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**A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA**

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## A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

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Let  $f$  be a holomorphic self map of a compact complex analytic manifold  $X$ . The differential of  $f$  commutes with  $\bar{\partial}$  and, hence, induces an endomorphism of the  $\bar{\partial}$ -complex of  $X$ . If  $f$  has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map  $f$  near the fixed points. For example, if  $X$  is a curve, this Lefschetz number is the sum of the residues of  $(z - f(z))^{-1}$  at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let  $E$  be the  $\bar{\partial}$ -complex of a compact complex analytic manifold  $X$  of dimension  $n$ .

$$E: 0 \longrightarrow \Gamma(A^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(A^{0,1}) \longrightarrow \dots \xrightarrow{\bar{\partial}} \Gamma(A^{0,n}) \longrightarrow 0.$$

Since  $E$  is elliptic,  $H^i(X) = \ker \bar{\partial}_i / \text{im } \bar{\partial}_{i-1}$  is finite dimensional. Denote by  $T = \{T_i\}$  the endomorphism induced on  $E$  by the holomorphic map  $f$ , and by  $H^i T$  the resulting endomorphism on  $H^i(X)$ .

The Lefschetz number of  $f$  is then defined by

$$L(f) = \sum_{i=0}^n (-1)^i \text{tr } H^i T$$

and the finite dimensionality of the spaces  $H^i(X)$  insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating  $L(f)$  to that of finding a good parametrix for the  $\bar{\partial}$ -operator. In fact, let us suppose we can find operators  $P_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1})$ ,  $i = 1, \dots, n$ , having the property that

$$(1) \quad P_{i+1} \bar{\partial}_i + \bar{\partial}_{i-1} P_i = I - S_i$$

where  $S_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i})$  are integral operators with sufficiently smooth kernels. Observe that if  $\omega \in \Gamma(A^{0,i})$  is in the kernel of  $\bar{\partial}_i$ , then the left-hand side of (1) is a co-boundary. Hence,  $H^i I - H^i S$  is the zero-endomorphism on homology. Similarly, since  $T$  commutes

with  $\bar{\partial}$

$$T_i(P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i) = T_iP_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}T_{i-1}P_i = T_i - T_iS_i$$

so that  $H^i T = H^i TS$ . Therefore,

$$(2) \quad L(f) = \sum_{i=0}^n (-1)^i \text{tr} H^i(TS).$$

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

$$(3) \quad L(f) = \sum_{i=0}^n (-1)^i \text{tr} H^i TS = \sum_{i=0}^n (-1)^i \text{tr} T_i S_i$$

provided the right-hand side is finite. This will be the case if the kernels of the operators  $S_i$  are sufficiently smooth along the graph of  $f$ .

To carry out the above procedure and evaluate  $L(f)$  we make an explicit choice of the operators  $P_i$ .

2. The most natural way to choose a parametrix on  $X$  is to glue together the local fundamental solutions of the  $\bar{\partial}$ -operator using partitions of unity. Given any finite open covering  $\{U_\alpha\}$  of  $X$ , there are, in each  $U_\alpha$ , integral operators  $Q_{\alpha,i}: \Gamma(A^{0,i}(U_\alpha)) \rightarrow \Gamma(A^{0,i-1}(U_\alpha))$   $i = 1, \dots, n$  such that for  $\omega \in C_0^\infty(U_\alpha)$

$$(4a) \quad \bar{\partial}Q_{\alpha,i}(\omega) = \omega - Q_{\alpha,i+1}(\bar{\partial}\omega)$$

$$(4b) \quad (Q_{\alpha,i}\omega)(z^\alpha) = \int_{U_\alpha} \omega(\zeta^\alpha) \wedge \Omega_i(z^\alpha, \zeta^\alpha)$$

where  $\Omega_i(z^\alpha, \zeta^\alpha) \in \Gamma(A^{0,i-1}(U_\alpha) \otimes A^{n,n-i}(U_\alpha))$  is a  $C^\infty$ -section off the diagonal and has an absolutely integrable singularity.

Let  $\Omega(z^\alpha, \zeta^\alpha) = \sum_{i=1}^n (-1)^i \Omega_i(z^\alpha, \zeta^\alpha)$ . This is an  $(n, n - 1)$  form on  $U_\alpha \times U_\alpha$  satisfying

$$(4c) \quad \bar{\partial}\Omega = 0.$$

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for  $\Omega$  appears near the end of § 3.

Suppose  $f$  has  $m$  isolated fixed points,  $P_1, \dots, P_m$ . Let  $U_k$  be a coordinate neighborhood containing  $P_k$ , chosen so that the sets  $U_k$  are mutually disjoint. Let  $N_k$  be a neighborhood of  $P_k$ , sufficiently small so that  $f^{-1}(N_k) \subset U_k$  ( $f$  is continuous and  $f(P_k) = P_k$ ). The collection  $U_1, \dots, U_m$  can be extended to a covering  $\{U_\alpha\}$  and a partition of unity  $\{\lambda_\alpha\}$  subordinate to this covering can be chosen such

that (for  $k = 1, \dots, m$ )

- (i)  $\text{supp } \lambda_k \subset N_k$
- (ii)  $\lambda_k = 1$  in a neighborhood of  $P_k$ .

Then  $\text{supp } \lambda_k \circ f \subset f^{-1}(N_k) \subset U_k$  and  $\lambda_k \circ f = 1$  in some (other) neighborhood of  $P_k$ .

Now choose nonnegative functions  $\sigma_\alpha \in C_0^\infty(U_\alpha)$  such that

- (iii)  $\sigma_\alpha = 1$  on  $\text{supp } \lambda_\alpha$   $\alpha \neq 1, \dots, m$
- (iv)  $\sigma_\alpha = 1$  on  $\{\text{supp } \lambda_\alpha\} \cup \{\text{supp } \lambda_\alpha \circ f\}$   $\alpha = 1, \dots, m$ .

Define  $P_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1})$  by

$$(5) \quad \begin{aligned} P_i \omega &= \sum_\alpha \lambda_\alpha Q_{\alpha,i}(\alpha_\alpha \omega) & i = 1, \dots, n \\ P_0 \omega &= 0. \end{aligned}$$

From (4a) we obtain

$$(6) \quad \begin{aligned} \bar{\partial} P_i \omega + P_{i+1} \bar{\partial} \omega &= \omega + \sum_\alpha \bar{\partial} \lambda_\alpha Q_{\alpha,i}(\sigma_\alpha \omega) - \sum_\alpha \lambda_\alpha Q_{\alpha,i+1}(\bar{\partial} \sigma_\alpha \wedge \omega) \\ &= \omega - S_i \omega & i = 0, \dots, n \end{aligned}$$

where

$$\begin{aligned} S_i \omega(z) &= - \sum_\alpha \bar{\partial} \lambda_\alpha(z) \int_{U_\alpha} \sigma_\alpha(\zeta) \omega(\zeta) \wedge \Omega_i(z, \zeta) \\ &\quad + \sum_\alpha \lambda_\alpha(z) \int_{U_\alpha} \bar{\partial} \sigma_\alpha(\zeta) \wedge \omega(\zeta) \wedge \Omega_{i+1}(z, \zeta). \end{aligned}$$

(We consistently suppress the coordinate superscript when possible: writing, for example,  $\sigma_\alpha(\zeta)$  for  $\sigma_\alpha(\zeta^\alpha)$ .)

3. Because of the construction of the covering and the patching functions, the kernel of  $S_i$  is smooth in a neighborhood of the graph of  $f$ . In fact, if  $\alpha > m$ , then  $f$  has no fixed points in  $U_\alpha$  and therefore,  $\zeta - f(\zeta)$  is bounded away from zero so that  $\Omega_i(f(\zeta), \zeta)$  is a  $C^\infty$ -function in  $U_\alpha$ . Furthermore, in  $U_k, k \leq m$ , we have chosen  $\lambda_k$  so that  $\lambda_k(f(\zeta)) \equiv 1$  in a neighborhood of  $P_k$ . Then,  $\bar{\partial} \lambda_k(f(\zeta)) = 0$  near  $\zeta = f(\zeta)$ . Also, since  $\sigma_k(\zeta) \equiv 1$  on the support of  $\lambda_k(f(\zeta))$ , we have  $\bar{\partial} \sigma_\alpha(\zeta) = 0$  near  $\zeta = f(\zeta)$ . Thus, the kernel of  $S_i$  may be evaluated along the graph of  $f$  to obtain:

$$\begin{aligned} \sum_0^n (-1)^i \text{tr}(T_i S_i) &= \sum_\alpha \left\{ \sum_1^n (-1)^{i+1} \int_{U_\alpha} \bar{\partial} \lambda_\alpha(f(\zeta)) \wedge \sigma_\alpha(\zeta) \Omega_i(f(\zeta), \zeta) \right\} \\ &\quad + \sum_\alpha \left\{ \sum_0^{n-1} (-1)^i \int_{U_\alpha} \lambda_\alpha(f(\zeta)) \bar{\partial} \sigma_\alpha(\zeta) \wedge \Omega_{i+1}(f(\zeta), \zeta) \right\} \\ &= - \sum_\alpha \int_{U_\alpha} \bar{\partial} \{ \lambda_\alpha(f(\zeta)) \sigma_\alpha(\zeta) \} \wedge \sum_1^n (-1)^i \Omega_i(f(\zeta), \zeta) \end{aligned}$$

from which

$$(7) \quad L(f) = -\sum_{\alpha} \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\} \wedge \Omega(f(\zeta), \zeta).$$

In  $U_{\alpha}$ , for  $\alpha > m$ ,  $f$  has no fixed points. Using (4c), integrating by parts, and making use of the fact that  $\sigma_{\alpha}$  has compact support in  $U_{\alpha}$ , we have

$$\begin{aligned} \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\} \wedge \Omega(f(\zeta), \zeta) &= \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\Omega(f(\zeta), \zeta)\} \\ &= \int_{\partial U_{\alpha}} \lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\Omega(f(\zeta), \zeta) \equiv 0. \end{aligned}$$

For  $\alpha = k \leq m$ , let  $B_k$  be a ball around  $P_k$  on which  $\lambda_k(f(\zeta)) \equiv 1$ . Since  $\sigma_k(\zeta) \equiv 1$  on the support of  $\lambda_k(f(\zeta))$ ,

$$(8) \quad \begin{aligned} L(f) &= -\sum_{k=1}^m \int_{U_{k-B_k}} \bar{\partial}\{\lambda_k(f(\zeta))\Omega(f(\zeta), \zeta)\} = \sum_{k=1}^m \int_{\partial B_k} \lambda_k(f(\zeta))\Omega(f(\zeta), \zeta) \\ &= \sum_{k=1}^m \int_{\partial B_k} \Omega(f(\zeta), \zeta). \end{aligned}$$

Using local coordinates in  $B_k$ , let  $g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k)$ ,  $i = 1, \dots, n$ . Then, for  $n > 1$ ,

$$\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} |\zeta^k - z^k|^{-2n} \sum_{i=1}^n (-1)^{i+1} \overline{\zeta_i^k - z_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^n \overline{d\zeta_j^k - dz_j^k} \bigwedge_{l=1}^n d\zeta_l^k$$

and

$$(9) \quad L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^m \int_{\partial B_k} (\sum |g_i^k|^2)^{-n} \sum_{i=1}^n (-1)^{i+1} \overline{g_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^n \overline{dg_j^k} \bigwedge_{l=1}^n d\zeta_l^k$$

which is the desired formula.

For  $n = 1$ ,  $\Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k)$  and

$$L(f) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\partial B_k} \frac{d\zeta^k}{\zeta^k - f(\zeta^k)} = \sum_{f(\zeta)=z} \text{Res}(\zeta - f(\zeta))^{-1}.$$

NOTE. Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

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