A NOTE ON STARSHAPED SETS, $(k)$-EXTREME POINTS AND THE HALF RAY PROPERTY

N. STAVRAKAS
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Let \(S\) be a compact subset of \(\mathbb{R}^d\), \(d \geq 2\). \(S\) is said to have the half-ray property if for each point \(x\) of the complement of \(S\) there exists a half line with \(x\) as vertex having empty intersection with \(S\). It is proven that \(S\) is starshaped if \(S\) has the half-ray property and the intersection of the stars of the \((d-2)\)-extreme points is not empty.

Let \(S \subset \mathbb{R}^d\). We say \(x \in S\) is a \((k)\)-extreme point of \(S\) provided for every \(k+1\) dimensional simplex \(D \subset S\), \(x \notin \text{relint } D\) where relint \(D\) denotes the interior of \(D\) relative to the \(k+1\) dimensional space \(D\) generates. If \(y \in S\) the symbol \(S(y)\) is defined as \(S(y) = \{z \mid z \in S\text{ and }[yz] \subset S\}\), where \([yz]\) denotes the closed line segment from \(y\) to \(z\). The symbol \(E(S)\) denotes the set of all \((d-2)\)-extreme points of \(S\). We say \(S\) is starshaped if \(\ker S = \emptyset\), where \(\ker S = \bigcap_{y \in S} S(y)\).

In [1] the following is proved:

**Theorem 1.** Let \(S \subset \mathbb{R}^d\), \(d \geq 2\), be compact and starshaped. Then \(\ker S = \bigcap_{x \in E(S)} S(x)\).

Theorem 1 certainly yields information about the structure of a starshaped set but at the same time raises several questions. First, has Theorem 1 a converse? Specifically, given that \(\bigcap_{x \in E(S)} S(x) = \emptyset\), under what hypothesis will \(S\) be starshaped? Secondly, can the hypothesis of starshaped be replaced with a seemingly more general hypothesis? We answer the latter question in Theorem 2.

**Definition 1.** Let \(S \subset \mathbb{R}^d\) and let \(S^c\) be the complement of \(S\). We say \(S\) has the half-ray property if and only if for every \(x \in S^c\) there exists a half line \(l\) with \(x\) as vertex such that \(l \cap S = \emptyset\).

**Theorem 2.** Let \(S \subset \mathbb{R}^d\), \(d \geq 2\), be compact and suppose \(\bigcap_{x \in E(S)} S(x) = \emptyset\). Then the following are equivalent:

1. \(S\) has the half-ray property.
2. \(\ker S = \bigcap_{x \in E(S)} S(x)\).

Since for any starshaped set \(S\), \(S\) has the half-ray property and \(\bigcap_{x \in E(S)} S(x) = \emptyset\), the implication (1) \(\implies\) (2) generalizes Theorem 1. Further, the implication (1) \(\implies\) (2) is a type of converse since we assume \(\bigcap_{x \in E(S)} S(x) = \emptyset\) and obtain as a conclusion, rather than a hypothesis, that \(S\) is starshaped. As a corollary to Theorem 2,
we obtain a new characterization for starshaped sets.

**COROLLARY 1.** Let \( S \subset \mathbb{R}^d \), \( d \geq 2 \), be compact. Then the following are equivalent:

1. \( S \) is starshaped.
2. \( \bigcap_{x \in E(S)} S(x) \neq \emptyset \) and \( S \) has the half-ray property.

**2. Proof of Theorem 2.** In the proof the symbol \( \| \| \) denotes the Euclidean norm and the symbol \([ab_\infty)\) denotes the half line determined by the points \( a \) and \( b \) with \( a \) as vertex.

(2) \( \Rightarrow \) (1). This follows immediately since any starshaped set has the half-ray property.

(1) \( \Rightarrow \) (2). Let \( y \in \bigcap_{x \in E(S)} S(x) \) and we show \( y \in \text{Ker} \ S \). Suppose \( y \not\in \text{Ker} \ S \). Then there exists \( z \in S \) such that \([yz] \not\subset S \). Let \( a \in [yz] \sim S \). Without loss of generality, suppose \( a \) is the origin, \( O \). By hypothesis there exists a half line \( l = [0,b_\infty) \) with \([0,b_\infty) \cap S = \emptyset \). Let \( Q \) be the two dimensional subspace spanned by \( y \) and \( b \). Now rotate \( l \) in \( Q \) so that the angle between \( l \) and \([0,z_\infty) \) (which is already less than \( \pi \)) decreases. Cease the rotation when \( S \) is intersected and let the rotated half line be \( l^* \). Note \( l^* \cap S \) is compact and hence \( \theta = \sup \{ \| x \| : x \in l^* \cap S \} \) exists. Let \( x \in l^* \cap S \) be such that \( \| x \| = \theta \). We claim \( x \in E(S) \). Suppose not. Then \( x \in \text{relint} \ D \) where \( D \) is a \( d-1 \) dimensional simplex in \( S \). Since \( x \in D \cap Q \), \( \dim (D \cap Q) \geq 1 \). For each \( z \in D \), \( z \neq x \) let \([zze_\infty) \cap D \) be \([ze_\infty) \) and note \( x \in (ze_\infty) \). Let \( w \in D \cap Q \), \( w \neq x \). Note \([we_\infty) \subset Q \). Now, if \([we_\infty) \subset l^* \), we contradict the definition of \( x \) since \( x \in (we_\infty) \) and if \([we_\infty) \not\subset l^* \), we contradict the definition of \( l^* \). Thus, \( x \in E(S) \). Then \([xy] \subset S \) and this contradicts the definition of \( l^* \). Thus, \( y \in \text{Ker} \ S \) and we are done.

In conclusion, we remark that a triangle in \( E^2 \) is an example of a nonstarshaped set for which \( \bigcap_{x \in E(S)} S(x) \neq \emptyset \) and which does not have the half-ray property. The latter shows that in the implication (1) \( \Rightarrow \) (2) of Theorem 2 the hypothesis of \( S \) having the half-ray property cannot be deleted.

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