A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

Lesley Millman Sibner and Robert Jules Sibner
A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA
L. M. SIBNER and R. J. SIBNER

Let $f$ be a holomorphic self map of a compact complex analytic manifold $X$. The differential of $f$ commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$-complex of $X$. If $f$ has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map $f$ near the fixed points. For example, if $X$ is a curve, this Lefschetz number is the sum of the residues of $(z - f(z))^{-1}$ at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let $E$ be the $\bar{\partial}$-complex of a compact complex analytic manifold $X$ of dimension $n$.

$$E: 0 \longrightarrow \Gamma(A^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(A^{0,1}) \longrightarrow \cdots \xrightarrow{\bar{\partial}} \Gamma(A^{0,n}) \longrightarrow 0.$$ Since $E$ is elliptic, $H^i(X) = \ker \bar{\partial}/\text{im} \bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on $E$ by the holomorphic map $f$, and by $H^i T$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of $f$ is then defined by

$$L(f) = \sum_{i=0}^{n} (-1)^i tr H^i T$$

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$-operator. In fact, let us suppose we can find operators $P_i: \Gamma(A^{0,i}) \to \Gamma(A^{0,i-1})$, $i = 1, \cdots, n$, having the property that

$$P_{i+1} \bar{\partial}_i + \bar{\partial}_{i-1} P_i = I - S_i \quad (1)$$

where $S_i: \Gamma(A^{0,i}) \to \Gamma(A^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(A^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^i I - H^i S$ is the zero-endomorphism on homology. Similarly, since $T$ commutes
with \( \bar{\partial} \)

\[
T_i(P_{t+1}\bar{\partial}_t + \bar{\partial}_{t-1}P_t) = T_iP_{t+1}\bar{\partial}_t + \bar{\partial}_{t-1}T_iP_t = T_i - T_iS_t
\]

so that \( H^iT = H^iTS \). Therefore,

\[
L(f) = \sum_{i=0}^{n} (-1)^i trH^i(TS).
\]

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

\[
L(f) = \sum_{i=0}^{n} (-1)^i trH^i(TS) = \sum_{i=0}^{n} (-1)^i trT_iS_t
\]

provided the right-hand side is finite. This will be the case if the kernels of the operators \( S_t \) are sufficiently smooth along the graph of \( f \).

To carry out the above procedure and evaluate \( L(f) \) we make an explicit choice of the operators \( P_t \).

2. The most natural way to choose a parametrix on \( X \) is to glue together the local fundamental solutions of the \( \bar{\partial} \)-operator using partitions of unity. Given any finite open covering \( \{U_a\} \) of \( X \), there are, in each \( U_a \), integral operators \( Q_{a,i}: \Gamma(A^{0,i}(U_a)) \to \Gamma(A^{0,i-1}(U_a)) \) \( i = 1, \ldots, n \) such that for \( \omega \in C_0^\infty(U_a) \)

\[
\bar{\partial}Q_{a,i}(\omega) = \omega - Q_{a,i+1}(\bar{\partial}\omega)
\]

\[
(Q_{a,i}\omega)(z^a) = \int_{U_a} \omega(\zeta^a) \wedge \Omega_i(z^a, \zeta^a)
\]

where \( \Omega_i(z^a, \zeta^a) \in \Gamma(A^{0,i-1}(U_a) \otimes A^{n,n-i}(U_a)) \) is a \( C^\infty \)-section off the diagonal and has an absolutely integrable singularity.

Let \( \Omega(z^a, \zeta^a) = \sum_{i=1}^{n} (-1)^i \Omega_i(z^a, \zeta^a) \). This is an \((n, n-1)\) form on \( U_a \times U_a \) satisfying

\[
\bar{\partial}\Omega = 0.
\]

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for \( \Omega \) appears near the end of § 3.

Suppose \( f \) has \( m \) isolated fixed points, \( P_1, \ldots, P_m \). Let \( U_k \) be a coordinate neighborhood containing \( P_k \), chosen so that the sets \( U_k \) are mutually disjoint. Let \( N_k \) be a neighborhood of \( P_k \), sufficiently small so that \( f^{-1}(N_k) \subset U_k \) (\( f \) is continuous and \( f(P_k) = P_k \)). The collection \( U_1, \ldots, U_m \) can be extended to a covering \( \{U_a\} \) and a partition of unity \( \{\lambda_a\} \) subordinate to this covering can be chosen such
that (for \( k = 1, \ldots, m \))

( i ) \( \text{supp} \lambda_k \subseteq N_k \)

( ii ) \( \lambda_k = 1 \) in a neighborhood of \( P_k \).

Then \( \text{supp} \lambda_k \circ f \subseteq f^{-1}(N_k) \subseteq U_k \) and \( \lambda_k \circ f = 1 \) in some (other) neighborhood of \( P_k \).

Now choose nonnegative functions \( \sigma_a \in C_0^\infty(U_a) \) such that

(iii) \( \sigma_a = 1 \) on \( \text{supp} \lambda_a \alpha \neq 1, \ldots, m \)

(iv) \( \sigma_a = 1 \) on \( \{ \text{supp} \lambda_a \} \cup \{ \text{supp} \lambda_a \circ f \} \alpha = 1, \ldots, m \).

Define \( P_i : \Gamma(A^{0,1}) \rightarrow \Gamma(A^{0,1}) \) by

(5) \[
P_i \omega = \sum_a \lambda_a Q_{a,i}(\alpha_a \omega) \quad i = 1, \ldots, n
\]

\( P_i \omega = 0 \).

From (4a) we obtain

(6) \[
\bar{\partial} P_i \omega + P_{i+1} \bar{\partial} \omega = \omega + \sum_a \bar{\partial} \lambda_a Q_{a,i}(\alpha_a \omega) - \sum_a \lambda_a Q_{a,i+1}(\partial \sigma_a \land \omega)
\]

\[
= \omega - S_i \omega \quad i = 0, \ldots, n
\]

where

\[
S_i \omega(z) = - \sum_a \bar{\partial} \lambda_a(z) \int_{U_a} \sigma_a(\zeta) \omega(\zeta) \land \Omega_i(z, \zeta)
\]

\[
+ \sum_a \lambda_a(z) \int_{U_a} \partial \sigma_a(\zeta) \land \omega(\zeta) \land \Omega_{i+1}(z, \zeta).
\]

(We consistently suppress the coordinate superscript when possible: writing, for example, \( \sigma_a(\zeta) \) for \( \sigma_a(\zeta_0) \).)

3. Because of the construction of the covering and the patching functions, the kernel of \( S_i \) is smooth in a neighborhood of the graph of \( f \). In fact, if \( \alpha > m \), then \( f \) has no fixed points in \( U_a \) and therefore, \( \zeta - f(\zeta) \) is bounded away from zero so that \( \Omega_i(f(\zeta), \zeta) \) is a \( C^\infty \)-function in \( U_a \). Furthermore, in \( U_k, k \leq m \), we have chosen \( \lambda_k \) so that \( \lambda_k(f(\zeta)) = 1 \) in a neighborhood of \( P_k \). Then, \( \bar{\partial} \lambda_k(f(\zeta)) = 0 \) near \( \zeta = f(\zeta) \). Also, since \( \sigma_a(\zeta) = 1 \) on the support of \( \lambda_a(f(\zeta)) \), we have \( \bar{\partial} \sigma_a(\zeta) = 0 \) near \( \zeta = f(\zeta) \). Thus, the kernel of \( S_i \) may be evaluated along the graph of \( f \) to obtain:

\[
\sum_a (-1)^i tr(T_i S_i) = \sum_a \left\{ \sum_i (-1)^{i+1} \int_{U_a} \bar{\partial} \lambda_a(f(\zeta)) \land \sigma_a(\zeta) \Omega_i(f(\zeta), \zeta) \right\}
\]

\[
+ \sum_a \left\{ \sum_i (-1)^i \int_{U_a} \lambda_a(f(\zeta)) \bar{\partial} \sigma_a(\zeta) \land \Omega_{i+1}(f(\zeta), \zeta) \right\}
\]

\[
= - \sum_a \int_{U_a} \bar{\partial} \lambda_a(f(\zeta)) \sigma_a(\zeta) \land \sum_i (-1)^i \Omega_i(f(\zeta), \zeta)
\]

from which
In $U_a$, for $\alpha > m$, $f$ has no fixed points. Using (4c), integrating by parts, and making use of the fact that $\sigma_\alpha$ has compact support in $U_a$, we have

$$
\int_{U_a} \delta(\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)) \land \Omega(f(\zeta), \zeta) = \int_{U_a} \delta(\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta))
$$

$$
= \int_{U_a} \lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta) = 0 .
$$

For $\alpha = k \leq m$, let $B_k$ be a ball around $P_k$ on which $\lambda_k(f(\zeta)) \equiv 1$. Since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$,

$$
L(f) = - \sum_{k=1}^{\infty} \int_{B_k} \delta(\lambda_k(f(\zeta))\sigma_k(\zeta)) \land \Omega(f(\zeta), \zeta) = \sum_{k=1}^{\infty} \int_{B_k} \lambda_k(f(\zeta))\Omega(f(\zeta), \zeta)
$$

Using local coordinates in $B_k$, let $g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k)$, $i = 1, \ldots, n$. Then, for $n > 1$,

$$
\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} \left| \zeta^k - z^k \right|^{-2n} \sum_{i=1}^{n} (-1)^{i+1} \zeta_i^k \wedge \frac{d\zeta_j^k - d\zeta_j^k}{\zeta_j^k - f(\zeta_j^k)}
$$

and

$$
L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} \int_{B_k} (\Sigma |g_i^k|^2)^{-n} \sum_{i=1}^{n} (-1)^{i+1} g_i^k \wedge \frac{d\zeta_i^k}{\zeta_i^k - f(\zeta_i^k)}
$$

which is the desired formula.

For $n = 1$, $\Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k)$ and

$$
L(f) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{B_k} \frac{d\zeta^k}{\zeta^k - f(\zeta^k)} = \sum_{f(\zeta) = \zeta} \text{Res}(\zeta - f(\zeta))^{-1} .
$$

**Note.** Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

**References**


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