]$  by

$$g(t) = \rho(p(t)) \quad \text{and} \quad \bar{g}(t) = \rho(\bar{p}(t)).$$

Since  $\bar{g}(s) \leq g(s) \leq \inf \bar{g}(t)$  it follows from well-known properties of Riemann-Stieltjes integrals that

$$\int f(t) d\bar{g}(t) = \int f(t) dg(t)$$

for every continuous function  $f$  on  $[0, \|\delta\|]$ .

Thus

$$\rho\left(\int_0^{\|\delta\|} f(t) d\bar{p}(t)\right) = \rho\left(\int_0^{\|\delta\|} f(t) dp(t)\right),$$

and since this holds for all  $\rho$  on  $A''$  we have

$$\int_0^{\|\delta\|} f(t) d\bar{p}(t) = \int_0^{\|\delta\|} f(t) dp(t).$$

In particular

$$h = \int_0^{\|\delta\|} t dp(t).$$

For fixed  $t$  let  $\Lambda$  denote the net (under inclusion) of finite subsets of  $M_a[t, \infty]$ , and for  $\lambda$  in  $\Lambda$  let  $|\lambda|$  denote the cardinality of  $\lambda$ . Then the net in  $A_+$  with elements

$$x_\lambda = \left(|\lambda|^{-1} + \sum_{x \in \lambda} xx^*\right)^{-1} \sum_{x \in \lambda} xx^*$$

increases to a projection  $q(t)$  in  $(A_{s.a.})^m$ . Since

$$q(t) \geq (|\lambda|^{-1} + xx^*)^{-1}xx^*$$

for each  $x$  in  $\lambda$  we see that  $q(t)$  majorizes the range projection of each  $x$  in  $M_a[t, \infty]$ . Thus if  $H$  is the universal Hilbert space on which  $A''$  acts we conclude that  $q(t)$  is the projection on the closure of  $M_a[t, \infty]H$ . It follows that  $q(t) = 1 - p(t)$ . Put

$$h_n = n^{-1} \sum_{k=1}^n q(kn^{-1} \|\delta\|).$$

Then  $h_n \in (A_{s.a.})^m$  and  $0 \leq h - h_n \leq n^{-1}$ ; so that  $h \in ((A_{s.a.})^m)^-$  which is precisely what we wanted.

**PROPOSITION 2.2.** *For each derivation  $\delta$  of a  $C^*$ -algebra  $A$  the function  $\pi \rightarrow \|\pi \circ \delta\|$  is lower semi-continuous on  $\hat{A}$ .*

*Proof.* Given  $\pi_0$  in  $\hat{A}$  let  $t$  be the  $\liminf$  of  $\|\pi \circ \delta\|$  when  $\pi$  ranges over the neighborhood system of  $\pi_0$ . We shall prove that  $\|\pi_0 \circ \delta\| \leq t$ . Choose a net  $\{\pi_i\}$  in  $\hat{A}$  converging to  $\pi_0$  such that  $\|\pi_i \circ \delta\| < t + \varepsilon$  for all  $i$ . Then

$$\bigcap \ker \pi_i \subset \ker \pi_0 .$$

If therefore  $\rho$  denotes the representation  $\Sigma^{\oplus} \pi_i$  then  $\pi_0(A)$  is a quotient of  $\rho(A)$  so that  $\|\pi_0 \circ \delta\| \leq \|\rho \circ \delta\|$ . But

$$\|\rho \circ \delta\| = \sup \|\pi_i \circ \delta\| \leq t + \varepsilon$$

and consequently  $\|\pi_0 \circ \delta\| \leq t + \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary the proposition follows.

**REMARK 2.3.** If  $\delta^* = -\delta$  and  $h$  is the minimal positive generator for  $\delta$  then since each representation  $\pi$  of  $A$  is quasi-equivalent to a representation of the form  $x \rightarrow zx$  for some central projection  $z$  in  $A''$  we have

$$\|\pi \circ \delta\| = \|\bar{\delta} | A'' z \| = \|hz\| = \|\pi(h)\| .$$

Note that since  $h \in ((A_s)_a)^m$  the function  $\pi \rightarrow \|\pi(h)\|$  is lower semi-continuous on  $\hat{A}$  by [15, Theorem 4.6] in accordance with Proposition 2.2.

The next result is proved in [2] by an entirely different method.

**THEOREM 2.4.** *For each derivation  $\delta$  of a  $C^*$ -algebra  $A$  such that  $\delta^* = -\delta$ , the function  $\pi \rightarrow \|\pi \circ \delta\|$  is continuous on  $\hat{A}$  if and only if the minimal positive generators for  $\delta$  and  $-\delta$  both belong to  $M(A)$ .*

*Proof.* Without loss of generality assume that  $\|\delta\| = 1$ , and let  $h$  and  $k$  be the minimal positive generators for  $\delta$  and  $-\delta$ , respectively. Since  $1 - k$  is a positive generator for  $\delta$  we have  $h \leq 1 - k$ . Moreover,  $(1 - k) - h$  belongs to the center of  $A''$ . Put  $a = h + k$ . We claim that  $\|\pi(h)\| = \|\pi(a)\|$  for each irreducible representation  $\pi$  of  $A$ . For if  $\|\pi(h)\| + \varepsilon \leq \|\pi(a)\|$  for some  $\varepsilon > 0$  then since  $\pi(a)$  is a multiple of the identity we get

$$\pi(a) - \pi(k) = \pi(h) \leq \pi(a) - \varepsilon$$

so that  $\varepsilon \leq \pi(k)$ . But this is impossible as  $\pi(k)$  is the minimal positive generator for  $-\pi \circ \delta$ .

By the Dauns-Hofmann Theorem the central, positive element  $a$

in  $((A_{s.a.})^m)^-$  belongs to  $M(A)$  if and only if the function  $\pi \rightarrow \|\pi(a)\|$  ( $=\|\pi \circ \delta\|$ ) is continuous on  $\hat{A}$  (see [15, Corollary 4.7]). If both  $h$  and  $k$  belongs to  $M(A)$  then of course  $a \in M(A)$ . But if  $a \in M(A)$  then in particular  $a \in (\tilde{A}_{s.a.})_m$  and since  $-(\tilde{A}_{s.a.})^m = (\tilde{A}_{s.a.})_m$

$$h = a - k \in (\tilde{A}_{s.a.})_m .$$

Thus by Theorem 2.1

$$h \in (\tilde{A}_{s.a.})^m \cap (\tilde{A}_{s.a.})_m = M(A)_{s.a.} .$$

This completes the proof.

**COROLLARY 2.5.** (Sakai [17]). *Every derivation of a simple  $C^*$ -algebra is given by a multiplier of the algebra.*

*Proof.* Each nonzero representation of  $A$  is an isometry so that the function  $\pi \rightarrow \|\pi \circ \delta\|$  is constant, hence continuous.

**3. Derivations of sequentially closed  $C^*$ -algebras.** A monotone sequentially closed  $C^*$ -algebra  $B$  is a  $C^*$ -algebra in which every norm-bounded increasing sequence of self-adjoint elements has a least upper bound in the algebra. Basically these algebras are the non-commutative algebraic counterpart of abstract measure spaces, a point of view which has been successfully exploited in [5]. A monotone sequentially closed  $C^*$ -algebra which admits a faithful  $\sigma$ -normal representation on a Hilbert space (sometimes known as a Baire\* algebra) is a reasonable non-commutative analogue of the Baire functions on a locally compact space. These algebras are studied in [6], [11], [13], [14], and [16].

We say that the monotone sequentially closed  $C^*$ -algebra  $B$  is countably generated if it contains a sequence  $\{b_n\}$  such that the smallest monotone sequentially closed  $C^*$ -subalgebra of  $B$  containing  $\{b_n\}$  is equal to  $B$ . In this case  $B$  has a unit—the supremum of all range projections of the  $b_n$ 's.

**THEOREM 3.1.** *Every derivation of a countably generated monotone sequentially closed  $C^*$ -algebra is inner.*

*Proof.* We may assume that  $\delta^* = -\delta$ . Let  $A$  be the separable  $C^*$ -subalgebra of  $B$  generated by elements of the form  $\delta^m(b_n)$ ,  $m \geq 0$ , where  $\{b_n\}$  is a generating sequence for  $B$  containing 1. Then  $\delta(A) \subset A$  so that  $\delta_1 = \delta|_A$  is a derivation of  $A$ . By Theorem 2.1  $\delta_1 = ad h$  for some  $h$  in  $(A_+)^m$  (since  $1 \in A$  the subset  $(A_{s.a.})^m$  is norm-closed and  $((A_{s.a.})^m)_+ = (A_+)^m$ ). The separability of  $A$  implies that  $Q$  is metrizable so that we can find a sequence  $\{h_k\}$  in  $A_+$  with  $h_k \nearrow h$ .

Let  $\{u_n\}$  be a countable group of unitaries in  $A$  which generate



$A$  as a  $C^*$ -algebra. Note that

$$u_n^* h u_n - h = u_n^* \delta_1(u_n) \in A .$$

For fixed  $n_0$  put  $X = \sum_{n < n_0}^{\oplus} A$ . Then the sequence in  $X$  with elements

$$x_k = \sum_{n \geq n_0} (u_n^* h_k u_n - h_k - u_n^* \delta_1(u_n))$$

converges weakly to zero in  $X^{**}$ . Thus for every  $\varepsilon > 0$  and  $k_0$  there exists  $\{x_k \mid k_0 \leq k \leq k_1\}$  such that

$$\bigcap_k \{\rho \in X_1^* \mid |\rho(x_k)| \geq \varepsilon\} = \emptyset .$$

It follows that  $\|\sum \lambda_k x_k\| < \varepsilon$  for some convex combination of the  $x_k$ 's. Using this we can inductively find a sequence  $\{a_m\}$  in  $A_+$  such that

(i) Each  $a_m$  is a convex combination of elements from  $\{h_k\}$ .

(ii) The elements  $h_k$  occurring in the combination of  $a_{m+1}$  all have higher index than those occurring in  $a_m$ .

(iii)  $\|u_n^* a_m u_n - a_m - u_n^* \delta_1(u_n)\| \leq \frac{1}{m}$  for  $n \leq m$ .

By condition (i)  $a_m \leq \|h\|$  for all  $m$  and by condition (ii) the sequence  $\{a_m\}$  is increasing. Let  $a$  denote the least upper bound of  $\{a_m\}$  in  $B$ . Then  $u_n^* a u_n$  is the least upper bound for  $\{u_n^* a_m u_n\}$ . Since  $\{u_n^* a_m u_n - a_m\}$  is norm-convergent to  $u_n^* \delta_1(u_n)$  we conclude from [10, Lemma 2.2] that

$$u_n^* a u_n - a = u_n^* \delta_1(u_n)$$

for all  $u_n$  (the additional hypothesis in [10, Lemma 2.2] that  $B$  is (unrestrictedly) monotone complete is not needed for the proof). Thus  $\delta_1(u_n) = a u_n - u_n a$  for all  $u_n$ . The elements in  $B$  on which the two derivations  $\delta$  and  $ad a$  coincide form a  $C^*$ -algebra containing  $A$ . Since  $\delta(A) \subset A$  we see that the elements in  $B$  on which  $\delta^n$  and  $(ad a)^n$  coincide for every  $n$  form a  $C^*$ -algebra  $B_0$  containing  $A$ . If  $\{c_n\}$  is an increasing sequence of self-adjoint elements in  $B_0$  with least upper bound  $c$  in  $B$  then

$$\exp(it\delta)c_n = \exp(ita)c_n \exp(-ita)$$

for every  $n$  and all real  $t$ . Since  $*$  automorphisms are order-preserving this implies that

$$\exp(it\delta)c = \exp(ita)c \exp(-ita) .$$

Successive differentiations show that  $\delta^n(c) = (ad a)^n(c)$ ; hence  $c \in B_0$ . It follows that  $B_0$  is monotone sequentially closed and therefore  $B_0 = B$ . This completes the proof.

**COROLLARY 3.2.** *If  $\delta$  is a derivation of a countably generated monotone sequentially closed  $C^*$ -algebra  $B$  such that  $\delta^* = -\delta$  then there is a minimal positive generator  $a$  in  $B$  for  $\delta$  characterized by*

$$\|az\| = \|\delta | Bz\|$$

for every central projection  $z$  in  $B$ .

*Proof.* Since  $B$  is countably generated every projection in  $B$  has a central cover in  $B$ , so that  $B$  is well supplied with central projections. With the notation as in the proof of Theorem 3.1 note that each central projection  $z$  in  $B$  determines a representation  $\pi$  of  $A$  given by  $\pi(b) = bz$ . Since  $h$  is the minimal positive generator for  $\delta_1$ , this implies that  $\|\pi(h)\| = \|\pi \circ \delta_1\|$ . Now  $0 \leq a_m \leq h$  so that

$$\|a_m z\| = \|\pi(a_m)\| \leq \|\pi(h)\|.$$

Since  $az$  is the least upper bound in  $B$  of  $\{a_m z\}$  we conclude that  $0 \leq az \leq \|\pi(h)\|$ , hence  $\|az\| \leq \|\pi(h)\|$ . Finally

$$\|\pi \circ \delta_1\| = \|\delta | Az\| \leq \|\delta | Bz\|$$

so that  $\|az\| \leq \|\delta | Bz\|$ . The reverse inequality is obvious and it follows as in the proof of [12, Proposition 3] that  $a$  is uniquely characterized by these norm conditions and that it is the minimal positive generator for  $\delta$  in  $B$ .

**REMARK 3.3.** For a nonseparable Hilbert space  $H$  let  $S(H)$  denote the set of operators in  $B(H)$  with separable range. Then  $S(H)$  is a monotone sequentially closed  $C^*$ -algebra but since  $S(H)$  is an ideal in  $B(H)$  it is easy to find outer derivations for  $S(H)$ . Thus the condition of being countably generated can not be deleted from Theorem 3.1.

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