

Pacific Journal of Mathematics

A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

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Let f be a holomorphic self map of a compact complex analytic manifold X . The differential of f commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$ -complex of X . If f has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map f near the fixed points. For example, if X is a curve, this Lefschetz number is the sum of the residues of $(z - f(z))^{-1}$ at the fixed points.

Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.

1. Let E be the $\bar{\partial}$ -complex of a compact complex analytic manifold X of dimension n .

$$E: 0 \longrightarrow \Gamma(A^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(A^{0,1}) \longrightarrow \dots \xrightarrow{\bar{\partial}} \Gamma(A^{0,n}) \longrightarrow 0.$$

Since E is elliptic, $H^i(X) = \ker \bar{\partial}_i / \text{im } \bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on E by the holomorphic map f , and by $H^i T$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of f is then defined by

$$L(f) = \sum_{i=0}^n (-1)^i \text{tr } H^i T$$

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$ -operator. In fact, let us suppose we can find operators $P_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1})$, $i = 1, \dots, n$, having the property that

$$(1) \quad P_{i+1} \bar{\partial}_i + \bar{\partial}_{i-1} P_i = I - S_i$$

where $S_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(A^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^i I - H^i S$ is the zero-endomorphism on homology. Similarly, since T commutes

with $\bar{\partial}$

$$T_i(P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i) = T_iP_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}T_{i-1}P_i = T_i - T_iS_i$$

so that $H^i T = H^i TS$. Therefore,

$$(2) \quad L(f) = \sum_{i=0}^n (-1)_i \text{tr} H^i(TS).$$

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

$$(3) \quad L(f) = \sum_{i=0}^n (-1)^i \text{tr} H^i TS = \sum_{i=0}^n (-1)^i \text{tr} T_i S_i$$

provided the right-hand side is finite. This will be the case if the kernels of the operators S_i are sufficiently smooth along the graph of f .

To carry out the above procedure and evaluate $L(f)$ we make an explicit choice of the operators P_i .

2. The most natural way to choose a parametrix on X is to glue together the local fundamental solutions of the $\bar{\partial}$ -operator using partitions of unity. Given any finite open covering $\{U_\alpha\}$ of X , there are, in each U_α , integral operators $Q_{\alpha,i}: \Gamma(A^{0,i}(U_\alpha)) \rightarrow \Gamma(A^{0,i-1}(U_\alpha))$ $i = 1, \dots, n$ such that for $\omega \in C_0^\infty(U_\alpha)$

$$(4a) \quad \bar{\partial}Q_{\alpha,i}(\omega) = \omega - Q_{\alpha,i+1}(\bar{\partial}\omega)$$

$$(4b) \quad (Q_{\alpha,i}\omega)(z^\alpha) = \int_{U_\alpha} \omega(\zeta^\alpha) \wedge \Omega_i(z^\alpha, \zeta^\alpha)$$

where $\Omega_i(z^\alpha, \zeta^\alpha) \in \Gamma(A^{0,i-1}(U_\alpha) \otimes A^{n,n-i}(U_\alpha))$ is a C^∞ -section off the diagonal and has an absolutely integrable singularity.

Let $\Omega(z^\alpha, \zeta^\alpha) = \sum_{i=1}^n (-1)^i \Omega_i(z^\alpha, \zeta^\alpha)$. This is an $(n, n-1)$ form on $U_\alpha \times U_\alpha$ satisfying

$$(4c) \quad \bar{\partial}\Omega = 0.$$

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for Ω appears near the end of §3.

Suppose f has m isolated fixed points, P_1, \dots, P_m . Let U_k be a coordinate neighborhood containing P_k , chosen so that the sets U_k are mutually disjoint. Let N_k be a neighborhood of P_k , sufficiently small so that $f^{-1}(N_k) \subset U_k$ (f is continuous and $f(P_k) = P_k$). The collection U_1, \dots, U_m can be extended to a covering $\{U_\alpha\}$ and a partition of unity $\{\lambda_\alpha\}$ subordinate to this covering can be chosen such

that (for $k = 1, \dots, m$)

(i) $\text{supp } \lambda_k \subset N_k$

(ii) $\lambda_k = 1$ in a neighborhood of P_k .

Then $\text{supp } \lambda_k \circ f \subset f^{-1}(N_k) \subset U_k$ and $\lambda_k \circ f = 1$ in some (other) neighborhood of P_k .

Now choose nonnegative functions $\sigma_\alpha \in C_0^\infty(U_\alpha)$ such that

(iii) $\sigma_\alpha = 1$ on $\text{supp } \lambda_\alpha$ $\alpha \neq 1, \dots, m$

(iv) $\sigma_\alpha = 1$ on $\{\text{supp } \lambda_\alpha\} \cup \{\text{supp } \lambda_\alpha \circ f\}$ $\alpha = 1, \dots, m$.

Define $P_i: \Gamma(A^{0,i}) \rightarrow \Gamma(A^{0,i-1})$ by

$$(5) \quad \begin{aligned} P_i \omega &= \sum_\alpha \lambda_\alpha Q_{\alpha,i}(\alpha_\alpha \omega) & i = 1, \dots, n \\ P_0 \omega &= 0. \end{aligned}$$

From (4a) we obtain

$$(6) \quad \begin{aligned} \bar{\partial} P_i \omega + P_{i+1} \bar{\partial} \omega &= \omega + \sum_\alpha \bar{\partial} \lambda_\alpha Q_{\alpha,i}(\sigma_\alpha \omega) - \sum_\alpha \lambda_\alpha Q_{\alpha,i+1}(\bar{\partial} \sigma_\alpha \wedge \omega) \\ &= \omega - S_i \omega & i = 0, \dots, n \end{aligned}$$

where

$$\begin{aligned} S_i \omega(z) &= - \sum_\alpha \bar{\partial} \lambda_\alpha(z) \int_{U_\alpha} \sigma_\alpha(\zeta) \omega(\zeta) \wedge \Omega_i(z, \zeta) \\ &\quad + \sum_\alpha \lambda_\alpha(z) \int_{U_\alpha} \bar{\partial} \sigma_\alpha(\zeta) \wedge \omega(\zeta) \wedge \Omega_{i+1}(z, \zeta). \end{aligned}$$

(We consistently suppress the coordinate superscript when possible: writing, for example, $\sigma_\alpha(\zeta)$ for $\sigma_\alpha(\zeta^\alpha)$.)

3. Because of the construction of the covering and the patching functions, the kernel of S_i is smooth in a neighborhood of the graph of f . In fact, if $\alpha > m$, then f has no fixed points in U_α and therefore, $\zeta - f(\zeta)$ is bounded away from zero so that $\Omega_i(f(\zeta), \zeta)$ is a C^∞ -function in U_α . Furthermore, in $U_k, k \leq m$, we have chosen λ_k so that $\lambda_k(f(\zeta)) \equiv 1$ in a neighborhood of P_k . Then, $\bar{\partial} \lambda_k(f(\zeta)) = 0$ near $\zeta = f(\zeta)$. Also, since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$, we have $\bar{\partial} \sigma_\alpha(\zeta) = 0$ near $\zeta = f(\zeta)$. Thus, the kernel of S_i may be evaluated along the graph of f to obtain:

$$\begin{aligned} \sum_0^n (-1)^i \text{tr}(T_i S_i) &= \sum_\alpha \left\{ \sum_1^n (-1)^{i+1} \int_{U_\alpha} \bar{\partial} \lambda_\alpha(f(\zeta)) \wedge \sigma_\alpha(\zeta) \Omega_i(f(\zeta), \zeta) \right\} \\ &\quad + \sum_\alpha \left\{ \sum_0^{n-1} (-1)^i \int_{U_\alpha} \lambda_\alpha(f(\zeta)) \bar{\partial} \sigma_\alpha(\zeta) \wedge \Omega_{i+1}(f(\zeta), \zeta) \right\} \\ &= - \sum_\alpha \int_{U_\alpha} \bar{\partial} \{ \lambda_\alpha(f(\zeta)) \sigma_\alpha(\zeta) \} \wedge \sum_1^n (-1)^i \Omega_i(f(\zeta), \zeta) \end{aligned}$$

from which

$$(7) \quad L(f) = -\sum_{\alpha} \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\} \wedge \Omega(f(\zeta), \zeta).$$

In U_{α} , for $\alpha > m$, f has no fixed points. Using (4c), integrating by parts, and making use of the fact that σ_{α} has compact support in U_{α} , we have

$$\begin{aligned} \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\} \wedge \Omega(f(\zeta), \zeta) &= \int_{U_{\alpha}} \bar{\partial}\{\lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\Omega(f(\zeta), \zeta)\} \\ &= \int_{\partial U_{\alpha}} \lambda_{\alpha}(f(\zeta))\sigma_{\alpha}(\zeta)\Omega(f(\zeta), \zeta) \equiv 0. \end{aligned}$$

For $\alpha = k \leq m$, let B_k be a ball around P_k on which $\lambda_k(f(\zeta)) \equiv 1$. Since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$,

$$(8) \quad \begin{aligned} L(f) &= -\sum_{k=1}^m \int_{U_k - B_k} \bar{\partial}\{\lambda_k(f(\zeta))\Omega(f(\zeta), \zeta)\} = \sum_{k=1}^m \int_{\partial B_k} \lambda_k(f(\zeta))\Omega(f(\zeta), \zeta) \\ &= \sum_{k=1}^m \int_{\partial B_k} \Omega(f(\zeta), \zeta). \end{aligned}$$

Using local coordinates in B_k , let $g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k)$, $i = 1, \dots, n$. Then, for $n > 1$,

$$\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} |z^k - \zeta^k|^{-2n} \sum_{i=1}^n (-1)^{i+1} \overline{\zeta_i^k - z_i^k} \prod_{\substack{j=1 \\ j \neq i}}^n \overline{d\zeta_j^k} - \overline{dz_j^k} \prod_{l=1}^n d\zeta_l^k$$

and

$$(9) \quad L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^m \int_{\partial B_k} (\sum |g_i^k|^2)^{-n} \sum_{i=1}^n (-1)^{i+1} \overline{g_i^k} \prod_{\substack{j=1 \\ j \neq i}}^n \overline{dg_j^k} \prod_{l=1}^n d\zeta_l^k$$

which is the desired formula.

For $n = 1$, $\Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k)$ and

$$L(f) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\partial B_k} \frac{d\zeta^k}{\zeta^k - f(\zeta^k)} = \sum_{f(\zeta)=\zeta} \text{Res}(\zeta - f(\zeta))^{-1}.$$

NOTE. Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

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Received May 29, 1973. The first author was supported in part by National Science Foundation grant GP-27960. The second author was supported in part by National Science Foundation grant GP-7952X3.

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The *Pacific of Journal Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

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