ON CONJUGATION COBORDISM

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An almost-complex manifold supports an involution if there is a differentiable self-map on the manifold of period two. The differential of the map acts on the coset space of the almost-complex structures on $M$ by inner automorphism. This action is also of period two. If the almost-complex structure is sent to its conjugate, the manifold with structure, together with the given involution is called a conjugation. Any linear involution of Euclidean space may be used to stabilize this situation, giving a cobordism theory of exotic conjugations. The question considered here is: What is the image in complex cobordism of the functor which forgets equivariance. The result shown in the next section is: If a stably almost-complex manifold supports an exotic conjugation, every characteristic number is even.

The first cobordism results on conjugations are due to Conner and Floyd [3] (§ 24). In [4], Landweber established the equivariant analogues of the Thom theorems. Certain examples have been considered by Landweber, [5] (§ 3), and together with the result here the image of the forgetful functor can be seen to be maximal, in some cases.

2. Proof of the theorem. It is well-known from the work of Thom and Milnor that the unoriented bordism ring $\mathcal{N}_*$, with spectrum $MO$, is a polynomial ring over $\mathbb{Z}_2$ on manifold classes $n_t$, $t + 1$ any positive integer not a power of two ($t$ nondyadic). Also $\mathcal{U}_*$, the complex bordism ring with spectrum $MU$, is a polynomial ring over $\mathbb{Z}$ on manifold classes $u_t$, $t = 0, 1, \ldots$. Representatives for the dyadic generators $u_{2^i}$, $t + 1 = 2^i$, may be chosen so that every normal characteristic number is even. The principal ideal in $\mathcal{U}_*$ generated by dyadic generators is the graded Milnor ideal associated to 2, $I$. $I_{2k} = I \cap \mathcal{U}_{2k}$.

If a partition of $k$ contains a dyadic integer the partition will be called dyadic. Let $d(k)$ denote the dyadic partitions of $k$, $n(k)$ the nondyadic partitions of $k$. If $\alpha = a_1a_2\cdots a_r$ is a partition of $k$ then the group generator $u_{a_1}\cdots u_{a_r} \in \mathcal{U}_{2k}$ will be denoted $u_{\alpha}$. Similarly for $n_{\alpha} \in \mathcal{N}_{2k}$.

If $MU(n)$ is given the involution defined in [4] then it is a $G$-complex, $G = \mathbb{Z}_2$, in the sense of Bredon. Note that $\bar{\omega}(MU(n)) = \bar{\omega}(MU(n)) = 0$. The construction given in the next section produces, for each partition of $k$, $\alpha$, and sufficiently large $n$, an equivariant
inclusion and a $G$-complex $\varepsilon^\alpha: MU(n) \to Y^\alpha$ such that

- $(c\ i) \quad \tilde{\omega}_{n+k}(Y^\alpha) = \begin{cases} (\mathbb{Z} \to 0) & \text{if } \alpha \in n(k) \\ 0 & \text{if } \alpha \in d(k) \end{cases}$
- $(c\ ii) \quad \tilde{\omega}_{2n+2k}(Y^\alpha) = (0 \to (\mathbb{Z}, (-1)^{n+k})])$
- $(c\ iii) \quad \omega_t(Y^\alpha) = 0 \text{ if } t \neq n + k, 2n + 2k$
- $(c\ iv) \quad e^\alpha(G) : \tilde{\omega}_{2n+2k}(MU(n))(G) \cong \mathbb{Z}_{2k} \to \tilde{\omega}_{2n+2k}(Y^\alpha)(G) \equiv \mathbb{Z}$ maps $u_\alpha$ to an odd multiple of the generator $\alpha \in n(k)$.

Let the $r + s$ sphere with the orthogonal involution fixing an equatorial $s$-sphere be denoted $S^{r,s}$. The $G$-complex formed by attaching the cone over $S^{r,s}$ in $S^{r,s}$ will be denoted $S^{r,s}/S^{0,s}$. Let the equivariant homotopy groups

$$\left[\frac{S^{n+a,n+b}}{S^{0,a+b}}, MU(n)\right] \text{ and } \left[\frac{S^{n+a,n+b}}{S^{0,a+b}}, Y^\alpha\right]$$

be denoted $\lambda_{Z_{a,b}}$ and $\lambda_{Y_{a,b}}$ respectively. It is understood that $a + b$ is much less than $n$ whenever this is used.

It is easy to see, from the cochain complex, [1] I § 6, of $S^{r,s}/S^{0,s}$ that if $\tilde{\omega}$ is any generic coefficient system with a $G$-action $g$ on $\tilde{\omega}(G)$ then

$$H^k(\frac{S^{r,s}}{S^{0,s}}; \tilde{\omega}) \cong \begin{cases} 0 & \text{if } 0 < k \leq s \text{ or } r + s < k \\ \text{Ker}(1 + (-1)^{k-s}g) & \text{if } s < k < r \\ \text{Im}(1 + (-1)^{k-s}g) & \text{if } k = r + s \end{cases}$$

Note that the groups $\lambda_{Y_{a,b}}$ are the same for all partitions $\alpha$ of $k$. I.e., by Bredon's classification theorem [1] II (2.11)

$$\lambda_{Y_{k+q,k-q}} \equiv \frac{Z}{(1 + (-1)^{q+1})Z}$$

$$\lambda_{Y_{h+q+t,k-t}} \equiv \begin{cases} 0 & \text{even} \\ Z_2 & \text{odd} \end{cases}$$

$$\lambda_{Y_{l,m}} = 0 \text{ if } l + m < 2k.$$

From this computation the main result may now be deduced. Let $\psi$ denote the forgetful functor.

**Theorem.** $u_\alpha \in \text{Image } \{\psi: \lambda U_{k+q,k-q} \to \mathbb{Z}_{2k}\}$ only if $\alpha \in d(k)$.

**Proof.** Suppose $u_\alpha$ is in the image of $\psi$. Consider the com-
mutative diagram with exact row (see [3], p. 286 for definitions of \(\alpha\), \(\beta\), and \(\psi\)):

\[
\begin{array}{c}
\ldots \longrightarrow \lambda Y_{k+q+1,k-q} \longrightarrow \lambda Y_{k+q,k-q} \longrightarrow \pi_{2n+2k}(Y^a) \longrightarrow \lambda Y_{k+q+1,k-q-1} \\
\end{array}
\]

If \(q\) were odd, the lower \(\psi\) is zero. By (c iv) the upper \(\psi\) is zero and \(u_a = 0\), a contradiction. Now suppose \(q\) is even. The exact row then is 0 \(\rightarrow\) \(Z \rightarrow Z \rightarrow Z \rightarrow 0\), so that \(e^a(G)\) maps \(u_a\) to an even multiple of the generator and by (c iv), \(\alpha \in d(k)\).

**Corollary.** Image \(\psi \subseteq I\).

**Proof.** By ([4], (4.1)), \(2u_a \in \text{Image } \psi\) for every \(\alpha\).

Then if \(w \in \text{Image } \psi\), subtract off even multiples of group generators until we have \(w = 2w' + u_{\alpha_1} + u_{\alpha_2} + \cdots + u_{\alpha_t}\). Now construct diagram (2.1) for \(\alpha\) successively equal to \(\alpha_1, \ldots, \alpha_t\). This shows that \(\alpha_i \in d(k), \ldots, \alpha_t \in d(k)\), and the corollary is proved.

As a corollary of the construction in [5] § 3 there are free exotic conjugations on representatives \(\bar{u}_i\), \(t = 2^j - 1\), showing that Image \(\{\psi: \lambda U_{t+q,t-q} \rightarrow Z_{2k}\}\) contains \(u_i\) provided \(q\) divisible by \(2^{k(t+2)}\). Since the image of a forgetful functor is an ideal in \(\mathbb{Z}\), this shows:

**Corollary.** Image \(\{\psi: \lambda U_{t+q,t-q} \rightarrow Z_{2k}\} = I_{2k}\) if \(t = 2^j - 1 \leq k < 2^{j+1} - 1\) and \(q\) divisible by \(2^{k(t+2)}\). \(\phi(m)\) is the familiar number equal to the number of integers \(s, 0 < s \leq m\) with \(s \equiv 0, 1, 2, 4 \pmod{8}\).

3. The construction. Recall Bredon's procedure for killing the homotopy groups of a \(G\)-space \(X\), with \(\bar{\omega}_t(X, x_0) = \bar{\omega}_t(X, x_0) = 0\). Let \(T\) be some \(G\)-set and \(F(T)\) the free abelian \(G\)-module on \(T\) such that \(\text{Hom}(F(T), \bar{\omega}_t(X))\) contains an epimorphism \(A_r\). By use of [2], Chapter II, (2.11), take a representative \(a_r: S^r(T^+) \rightarrow X\) and define \(X_{r+1}\) by the equivariant Puppe sequence,

\[
S^r(T^+) \xrightarrow{a_r} X \xrightarrow{j} X_{r+1} \longrightarrow S^{r+1}(T^+) \longrightarrow \ldots
\]

Bredon shows, [2], (6.6), that

\[
j_\ast: \bar{\omega}_t(X) \longrightarrow \bar{\omega}_t(X_{r+1})\]

is an isomorphism for

\(0 \leq t \leq r - 1\) and \(\bar{\omega}_t(X_{r+1}) = 0\).
In this construction of $Y^\alpha$ there are at most two $r$ where $A_r$ is not taken to be an epimorphism. To begin, let $\alpha$ be a partition of $k \geq 0$ and take $n > 2k - 1$ so that $\pi_{n+k}(MO(n)) = \tilde{\omega}_{n+k}(MU(n))(\frac{G}{G}) \cong \mathcal{N}_k$ and $\pi_{2n+2k}(MU(n)) = \tilde{\omega}_{2n+2k}(MU(n))(\frac{G}{G}) \cong \mathbb{Z}_{2k}$. If $\alpha$ is dyadic let $n_\alpha \in \mathcal{N}_k$ denote the zero element. Regard $n\alpha$ and $u\alpha$ as elements of $Y_o = MU(n)$ and let all $A_r$ be epimorphisms $0 < r < n + k$.

Let $Y_o = MU(n)$ and let all $A_r$ be epimorphisms $0 < r < n + k$. Denote the composition of the inclusions by $E_r: MU(n) = Y_o \subset \cdots \subset Y_r$. If $\alpha$ is dyadic, let $A_r$ be epimorphisms $0 < r < 2n + 2k$; if not let $A_{n+k}$ be defined as follows. Let $T_{2n+2k}(Y_{2n+2k-1})$ and $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G})$ be denoted $F$. Define $T_{2n+2k}$ to be the $G$-set of elements in the union of the sets $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G})$ except $E_{n+k}(n\alpha)$ and all elements in $\tilde{\omega}_{n+k}(Y_{n+k-1}) \times (\frac{G}{G})$. Take $A_{n+k}$ to be the natural homomorphism defined by extending the $G$-set inclusion $T_{n+k} \cong \tilde{\omega}_{n+k}(Y_{n+k-1})$. Now let $A_r$, $n + k < r < 2n + 2k$, be epimorphisms. Let the free cyclic summand containing $E_{2n+2k-1}(u\alpha)$ in $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G})$ be denoted $F$. Define $T_{2n+2k}$ to be the $G$-set of elements in the union of the sets $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G})$ and $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G}) - F$, and define $A_{2n+2k}$ to be the natural induced homomorphism. To define $Y_r$, $2n + 2k < r$, let $A_r$ be epimorphisms. This defines $Y^\alpha$ as a limit of $G$-complexes $MU(n) = Y_o \subset Y_1 \subset \cdots$. Let $e^\alpha: MU(n) \rightarrow Y^\alpha$ be the inclusion.

It is clear that (c i) and (iii) are satisfied by this construction. To check the others some notation will be required. Let $g: S^{2n+2k} \rightarrow MU(n)$ be some representative for $u\alpha$, transverse regular on $BU(n) \subset MU(n)$ and let $M\alpha = g^{-1}(BU(n))$. Let $v\alpha \in \tilde{H}^*(MU(n); Z)$ denote the universal Thom class and $s\alpha \in \tilde{H}^{2k}(BU(n); Z)$ the symmetric function associated to $\alpha$ in the universal Chern classes $c_1, c_2, \cdots$. Let $f: MU(n) \rightarrow K(Z, 2n + 2k)$ represent $s\alpha \cup v\alpha \in \tilde{H}^{2n+2k}(MU(n); Z)$. It is well-known that the degree defined by $f \circ g$ is the normal characteristic number of $M\alpha, s\alpha(v\alpha)$.

The $G$-action of conjugation sends $c_i$ to $-c_i$, so by the splitting principle $c_i$ is sent to $(-1)c_i, v\alpha$ to $(-1)v\alpha$ and $s\alpha \cup v\alpha$ to $(-1)^{i+k}s\alpha \cup v\alpha$. However, this determines the $G$-action on homology which, through the Hurewicz isomorphism, gives the $G$-action on $\pi_{2n+2k}(MU(n))$. To check the remainder of (c ii) we attempt to extend the map $f$ to a map $h: Y^\alpha \rightarrow K(Z, 2n + 2k)$.

The preceding construction shows that an extension of $f$ to $f''': Y_{2n+2k-1} \rightarrow K(Z, 2n + 2k)$ exists for dimensional reasons. Thus there is an integer, $N \neq 0$, such that $N \cdot f''(E_{2n+2k-1}(u\alpha)) = f\alpha(u\alpha)$ in $\pi_{2n+2k}(K(Z, 2n + 2k))$. Note that this is justifies the preceding claim that $E_{2n+2k-1}(u\alpha)$ lies in an infinite cyclic summand in $\tilde{\omega}_{2n+2k}(Y_{2n+2k-1})(\frac{G}{G})$. The preceding construction shows that an extension of $f$ to $f'''$ exists for dimensional reasons.
Since \( n + k \) may be taken odd, \( F \) has only one fixed point, 0. Thus, in the construction, Image \( A_{2n+2k} \) and \( F \) have only 0 in common. But \( f^s'' \) lives on \( F \), so an extension \( f': Y_{2n+2k} K(Z, 2n + 2k) \) exists. The desired extension, \( h \), exists now by dimensional considerations and the following homotopy diagram commutes.

\[
\begin{array}{ccc}
\pi_{2n+2k}(S^{2n+2k}) & \xrightarrow{(e^s \circ g)^*} & \pi_{2n+2k}(Y^s) \\
g^s \downarrow & & \downarrow h^s \\
\pi_{2n+2k}(MU(n)) & \xrightarrow{f^s} & \pi_{2n+2k}(K(Z, 2n + 2k))
\end{array}
\]

Since \( f^s \) carries a generator to nonzero multiple of the generator, \( s_a(u_a) \cdot g \), we see that \( \pi_{2n+2k}(Y^s) \) cannot be finite. By construction, it is cyclic on one generator and this completes the verification of (c ii).

From this diagram, note that \( e^s \) carries \( u_a \) to some multiple of the generator, \( y \), of \( \pi_{2n+2k}(Y^s) \), \( e^s(u_a) = My \). By commutativity, \( M \) divides \( s_a(u_a) \). But if \( \alpha \in n(k) \), \( s_a(u_a) \) is odd; thus \( M \) is odd and (c iv) is verified.

**References**


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Kenneth Abernethy, *On characterizing certain classes of first countable spaces by open mappings* .......................................................... 319
Ross A. Beaumont and Donald Lawver, *Strongly semisimple abelian groups* .................. 327
Gerald A. Beer, *The index of convexity and parallel bodies* ........................................... 337
Victor P. Camilo and Kent Ralph Fuller, *On Loewy length of rings* ................................. 347
Stephen LaVern Campbell, *Linear operators for which $T^*T$ and $TT^*$ commute.* .......... 355
Charles Kam-Tai Chui and Philip Wesley Smith, *Characterization of a function by certain infinite series it generates* .................................................. 363
Allan L. Edelson, *Conjugations on stably almost complex manifolds* ............................. 373
Patrick John Fleury, *Hollow modules and local endomorphism rings* .............................. 379
Jack Tilden Goodykoontz, Jr., *Connectedness im kleinen and local connectedness in $2^X$ and $C(X)$* ................................................... 387
Robert Edward Jamison, II, *Functional representation of algebraic intervals* .......... 399
Athanassios G. Kartsatos, *Nonzero solutions to boundary value problems for nonlinear systems* .......................................................... 425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, *Torus group actions on simply connected manifolds* ................................................. 435
David Anthony Klarner and R. Rado, *Arithmetic properties of certain recursively defined sets* .......................................................... 445
Ray Alden Kunze, *On the Frobenius reciprocity theorem for square-integrable representations* ................................................. 465
John Lagnese, *Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space* ........................................ 473
Teck Cheong Lim, *A fixed point theorem for families on nonexpansive mappings* ........... 487
Lewis Lum, *A quasi order characterization of smooth continua* ....................................... 495
Andy R. Magid, *Principal homogeneous spaces and Galois extensions* .......................... 501
Charles Alan McCarthy, *The norm of a certain derivation* .............................................. 515
Louise Elizabeth Moser, *On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot* ................................................. 519
Gordon L. Nipp, *Quaternion orders associated with ternary lattices* .............................. 525
Anthony G. O’Farrell, *Equiconvergence of derivations* ............................................... 539
Dorte Olesen, *Derivations of AW*-algebras are inner* ................................................... 555
Dorte Olesen and Gert Kjærgaard Pedersen, *Derivations of C*-algebras have semi-continuous generators* ........................................... 563
Duane O’Neill, *On conjugation cobordism* ................................................................. 573
Chull Park and S. R. Paranjape, *Probabilities of Wiener paths crossing differentiable curves* .......................................................... 579
Edward Ralph Rozema, *Almost Chebyshev subspaces of $L^1(\mu; E)$* .......................... 585
Lesley Millman Sibner and Robert Jules Sibner, *A note on the Atiyah-Bott fixed point formula* .......................................................... 605
Betty Salzberg Stark, *Irreducible subgroups of orthogonal groups generated by groups of root type 1* ................................................... 611
N. Stavvakas, *A note on starshaped sets, (k)-extreme points and the half ray property* .......................................................... 627
Carl E. Swenson, *Direct sum subset decompositions of Z* ............................................. 629
Stephen Tefteller, *A two-point boundary problem for nonhomogeneous second order differential equations* ........................................... 635
Robert S. Wilson, *Representations of finite rings* .......................................................... 643