

# Pacific Journal of Mathematics

**PROBABILITIES OF WIENER PATHS CROSSING  
DIFFERENTIABLE CURVES**

CHULL PARK AND S. R. PARANJPE

## PROBABILITIES OF WIENER PATHS CROSSING DIFFERENTIABLE CURVES

C. PARK AND S. R. PARANJAPE

Let  $\{W(t); t \geq 0\}$  be the standard Wiener process. The probabilities  $P[\sup_{0 \leq t \leq T} W(t) \geq b]$  and  $P[\sup_{0 \leq t \leq T} W(t) - at \geq b]$  are well known. This paper gives the probabilities of the type  $P[\sup_{0 \leq t \leq T} W(t) - f(t) \geq b]$  for a large class of differentiable functions  $f(t)$  by the use of integral equation techniques.

1. Introduction. Let  $\{W(t), t \geq 0\}$  be the standard Wiener process such that (i)  $P[W(0) = 0] = 1$ , (ii)  $EW(t) = 0$  for all  $t \geq 0$ , and (iii)  $\text{Cov}[W(s), W(t)] = \min(s, t)$ . It is well known that for  $b \geq 0$

$$(1.1) \quad P[\sup_{0 \leq t \leq T} W(t) \geq b] = 2P[W(T) \geq b] = 2\psi(bT^{-1/2})$$

where

$$\psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2) du,$$

and that

$$(1.2) \quad \begin{aligned} &P[\sup_{0 \leq t \leq T} W(t) - at \geq b] \\ &= \psi[(aT + b)T^{-1/2}] + \exp(-2ab)\phi[(aT - b)T^{-1/2}], \end{aligned}$$

where  $\phi(x) = 1 - \psi(x)$ .

The identity (1.1) can be found in [2:392], [5:286], and [11:256] while the identity (1.2) can be found in [6], [7:348-349], and [9:80-82]. Doob [3:397-399] gives a very interesting proof of (1.2) for  $T = \infty$  case only. Shepp's proof for (1.2) is based on his transformation theorem in [7]. Cameron-Martin translation theorem in [1] also gives the same result using Shepp's argument.

The main purpose of this paper is to find the probability  $P[\sup_{0 \leq t \leq T} W(t) - f(t) \geq b]$  for a large class of functions  $f(t)$  differentiable in  $(0, T]$ , which is a generalization of the results (1.1) and (1.2). Durbin [4] gave an integral equation whose solution would be the required probability. However, it turned out to be that his integral equation could not be solved analytically, and hence he presented a numerical approximation method. After that Smith [8] introduced some new techniques to obtain an approximation for the probability. The present authors' integral equation gives explicit expression for the solution, while Durbin's and Smith's do not.

### 2. Statement of the result and proof.

**THEOREM.** For each  $T > 0$  let  $f(t)$  be continuous on  $[0, T]$ ,

*differentiable in  $(0, T)$ , and satisfy  $|f'(t)| \leq C/t^p$  ( $p < 1/2$ ) for some constant  $C$ . Then the probability  $P[\sup_{0 \leq t \leq T} W(t) - f(t) \geq b] \equiv F(T)$  is one if  $f(0) + b \leq 0$ , and otherwise it is given as the unique continuous solution of the integral equation*

$$(2.1) \quad F(T) = 2\Psi[(f(T) + b)T^{-1/2}] - 2 \int_0^T F(t)M(T, t)dt ,$$

where

$$\Psi(x) = (2\pi)^{-1/2} \int_x^\infty \exp(-u^2/2)du$$

and

$$(2.2) \quad M(z, t) = \begin{cases} (2\pi)^{-1/2} \frac{\partial}{\partial t} \int_{-\infty}^{[f(z)-f(t)](z-t)^{-1/2}} \exp(-u^2/2)du, & (0 \leq t < z \leq T) \\ 0, & (0 \leq z \leq t \leq T) . \end{cases}$$

More precisely for  $f(0) + b > 0$

$$(2.3) \quad \begin{aligned} & P[\sup_{0 \leq t \leq T} W(t) - f(t) \geq b] \\ &= h(T) + \sum_{n=1}^\infty 4^n \int_0^T K_n(T, t)h(t)dt , \end{aligned}$$

where

$$\begin{aligned} h(T) &= 2\Psi[(f(T) + b)T^{-1/2}] - 4 \int_0^T M(T, t)\Psi[(f(t) + b)t^{-1/2}]dt , \\ K_1(T, t) &= \int_t^T M(T, z)M(z, t)dz , \end{aligned}$$

and

$$K_{n+1}(T, t) = \int_t^T K_n(T, z)K_1(z, t)dz .$$

*Proof.* If  $f(0) + b \leq 0$ , then since  $W(0) = 0$  a.s., it is obvious that the probability is one. Now, let  $\tau = \tau(\omega)$  be the first hitting time of the curve  $f(t) + b$  by the sample path  $W(t, \omega)$ , that is to say that  $W(\tau, \omega) = f(\tau) + b$ , and if  $0 \leq t < \tau$ , then  $W(t, \omega) < f(t) + b$ . If  $W(t, \omega)$  never reaches the curve  $f(t) + b$ , then we simply set  $\tau = \infty$ . Thus

$$\begin{aligned} F(T) &= P[W(T) \geq f(T) + b] \\ &+ P[\sup_{0 \leq s \leq T} W(s) - f(s) \geq b, W(T) < f(T) + b] . \end{aligned}$$

Using the fact that  $P[\tau \leq t] = P[\sup_{0 \leq s \leq t} W(s) - f(s) \geq b] \equiv F(t)$  and the notation in the theorem, we obtain

$$\begin{aligned}
 F(T) &= \Psi[(f(T) + b)T^{-1/2}] \\
 &+ \int_0^T P[W(T) < f(T) + b \mid \tau = t]dF(t) \\
 &= \Psi[(f(T) + b)T^{-1/2}] \\
 &+ \int_0^T P[W(T) - W(t) < f(T) - f(t) \mid \tau = t]dF(t) .
 \end{aligned}$$

Since the increment  $W(T) - W(t)$  is independent of the condition  $\tau = t$ , it follows that

$$\begin{aligned}
 F(T) &= \Psi[(f(T) + b)T^{-1/2}] \\
 &+ \int_0^T \Phi[(f(T) - f(t))(T - t)^{-1/2}]dF(t) ,
 \end{aligned}$$

where  $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-u^2/2)du$ . As  $\lim_{t \uparrow T} [f(T) - f(t)](T - t)^{-1/2} = 0$ , integration by parts yields (interpreting the integral in improper sense)

$$F(T) = \Psi[(f(T) + b)T^{-1/2}] + \frac{1}{2}F(T) - \int_0^T F(t)M(T, t)dt ,$$

from which (2.1) follows.

To solve the integral equation (2.1) rewrite  $M(z, t)$  by the use of (2.2)

$$(2.4) \quad M(z, t) = \begin{cases} (2\pi)^{-1/2}(z - t)^{-1/2} \left[ -f'(t) + \frac{f(z) - f(t)}{2(z - t)} \right] \exp \left\{ -\frac{[f(z) - f(t)]^2}{2(z - t)} \right\} & \text{if } 0 \leq t < z \leq T , \\ 0 & \text{if } 0 \leq z \leq t \leq T . \end{cases}$$

Apparently  $M(z, t)$  is not square integrable on  $[0, T]^2$ . Hence the integral equation (2.1) can not be solved by usual methods for Volterra integral equations of the second kind (see Tricomi [10, pp.10-15]). However, using the expression (2.1) for  $F(t)$  in the right-hand side of (2.1), we can rewrite (2.1) as:

$$F(T) = G(T) - 2 \int_0^T M(T, z) \left[ G(z) - 2 \int_0^z F(t)M(z, t)dt \right] dz ,$$

where  $G(T) = 2\Psi[(f(T) + b)T^{-1/2}]$ . Thus the change of order of integration gives

$$(2.5) \quad \begin{aligned}
 F(T) &= G(T) - 2 \int_0^T M(T, t)G(t)dt \\
 &+ 4 \int_0^T F(t) \left[ \int_t^T M(T, z)M(z, t)dz \right] dt .
 \end{aligned}$$

Now, using the conditions on  $f(T)$  in the theorem and the Mean

Value Theorem, we obtain from (2.4) with suitable constants  $C_1$  and  $C_2$

$$\begin{aligned} & \left| \int_t^T M(T, z)M(z, t)dz \right| \\ & \leq C_1 \int_t^T (T-z)^{-1/2}(z-t)^{-1/2} \left[ |f'(z)| + \frac{C}{2}z^{-p} \right] \left[ |f'(t)| + \frac{C}{2}t^{-p} \right] dz \\ & \leq C_2 t^{-p} \int_t^T (T-z)^{-1/2}(z-t)^{-1/2} z^{-p} dz . \end{aligned}$$

The substitution  $z = t + (T-t)u$  in the above yields

$$\begin{aligned} \left| \int_t^T M(T, z)M(z, t)dz \right| & \leq C_2 t^{-p} \int_0^1 (1-u)^{-1/2} u^{-1/2} [uT + (1-u)t]^{-p} du \\ & \leq C_2 t^{-p} T^{-p} \int_0^1 (1-u)^{-1/2} u^{-1/2} u^{-p} du \\ & \leq (\text{const.}) t^{-p} T^{-p} . \end{aligned}$$

Thus the kernel  $\int_t^T M(T, z)M(z, t)dz$  in the integral equation (2.5) is indeed square integrable for any  $p < 1/2$ , and hence the integral equation has a unique continuous solution for  $F(T)$ , and the solution is given by (2.3) (see Tricomi [10, pp. 5-8]).

REMARK. In some special cases of  $f(t)$  the integral equation in the theorem can be solved more directly.

*Case 1.* If  $f(t) \equiv c$  in the theorem, then  $M(T, t) \equiv 0$  and hence  $F(T) = 2 \Psi[(c+b)T^{-1/2}]$  which agrees with (1.1).

*Case 2.* If  $f(t) = at$ , then

$$\begin{aligned} M(T, t) &= (2\pi)^{-1/2} \frac{\partial}{\partial t} \int_{-\infty}^{a\sqrt{T-t}} \exp(-u^2/2) du \\ &= \frac{-a}{2(2\pi)^{1/2}} (T-t)^{-1/2} \exp[-a^2(T-t)/2] \equiv N(T-t), \quad 0 \leq t < T . \end{aligned}$$

If we set  $G(T) \equiv 2\Psi[(aT+b)T^{-1/2}]$ , then the integral equation becomes

$$F(T) = G(T) - 2 \int_0^T F(t)N(T-t)dt .$$

Taking the Laplace transform ( $L[F(T)] = \int_0^\infty e^{-sT} F(T)dT$ ) of both sides, we get

$$L[F(T)] = L[G(T)] - 2L[F(T)]L[N(T)] ,$$

or

$$\begin{aligned} L[F(T)] &= L[G(T)]/\{1 + 2L[N(T)]\} \\ &= s^{-1} \exp[-ab - b(2s + a^2)^{1/2}]. \end{aligned}$$

Therefore,

$$F(T) = 1 - \Phi[(aT + b)T^{-1/2}] + \exp(-2ab)\Phi[(aT - b)T^{-1/2}]$$

which agrees with (1.2).

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