Pacific Journal of Mathematics

A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

LESLEY MILLMAN SIBNER AND ROBERT JULES SIBNER

Vol. 53, No. 2 April 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 2, 1974

A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

L. M. SIBNER and R. J. SIBNER

Let f be a holomorphic self map of a compact complex analytic manifold X. The differential of f commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$-complex of X. If f has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map f near the fixed points. For example, if X is a curve, this Lefschetz number is the sum of the residues of $(z - f(z))^{-1}$ at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let E be the $\bar{\partial}$-complex of a compact complex analytic manifold X of dimension n.

$$E: 0 \longrightarrow \Gamma(\Lambda^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,1}) \longrightarrow \cdots \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,n}) \longrightarrow 0 .$$

Since E is elliptic, $H^i(X) = \ker \bar{\partial}_i / im \bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on E by the holomorphic map f, and by $H^j T$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of f is then defined by

$$L(f) = \sum_{i=0}^{n} (-1)^i tr H^i T$$

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$-operator. In fact, let us suppose we can find operators $P_i: \Gamma(\Lambda^{0,i}) \to \Gamma(\Lambda^{0,i-1})$, $i = 1, \cdots, n$, having the property that

$$(1) \qquad P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i = I - S_i$$

where $S_i: \Gamma(\Lambda^{0,i}) \to \Gamma(\Lambda^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(\Lambda^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^i I - H^i S$ is the zero-endomorphism on homology. Similarly, since T commutes with $\bar{\partial}$

$$T_i(P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i) = T_iP_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}T_{i-1}P_i = T_i - T_iS_i$$

so that $H^iT = H^iTS$. Therefore,

$$L(f) = \sum_{i=0}^{n}(-1)_i trH^i(TS) . \tag{2}$$

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

$$L(f) = \sum_{i=0}^{n}(-1)^i trH^iTS = \sum_{i=0}^{n}(-1)^i trT_iS_i \tag{3}$$

provided the right-hand side is finite. This will be the case if the kernels of the operators S_i are sufficiently smooth along the graph of f.

To carry out the above procedure and evaluate $L(f)$ we make an explicit choice of the operators P_i.

2. The most natural way to choose a parametrix on X is to glue together the local fundamental solutions of the $\bar{\partial}$-operator using partitions of unity. Given any finite open covering $\{U_\alpha\}$ of X, there are, in each U_α, integral operators $Q_{\alpha,i}\colon \Gamma(\Lambda^{0,i}(U_\alpha)) \to \Gamma(\Lambda^{0,i-1}(U_\alpha))$ $i = 1, \cdots, n$ such that for $\omega \in C_0^\infty(U_\alpha)$

$$\bar{\partial}Q_{\alpha,i}(\omega) = \omega - Q_{\alpha,i+1}(\bar{\partial}\omega) \tag{4a}$$

$$(Q_{\alpha,i}\omega)(z^\alpha) = \int_{U_\alpha}\omega(\zeta^\alpha) \wedge \Omega_i(z^\alpha, \zeta^\alpha) \tag{4b}$$

where $\Omega_i(z^\alpha, \zeta^\alpha) \in \Gamma(\Lambda^{0,i-1}(U_\alpha) \otimes \Lambda^{n,n-i}(U_\alpha))$ is a C^∞-section off the diaganal and has an absolutely integrable singularity.

Let $\Omega(z^\alpha, \zeta^\alpha) = \sum_{i=1}^{n}(-1)^i\Omega_i(z^\alpha, \zeta^\alpha)$. This is an $(n, n-1)$ form on $U_\alpha \times U_\alpha$ satisfying

$$\bar{\partial}\Omega = 0 . \tag{4c}$$

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for Ω appears near the end of § 3.

Suppose f has m isolated fixed points, $P_1, \cdots, P_m$. Let U_k be a coordinate neighborhood containing P_k, chosen so that the sets U_k are mutually disjoint. Let N_k be a neighborhood of P_k, sufficiently small so that $f^{-1}(N_k) \subset U_k$ (f is continuous and $f(P_k) = P_k$). The collection $U_1, \cdots, U_m$ can be extended to a covering $\{U_\alpha\}$ and a partition of unity $\{\lambda_\alpha\}$ subordinate to this covering can be chosen such that (for $k = 1, \cdots, m$)

(i) $\operatorname{supp} \lambda_k \subset N_k$

(ii) $\lambda_k = 1$ in a neighborhood of P_k.

Then $\operatorname{supp} \lambda_k \circ f \subset f^{-1}(N_k) \subset U_k$ and $\lambda_k \circ f = 1$ in some (other) neighborhood of P_k.

Now choose nonnegative functions $\sigma_\alpha \in C_0^\infty(U_\alpha)$ such that

(iii) $\sigma_\alpha = 1$ on $\operatorname{supp} \lambda_\alpha$ $\alpha \neq 1, \cdots, m$

(iv) $\sigma_\alpha = 1$ on $\{\operatorname{supp} \lambda_\alpha\} \cup \{\operatorname{supp} \lambda_\alpha \circ f\}$ $\alpha = 1, \cdots, m$.

Define $P_i\colon \Gamma(\Lambda^{0,i}) \to \Gamma(\Lambda^{0,i-1})$ by

$$(5) \qquad P_i\omega = \sum_\alpha \lambda_\alpha Q_{\alpha,i}(\alpha_\alpha \omega) \qquad i = 1, \cdots, n$$

$$P_0\omega = 0 .$$

From (4a) we obtain

$$(6) \qquad \bar\partial P_i\omega + P_{i+1}\bar\partial\omega = \omega + \sum_\alpha \bar\partial\lambda_\alpha Q_{\alpha,i}(\sigma_\alpha\omega) - \sum_\alpha \lambda_\alpha Q_{\alpha,i+1}(\bar\partial\sigma_\alpha \wedge \omega)$$

$$= \omega - S_i\omega \qquad i = 0, \cdots, n$$

where

$$S_i\omega(z) = -\sum_\alpha \bar\partial\lambda_\alpha(z)\int_{U_\alpha} \sigma_\alpha(\zeta)\omega(\zeta) \wedge \Omega_i(z, \zeta)$$

$$+ \sum_\alpha \lambda_\alpha(z)\int_{U_\alpha} \bar\partial\sigma_\alpha(\zeta) \wedge \omega(\zeta) \wedge \Omega_{i+1}(z, \zeta) .$$

(We consistently suppress the coordinate superscript when possible: writing, for example, $\sigma_\alpha(\zeta)$ for $\sigma_\alpha(\zeta^\alpha)$.)

3. Because of the construction of the covering and the patching functions, the kernel of S_i is smooth in a neighborhood of the graph of f. In fact, if $\alpha > m$, then f has no fixed points in U_α and therefore, $\zeta - f(\zeta)$ is bounded away from zero so that $\Omega_i(f(\zeta), \zeta)$ is a C^∞-function in U_α. Furthermore, in U_k, $k \leqq m$, we have chosen λ_k so that $\lambda_k(f(\zeta)) \equiv 1$ in a neighborhood of P_k. Then, $\bar\partial\lambda_k(f(\zeta)) = 0$ near $\zeta = f(\zeta)$. Also, since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$, we have $\bar\partial\sigma_\alpha(\zeta) = 0$ near $\zeta = f(\zeta)$. Thus, the kernel of S_i may be evaluated along the graph of f to obtain:

$$\sum_0^n (-1)^i tr(T_iS_i) = \sum_\alpha\left\{\sum_1^n (-1)^{i+1}\int_{U_\alpha} \bar\partial\lambda_\alpha(f(\zeta)) \wedge \sigma_\alpha(\zeta)\Omega_i(f(\zeta), \zeta)\right\}$$

$$+ \sum_\alpha\left\{\sum_0^{n-1} (-1)^i\int_{U_\alpha} \lambda_\alpha(f(\zeta))\bar\partial\sigma_\alpha(\zeta) \wedge \Omega_{i+1}(f(\zeta), \zeta)\right\}$$

$$= -\sum_\alpha\int_{U_\alpha} \bar\partial\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \sum_1^n (-1)^i\Omega_i(f(\zeta^\alpha), \zeta^\alpha)$$

from which

$$L(f) = -\sum_{\alpha} \int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \Omega(f(\zeta), \zeta) . \tag{7}$$

In U_α, for $\alpha > m$, f has no fixed points. Using (4c), integrating by parts, and making use of the fact that σ_α has compact support in U_α, we have

$$\begin{aligned}\int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \Omega(f(\zeta), \zeta) &= \int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta)\} \\ &= \int_{\partial U_\alpha} \lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta) \equiv 0 .\end{aligned}$$

For $\alpha = k \leqq m$, let B_k be a ball around P_k on which $\lambda_k(f(\zeta)) \equiv 1$. Since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$,

$$\begin{aligned}L(f) = -\sum_{k=1}^{m} \int_{U_k - B_k} \bar{\partial}\{\lambda_k(f(\zeta))\Omega(f(\zeta), \zeta)\} &= \sum_{k=1}^{m} \int_{\partial B_k} \lambda_k(f(\zeta))\Omega(f(\zeta), \zeta) \\ &= \sum_{k=1}^{m} \int_{\partial B_k} \Omega(f(\zeta), \zeta) .\end{aligned} \tag{8}$$

Using local coordinates in B_k, let $g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k)$, $i = 1, \cdots, n$. Then, for $n > 1$,

$$\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} |\zeta^k - z^k|^{-2n} \sum_{i=1}^{n} (-1)^{i+1} \overline{\zeta_i^k - z_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} (\overline{d\zeta_j^k} - \overline{dz_j^k}) \bigwedge_{l=1}^{n} d\zeta_l^k$$

and

$$L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{m} \int_{\partial B_k} (\Sigma |g_i^k|^2)^{-n} \sum_{i=1}^{n} (-1)^{i+1} \overline{g_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \overline{dg_j^k} \bigwedge_{l=1}^{n} d\zeta_l^k \tag{9}$$

which is the desired formula.

For $n = 1$, $\Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k)$ and

$$L(f) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\partial B_k} \frac{d\zeta^k}{\zeta^k - f(\zeta^k)} = \sum_{f(\zeta)=\zeta} \operatorname{Res}(\zeta - f(\zeta))^{-1} .$$

NOTE. Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

REFERENCES

1. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes, I and II*, Annals of Math., **86** (1967), 374-407; **88** (1968), 451-491.

2. Walter Koppelman, *The Cauchy integral for differential forms*; Bull. Amer. Math. Soc., **73** (1967), 554-556.

3. Ingo Lieb, *Die Cauchy-Riemannschen Differentialgleichungen auf streng pseudokonvexen Gebieten*, Math. Ann., **190** (1970), 6-44.

4. Nils Øvrelid, *Integral representation formulas and L^p-estimates for the $\bar{\partial}$-equation*, preprint.

5. Domingo Toledo, *On the Atiyah-Bott formula for isolated fixed points*, J. Differential Geometry, **8** (1973), 401–436.
6. Yue Lin L. Tong, *de Rham's integrals and Lefschetz fixed point formula for d'' cohomology*, Bull. Amer. Math. Soc., **78** (1972), 420–422.

Received May 29, 1973. The first author was supported in part by National Science Foundation grant GP-27960. The second author was supported in part by National Science Foundation grant GP-7952X3.

THE INSTITUTE FOR ADVANCED STUDY

Current addresses: L. M. Sibner
Polytechnic Institute of New York
Brooklyn, NY 11201
R. J. Sibner
City University of New York
Brooklyn College
Brooklyn, NY 11210

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 53, No. 2 April, 1974