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# A NOTE ON STARSHAPED SETS, $(k)$-EXTREME POINTS AND THE HALF RAY PROPERTY

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# A NOTE ON STARSHAPED SETS, $(k)$-EXTREME POINTS AND THE HALF RAY PROPERTY

N. STAVRAKAS

**Let $S$ be a compact subset of $R^d$, $d \geqq 2$. $S$ is said to have the half-ray property if for each point $x$ of the complement of $S$ there exists a half line with $x$ as vertex having empty intersection with $S$. It is proven that $S$ is starshaped iff $S$ has the half-ray property and the intersection of the stars of the $(d-2)$-extreme points is not empty.**

Let $S \subset R^d$. We say $x \in S$ is a $(k)$-*extreme* point of $S$ provided for every $k+1$ dimensional simplex $D \subset S$, $x \notin \operatorname{relint} D$ where relint $D$ denotes the interior of $D$ relative to the $k+1$ dimensional space $D$ generates. If $y \in S$ the symbol $S(y)$ is defined as $S(y) = \{z \mid z \in S$ and $[yz] \subset S\}$, where $[yz]$ denotes the closed line segment from $y$ to $z$. The symbol $E(S)$ denotes the set of all $(d-2)$-extreme points of $S$. We say $S$ is *starshaped* if $\operatorname{Ker} S \neq \varnothing$, where $\operatorname{Ker} S = \bigcap_{y \in S} S(y)$. In [1] the following is proved:

THEOREM 1. *Let $S \subset R^d$, $d \geqq 2$, be compact and starshaped. Then* $\operatorname{Ker} S = \bigcap_{x \in E(S)} S(x)$.

Theorem 1 certainly yields information about the structure of a starshaped set but at the same time raises several questions. First, has Theorem 1 a converse? Specifically, given that $\bigcap_{x \in E(S)} S(x) \neq \varnothing$, under what hypothesis will $S$ be starshaped? Secondly, can the hypothesis of starshaped be replaced with a seemingly more general hypothesis? We answer the latter question in Theorem 2.

DEFINITION 1. Let $S \subset R^d$ and let $S^\sim$ be the complement of $S$. We say $S$ has the half-ray property if and only if for every $x \in S^\sim$ there exists a half line $l$ with $x$ as vertex such that $l \cap S = \varnothing$.

THEOREM 2. *Let $S \subset R^d$, $d \geqq 2$, be compact and suppose $\bigcap_{x \in E(S)} S(x) \neq \varnothing$. Then the following are equivalent:*

(1) *$S$ has the half-ray property.*

(2) $\operatorname{Ker} S = \bigcap_{x \in E(S)} S(x)$.

*Since for any starshaped set $S$, $S$ has the half-ray property and $\bigcap_{x \in E(S)} S(x) \neq \varnothing$, the implication $(1) \Rightarrow (2)$ generalizes Theorem 1. Further, the implication $(1) \Rightarrow (2)$ is a type of converse since we assume $\bigcap_{x \in E(S)} S(x) = \varnothing$ and obtain as a conclusion, rather than a hypothesis, that $S$ is starshaped. As a corollary to Theorem 2,* *we obtain a new characterization for starshaped sets.*

COROLLARY 1. *Let $S \subset R^d$, $d \geqq 2$, be compact. Then the following are equivalent:*

(1) *$S$ is starshaped.*

(2) *$\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and $S$ has the half-ray property.*

2. *Proof of Theorem* 2. In the proof the symbol $\| \ \|$ denotes the Euclidean norm and the symbol $[ab_\infty)$ denotes the half line determined by the points $a$ and $b$ with $a$ as vertex.

(2) $\Rightarrow$ (1). This follows immediately since any starshaped set has the half-ray property.

(1) $\Rightarrow$ (2). Let $y \in \bigcap_{x \in E(S)} S(x)$ and we show $y \in \operatorname{Ker} S$. Suppose $y \notin \operatorname{Ker} S$. Then there exists $z \in S$ such that $[yz] \not\subset S$. Let $a \in [yz] \sim S$. Without loss of generality, suppose $a$ is the origin, $O_v$. By hypothesis there exists a half line $l = [0_v b_\infty)$ with $[0_v b_\infty) \cap S = \emptyset$. Let $Q$ be the two dimensional subspace spanned by $y$ and $b$. Now rotate $l$ in $Q$ so that the angle between $l$ and $[0_v z_\infty)$ (which is already less than $\pi$) decreases. Cease the rotation when $S$ is intersected and let the rotated half line be $l^*$. Note $l^* \cap S$ is compact and hence $\theta = \sup\{\|x\| \mid x \in l^* \cap S\}$ exists. Let $x \in l^* \cap S$ be such that $\|x\| = \theta$. We claim $x \in E(S)$. Suppose not. Then $x \in \operatorname{relint} D$ where $D$ is a $d-1$ dimensional simplex in $S$. Since $x \in D \cap Q$, $\dim(D \cap Q) \geqq 1$. For each $z \in D$, $z \neq x$ let $[zx_\infty) \cap D$ be $[ze_z]$ and note $x \in (ze_z)$. Let $w \in D \cap Q$, $w \neq x$. Note $[we_w] \subset Q$. Now, if $[we_w] \subset l^*$, we contradict the definition of $x$ since $x \in (we_w)$ and if $[we_w] \not\subset l^*$, we contradict the definition of $l^*$. Thus, $x \in E(S)$. Then $[xy] \subset S$ and this contradicts the definition of $l^*$. Thus, $y \in \operatorname{Ker} S$ and we are done.

In conclusion, we remark that a triangle in $E^2$ is an example of a nonstarshaped set for which $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and which does not have the half-ray property. The latter shows that in the implication (1) $\Rightarrow$ (2) of Theorem 2 the hypothesis of $S$ having the half-ray property cannot be deleted.

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# Pacific Journal of Mathematics

Vol. 53, No. 2 April, 1974

Kenneth Abernethy, *On characterizing certain classses of first countable spaces by open mappings* . . . . . . 319
Ross A. Beaumont and Donald Lawver, *Strongly semisimple abelian groups* . . . . . . 327
Gerald A. Beer, *The index of convexity and parallel bodies* . . . . . . 337
Victor P. Camillo and Kent Ralph Fuller, *On Loewy length of rings* . . . . . . 347
Stephen LaVern Campbell, *Linear operators for which $T^*T$ and $TT^*$ commute. II* . . . . . . 355
Charles Kam-Tai Chui and Philip Wesley Smith, *Characterization of a function by certain infinite series it generates* . . . . . . 363
Allan L. Edelson, *Conjugations on stably almost complex manifolds* . . . . . . 373
Patrick John Fleury, *Hollow modules and local endomorphism rings* . . . . . . 379
Jack Tilden Goodykoontz, Jr., *Connectedness im kleinen and local connectedness in $2^X$ and $C(X)$* . . . . . . 387
Robert Edward Jamison, II, *Functional representation of algebraic intervals* . . . . . . 399
Athanassios G. Kartsatos, *Nonzero solutions to boundary value problems for nonlinear systems* . . . . . . 425
Soon-Kyu Kim, Dennis McGavran and Jingyal Pak, *Torus group actions on simply connected manifolds* . . . . . . 435
David Anthony Klarner and R. Rado, *Arithmetic properties of certain recursively defined sets* . . . . . . 445
Ray Alden Kunze, *On the Frobenius reciprocity theorem for square-integrable representations* . . . . . . 465
John Lagnese, *Existence, uniqueness and limiting behavior of solutions of a class of differential equations in Banach space* . . . . . . 473
Teck Cheong Lim, *A fixed point theorem for families on nonexpansive mappings* . . . . . . 487
Lewis Lum, *A quasi order characterization of smooth continua* . . . . . . 495
Andy R. Magid, *Principal homogeneous spaces and Galois extensions* . . . . . . 501
Charles Alan McCarthy, *The norm of a certain derivation* . . . . . . 515
Louise Elizabeth Moser, *On the impossibility of obtaining $S^2 \times S^1$ by elementary surgery along a knot* . . . . . . 519
Gordon L. Nipp, *Quaternion orders associated with ternary lattices* . . . . . . 525
Anthony G. O'Farrell, *Equiconvergence of derivations* . . . . . . 539
Dorte Olesen, *Derivations of $AW^*$-algebras are inner* . . . . . . 555
Dorte Olesen and Gert Kjærgaard Pedersen, *Derivations of $C^*$-algebras have semi-continuous generators* . . . . . . 563
Duane O'Neill, *On conjugation cobordism* . . . . . . 573
Chull Park and S. R. Paranjape, *Probabilities of Wiener paths crossing differentiable curves* . . . . . . 579
Edward Ralph Rozema, *Almost Chebyshev subspaces of $L^1(\mu;\ E)$* . . . . . . 585
Lesley Millman Sibner and Robert Jules Sibner, *A note on the Atiyah-Bott fixed point formula* . . . . . . 605
Betty Salzberg Stark, *Irreducible subgroups of orthogonal groups generated by groups of root type* 1 . . . . . . 611
N. Stavrakas, *A note on starshaped sets, $(k)$-extreme points and the half ray property* . . . . . . 627
Carl E. Swenson, *Direct sum subset decompositions of $Z$* . . . . . . 629
Stephen Tefteller, *A two-point boundary problem for nonhomogeneous second order differential equations* . . . . . . 635
Robert S. Wilson, *Representations of finite rings* . . . . . . 643