A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

L. M. SIBNER and R. J. SIBNER

Let $f$ be a holomorphic self map of a compact complex analytic manifold $X$. The differential of $f$ commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$-complex of $X$. If $f$ has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map $f$ near the fixed points. For example, if $X$ is a curve, this Lefschetz number is the sum of the residues of $(z - f(z))^{-1}$ at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let $E$ be the $\bar{\partial}$-complex of a compact complex analytic manifold $X$ of dimension $n$.

\[ E: 0 \rightarrow \Gamma(A^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(A^{0,1}) \rightarrow \cdots \xrightarrow{\bar{\partial}} \Gamma(A^{0,n}) \rightarrow 0. \]

Since $E$ is elliptic, $H^i(X) = \ker \bar{\partial}_i/\text{im}\bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on $E$ by the holomorphic map $f$, and by $H^iT$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of $f$ is then defined by

\[ L(f) = \sum_{i=0}^{n} (-1)^i \text{tr} H^iT \]

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$-operator. In fact, let us suppose we can find operators $P_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1})$, $i = 1, \ldots, n$, having the property that

\[ P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i = I - S_i \]

where $S_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(A^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^iI - H^iS$ is the zero-endomorphism on homology. Similarly, since $T$ commutes
with $\delta$

$$T_i(P_{i+1}\delta_i + \delta_{i-1}P_i) = T_iP_{i+1}\delta_i + \delta_{i-1}T_{i-1}P_i = T_i - T_iS_i$$

so that $H^i T = H^i TS$. Therefore,

$$L(f) = \sum_{i=0}^{n} (-1)^i tr H^i(TS).$$

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

$$L(f) = \sum_{i=0}^{n} (-1)^i tr T_i S_i$$

provided the right-hand side is finite. This will be the case if the kernels of the operators $S_i$ are sufficiently smooth along the graph of $f$.

To carry out the above procedure and evaluate $L(f)$ we make an explicit choice of the operators $P_i$.

2. The most natural way to choose a parametrix on $X$ is to glue together the local fundamental solutions of the $\delta$-operator using partitions of unity. Given any finite open covering $\{U_a\}$ of $X$, there are, in each $U_a$ integral operators $Q_{\alpha,i}: \Gamma(A^{0,i}(U_a)) \to \Gamma(A^{0,i-1}(U_a))$ $i = 1, \ldots, n$ such that for $\omega \in C_0^\infty(U_a)$

$$\bar{\partial}Q_{\alpha,i}(\omega) = \omega - Q_{\alpha,i+1}(\bar{\partial}\omega)$$

$$Q_{\alpha,i}(\omega)(z^\alpha) = \int_{U_\alpha} \omega(\zeta^\alpha) \wedge \Omega_i(z^\alpha, \zeta^\alpha)$$

where $\Omega_i(z^\alpha, \zeta^\alpha) \in \Gamma(A^{0,i-1}(U_a) \otimes A^{n-i-1}(U_a))$ is a $C^\infty$-section off the diagonal and has an absolutely integrable singularity.

Let $\Omega(z^\alpha, \zeta^\alpha) = \sum_{i=1}^{n} (-1)^i \Omega_i(z^\alpha, \zeta^\alpha)$. This is an $(n, n-1)$ form on $U_\alpha \times U_\alpha$ satisfying

$$\bar{\partial}\Omega = 0.$$  

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for $\Omega$ appears near the end of § 3.

Suppose $f$ has $m$ isolated fixed points, $P_1, \ldots, P_m$. Let $U_k$ be a coordinate neighborhood containing $P_k$, chosen so that the sets $U_k$ are mutually disjoint. Let $N_k$ be a neighborhood of $P_k$, sufficiently small so that $f^{-1}(N_k) \subset U_k$ ($f$ is continuous and $f(P_k) = P_k$). The collection $U_1, \ldots, U_m$ can be extended to a covering $\{U_a\}$ and a partition of unity $\{\lambda_a\}$ subordinate to this covering can be chosen such
that (for \( k = 1, \ldots, m \))

(i) \( \text{supp} \lambda_k \subset N_k \)

(ii) \( \lambda_k = 1 \) in a neighborhood of \( P_k \).

Then \( \text{supp} \lambda_k \circ f \subset f^{-1}(N_k) \subset U_k \) and \( \lambda_k \circ f = 1 \) in some (other) neighborhood of \( P_k \).

Now choose nonnegative functions \( \sigma_\alpha \in C_0^\infty(U_\alpha) \) such that

(iii) \( \sigma_\alpha = 1 \) on \( \text{supp} \lambda_\alpha \), \( \alpha \neq 1, \ldots, m \)

(iv) \( \sigma_\alpha = 1 \) on \( \{ \text{supp} \lambda_\alpha \} \cup \{ \text{supp} \lambda_\alpha \circ f \} \), \( \alpha = 1, \ldots, m \).

Define \( P_i : \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1}) \) by

\[
P_i \omega = \sum_\alpha \lambda_\alpha Q_{\alpha,i}(\alpha_\omega) \quad \text{for } i = 1, \ldots, n
\]

\[
P_0 \omega = 0.
\]

From (4a) we obtain

\[
(6) \quad \delta P_i \omega + P_{i+1} \delta \omega = \omega + \sum_\alpha \bar{\delta} \lambda_\alpha Q_{\alpha,i}(\sigma_\omega) - \sum_\alpha \lambda_\alpha Q_{\alpha,i+1} (\bar{\delta} \sigma_\alpha \wedge \omega)
\]

\[
= \omega - S_i \omega \quad \text{for } i = 0, \ldots, n
\]

where

\[
S_i \omega(z) = -\sum_\alpha \bar{\delta} \lambda_\alpha(z) \int_{U_\alpha} \sigma_\alpha(\zeta) \omega(\zeta) \wedge \Omega_i(z, \zeta)
\]

\[
+ \sum_\alpha \lambda_\alpha(z) \int_{U_\alpha} \bar{\delta} \sigma_\alpha(\zeta) \wedge \omega(\zeta) \wedge \Omega_{i+1}(z, \zeta).
\]

(We consistently suppress the coordinate superscript when possible: writing, for example, \( \sigma_\alpha(\zeta) \) for \( \sigma_\alpha(\zeta^\alpha) \).

3. Because of the construction of the covering and the patching functions, the kernel of \( S_i \) is smooth in a neighborhood of the graph of \( f \). In fact, if \( \alpha > m \), then \( f \) has no fixed points in \( U_\alpha \) and therefore, \( \zeta - f(\zeta) \) is bounded away from zero so that \( \Omega_i(f(\zeta), \zeta) \) is a \( C^\infty \)-function in \( U_\alpha \). Furthermore, in \( U_k, k \leq m \), we have chosen \( \lambda_k \) so that \( \lambda_k(f(\zeta)) \equiv 1 \) in a neighborhood of \( P_k \). Then, \( \bar{\delta} \lambda_k(f(\zeta)) = 0 \) near \( \zeta = f(\zeta) \). Also, since \( \sigma_\alpha(\zeta) \equiv 1 \) on the support of \( \lambda_k(f(\zeta)) \), we have \( \bar{\delta} \sigma_\alpha(\zeta) = 0 \) near \( \zeta = f(\zeta) \). Thus, the kernel of \( S_i \) may be evaluated along the graph of \( f \) to obtain:

\[
\sum_0^n (-1)^i \text{tr}(T_i S_i) = \sum_\alpha \left\{ \sum_1^n (-1)^{i+1} \int_{U_\alpha} \bar{\delta} \lambda_\alpha(f(\zeta)) \wedge \sigma_\alpha(\zeta) \Omega_i(f(\zeta), \zeta) \right\}
\]

\[
+ \sum_\alpha \left\{ \sum_0^{n-1} (-1)^i \int_{U_\alpha} \lambda_\alpha(f(\zeta)) \bar{\delta} \sigma_\alpha(\zeta) \wedge \Omega_{i+1}(f(\zeta), \zeta) \right\}
\]

\[
= -\sum_\alpha \int_{U_\alpha} \bar{\delta} \lambda_\alpha(f(\zeta)) \sigma_\alpha(\zeta) \wedge \sum_1^n (-1)^i \Omega_i(f(\zeta^\alpha), \zeta^\alpha)
\]

from which
(7) \[ L(f) = -\sum_{\alpha} \int_{U_{\alpha}} \Tilde{\lambda}(\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)) \wedge \Omega(f(\zeta), \zeta). \]

In \( U_a \), for \( \alpha > m \), \( f \) has no fixed points. Using (4c), integrating by parts, and making use of the fact that \( \sigma_{\alpha} \) has compact support in \( U_a \), we have

\[
\int_{U_{\alpha}} \Tilde{\lambda}(\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)) \wedge \Omega(f(\zeta), \zeta) = \int_{U_{\alpha}} \Tilde{\lambda}(\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\Omega(f(\zeta), \zeta)) = 0.
\]

For \( \alpha = k \leq m \), let \( B_k \) be a ball around \( P \) on which \( \lambda_{P}(f(\zeta)) \equiv 1 \). Since \( \sigma_{k}(\zeta) \equiv 1 \) on the support of \( \lambda_{k}(f(\zeta)) \),

(8) \[ L(f) = -\sum_{k=1}^{m} \int_{B_k} \Tilde{\lambda}(\lambda_{k}(f(\zeta))\Omega(f(\zeta), \zeta)) = \sum_{k=1}^{m} \int_{B_k} \lambda_{k}(f(\zeta))\Omega(f(\zeta), \zeta) = 0.
\]

Using local coordinates in \( B_k \), let \( g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k) \), \( i = 1, \ldots, n \). Then, for \( n > 1 \),

\[
\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} |z^k - \zeta^k|^{-2n} \sum_{i=1}^{n} (-1)^{i+1} \frac{\zeta_i}{z_i} \wedge \frac{d\zeta_i}{d\zeta_i} \wedge \frac{d\zeta_i}{d\zeta_i}
\]

and

(9) \[ L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{m} \int_{B_k} (\sum |g_i^k|^2)^{-n} \sum_{i=1}^{n} (-1)^{i+1} \frac{\zeta_i}{z_i} \wedge \frac{d\zeta_i}{d\zeta_i} \wedge \frac{d\zeta_i}{d\zeta_i},
\]

which is the desired formula.

For \( n = 1 \), \( \Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k) \) and

\[ L(f) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{B_k} \frac{d\zeta_i^k}{\zeta_i^k - f(\zeta^k)} = \sum_{f(\zeta^k) = \zeta} \text{Res}(\zeta - f(\zeta))^{-1}.
\]

Note. Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

References

4. Nils Øvrelid, Integral representation formulas and Lp-estimates for the \( \bar{\partial} \)-equation, preprint.

Received May 29, 1973. The first author was supported in part by National Science Foundation grant GP-27960. The second author was supported in part by National Science Foundation grant GP-7952X3.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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