

Pacific Journal of Mathematics

**A NOTE ON STARSHAPED SETS, (k) -EXTREME POINTS AND
THE HALF RAY PROPERTY**

N. STAVRAKAS

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Let S be a compact subset of R^d , $d \geq 2$. S is said to have the half-ray property if for each point x of the complement of S there exists a half line with x as vertex having empty intersection with S . It is proven that S is starshaped iff S has the half-ray property and the intersection of the stars of the $(d - 2)$ -extreme points is not empty.

Let $S \subset R^d$. We say $x \in S$ is a (k) -extreme point of S provided for every $k + 1$ dimensional simplex $D \subset S$, $x \notin \text{relint } D$ where $\text{relint } D$ denotes the interior of D relative to the $k + 1$ dimensional space D generates. If $y \in S$ the symbol $S(y)$ is defined as $S(y) = \{z \mid z \in S \text{ and } [yz] \subset S\}$, where $[yz]$ denotes the closed line segment from y to z . The symbol $E(S)$ denotes the set of all $(d - 2)$ -extreme points of S . We say S is starshaped if $\text{Ker } S \neq \emptyset$, where $\text{Ker } S = \bigcap_{y \in S} S(y)$. In [1] the following is proved:

THEOREM 1. *Let $S \subset R^d$, $d \geq 2$, be compact and starshaped. Then $\text{Ker } S = \bigcap_{x \in E(S)} S(x)$.*

Theorem 1 certainly yields information about the structure of a starshaped set but at the same time raises several questions. First, has Theorem 1 a converse? Specifically, given that $\bigcap_{x \in E(S)} S(x) \neq \emptyset$, under what hypothesis will S be starshaped? Secondly, can the hypothesis of starshaped be replaced with a seemingly more general hypothesis? We answer the latter question in Theorem 2.

DEFINITION 1. Let $S \subset R^d$ and let S^c be the complement of S . We say S has the half-ray property if and only if for every $x \in S^c$ there exists a half line l with x as vertex such that $l \cap S = \emptyset$.

THEOREM 2. *Let $S \subset R^d$, $d \geq 2$, be compact and suppose $\bigcap_{x \in E(S)} S(x) \neq \emptyset$. Then the following are equivalent:*

- (1) S has the half-ray property.
- (2) $\text{Ker } S = \bigcap_{x \in E(S)} S(x)$.

Since for any starshaped set S , S has the half-ray property and $\bigcap_{x \in E(S)} S(x) \neq \emptyset$, the implication (1) \Rightarrow (2) generalizes Theorem 1. Further, the implication (1) \Rightarrow (2) is a type of converse since we assume $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and obtain as a conclusion, rather than a hypothesis, that S is starshaped. As a corollary to Theorem 2,

we obtain a new characterization for starshaped sets.

COROLLARY 1. *Let $S \subset R^d$, $d \geq 2$, be compact. Then the following are equivalent:*

- (1) S is starshaped.
- (2) $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and S has the half-ray property.

2. *Proof of Theorem 2.* In the proof the symbol $\| \cdot \|$ denotes the Euclidean norm and the symbol $[ab_\infty)$ denotes the half line determined by the points a and b with a as vertex.

(2) \Rightarrow (1). This follows immediately since any starshaped set has the half-ray property.

(1) \Rightarrow (2). Let $y \in \bigcap_{x \in E(S)} S(x)$ and we show $y \in \text{Ker } S$. Suppose $y \notin \text{Ker } S$. Then there exists $z \in S$ such that $[yz] \not\subset S$. Let $a \in [yz] \sim S$. Without loss of generality, suppose a is the origin, O_v . By hypothesis there exists a half line $l = [0_v b_\infty)$ with $[0_v b_\infty) \cap S = \emptyset$. Let Q be the two dimensional subspace spanned by y and b . Now rotate l in Q so that the angle between l and $[0_v z_\infty)$ (which is already less than π) decreases. Cease the rotation when S is intersected and let the rotated half line be l^* . Note $l^* \cap S$ is compact and hence $\theta = \sup \{ \|x\| \mid x \in l^* \cap S \}$ exists. Let $x \in l^* \cap S$ be such that $\|x\| = \theta$. We claim $x \in E(S)$. Suppose not. Then $x \in \text{relint } D$ where D is a $d-1$ dimensional simplex in S . Since $x \in D \cap Q$, $\dim(D \cap Q) \geq 1$. For each $z \in D$, $z \neq x$ let $[zx_\infty) \cap D$ be $[ze_1]$ and note $x \in (ze_1)$. Let $w \in D \cap Q$, $w \neq x$. Note $[we_w) \subset Q$. Now, if $[we_w) \subset l^*$, we contradict the definition of x since $x \in (we_w)$ and if $[we_w) \not\subset l^*$, we contradict the definition of l^* . Thus, $x \in E(S)$. Then $[xy) \subset S$ and this contradicts the definition of l^* . Thus, $y \in \text{Ker } S$ and we are done.

In conclusion, we remark that a triangle in E^2 is an example of a nonstarshaped set for which $\bigcap_{x \in E(S)} S(x) \neq \emptyset$ and which does not have the half-ray property. The latter shows that in the implication (1) \Rightarrow (2) of Theorem 2 the hypothesis of S having the half-ray property cannot be deleted.

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REFERENCE

1. J. W. Kenelly, W. R. Hare et. al., *Convex components, extreme points, and the convex kernel*, Proc. Amer. Math. Soc., **21** (1969), 83-87.

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