

# Pacific Journal of Mathematics

## ON $\Lambda(p)$ SETS

GREGORY FRANK BACHELIS AND SAMUEL EBENSTEIN

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**In this note it is shown that if  $1 \leq p < 2$  and  $E$  is a set of type  $\Lambda(p)$  in the dual of a compact abelian group, then  $E$  is of type  $\Lambda(p + \varepsilon)$  for some  $\varepsilon > 0$ .**

**Introduction.** Let  $G$  be a compact abelian group with dual group  $\Gamma$ . For  $0 < p < \infty$ , we denote by  $L^p(G)$  the set of complex-valued measurable functions  $f$  on  $G$  such that

$$\|f\|_p = \left( \int_G |f(x)|^p dx \right)^{1/p}$$

is finite, where  $dx$  denotes normalized Haar measure on  $G$ . For  $f \in L^1(G)$ , the Fourier transform is defined by

$$\hat{f}(\gamma) = \int_G f(x) \overline{(x, \gamma)} dx, \quad \gamma \in \Gamma.$$

As in [5], we call a subset  $E \subset \Gamma$  a set of type  $\Lambda(p)$  if there exists a  $q < p$  and a constant  $K_q$  such that

$$(1) \quad \|P\|_p \leq K_q \|P\|_q$$

for all trigonometric polynomials  $P$  such that  $\hat{P} = 0$  outside  $E$ .

As shown in [5], if (1) holds for some  $q$ ,  $0 < q < p$ , then it holds for all such  $q$ . Also, if  $p > 1$ , then the definition of  $\Lambda(p)$  set is equivalent to the statement that  $L_E^q = L_E^p$  for some  $q$ ,  $1 \leq q < p$ , where  $L_E^q = \{f \in L^q: \hat{f} = 0 \text{ outside } E\}$ . For further details on  $\Lambda(p)$  sets, the reader is referred to [1] or [5].

In this note we apply results of [4] to show the following:

**THEOREM.** *Let  $1 \leq p < 2$ . If  $E$  is of type  $\Lambda(p)$ , then  $E$  is of type  $\Lambda(p + \varepsilon)$  for some  $\varepsilon > 0$ .*

This result is in contrast to the situation when  $p$  is an even integer,  $p \geq 4$ . In that case there are known to exist sets of type  $\Lambda(p)$  which are not of type  $\Lambda(p + \varepsilon)$  when  $G$  is the circle group [5], and also for a large class of compact abelian groups [2].

*The Main Result.* We shall proceed to the proof of the theorem after establishing two lemmas; these lemmas were communicated to the authors by Haskell Rosenthal.

**LEMMA 1.** *Suppose  $X$  is a nonreflexive subspace of  $L^1(\mu)$ , where*

$\mu$  is a probability measure on some measure space. Then given  $\delta > 0$  and  $M > 0$  there exists  $f \in X$  with  $\|f\|_1 = 1$  and

$$\int_S |f(x)| d\mu(x) > 1 - \delta,$$

where  $S = \{x: |f(x)| \geq M\}$ .

*Proof.* Suppose there exists  $M > 0$  and  $\delta > 0$  so that if  $f \in X$  and  $\|f\|_1 = 1$  then

$$\int_S |f(x)| d\mu(x) \leq 1 - \delta.$$

Choose  $\varepsilon > 0$  so that  $M\varepsilon < \delta/2$ . Since  $X$  is nonreflexive, it follows from Lemmas 6 and 7 of [4] that there exists  $f \in X$  and a measurable set  $F$  with  $\|f\|_1 = 1$ ,  $\mu(F) < \varepsilon$  and

$$\int_F |f(x)| d\mu(x) > 1 - \delta/2.$$

We have

$$\begin{aligned} 1 - \delta/2 &< \int_F |f(x)| d\mu(x) = \int_{F \cap S} |f(x)| d\mu(x) + \int_{F \cap S^c} |f(x)| d\mu(x) \\ &\leq \int_S |f(x)| d\mu(x) + \int_F M d\mu(x) \leq 1 - \delta + M\varepsilon \\ &< 1 - \delta + \delta/2 = 1 - \delta/2, \end{aligned}$$

a contradiction.

**LEMMA 2.** *If  $E$  is of type  $A(1)$ , then  $L_E^1$  is reflexive.*

*Proof.* Suppose  $L_E^1$  is nonreflexive. Let  $M, \delta > 0$  and let  $f \in L_E^1$  be as given by Lemma 1.

If  $0 < p < 1$ , then

$$1 \geq \int_S |f(x)| dx = \int_S |f(x)|^p |f(x)|^{1-p} dx \geq \left( \int_S |f(x)|^p dx \right) M^{1-p},$$

so

$$\int_S |f(x)|^p dx \leq 1/M^{1-p}.$$

But

$$\left( \int_{S^c} |f(x)|^p dx \right)^{1/p} \leq \int_{S^c} |f(x)| dx < \delta,$$

so

$$\begin{aligned} \|f\|_p &= \left( \int_S |f(x)|^p dx + \int_{S^c} |f(x)|^p dx \right)^{1/p} \\ &\leq (1/M^{1-p} + \delta^p)^{1/p}. \end{aligned}$$

Now this last quantity can be made arbitrarily small, so it follows from (1) that  $E$  is not of type  $\Lambda(1)$ .

*Proof of Theorem.* First suppose that  $p = 1$ . By Lemma 2,  $L_E^1$  is reflexive. It follows from Theorem 1 and Lemma 6 of [4] that there exists  $q > 1$  and a nonnegative function  $\phi \in L^1$  such that  $0 \neq \|\phi\|_1 \leq 1$  and

$$(2) \quad \left( \int_G |f(x)|^q \phi^{1-q}(x) dx \right)^{1/q} \leq K \int_G |f(x)| dx, \quad f \in L_E^1.$$

Letting  $f$  be some element of  $E$ , we see that  $\phi^{1-q} \in L^1$ . Let  $h = \phi^{1/q-1}$ . Then  $h^q = \phi^{1-q} \in L^1$ , so  $h \in L^q \subset L^1$  and  $\hat{h}(0) > 0$ .

For  $f \in L_E^1$ , let

$$Tf(x) = f(x)h(x).$$

It follows from (2) that  $Tf \in L^q$  and

$$\|Tf\|_q \leq K \|f\|_1.$$

If  $f \in L_E^1$  and  $x \in G$  then  $f_x \in L_E^1$ , where  $f_x(y) = f(x+y)$ , since  $L_E^1$  is a translation-invariant subspace of  $L^1$ .

The map  $x \rightarrow (T(f_x))_{-x}$  is continuous from  $G$  into  $L^q$ . Thus we may define  $\tilde{T}$  from  $L_E^1$  to  $L^q$  by the following vector-valued integral:

$$\tilde{T}(f) = \int_G (T(f_x))_{-x} dx, \quad f \in L_E^1,$$

(cf. [3], p. 154). Then

$$\|\tilde{T}(f)\|_q \leq \|T(f)\|_q \leq K \|f\|_1, \quad f \in L_E^1,$$

so  $\tilde{T}$  is a bounded linear operator from  $L_E^1$  to  $L^q$ . Now

$$\begin{aligned} \tilde{T}(f) &= \int_G (T(f_x))_{-x} dx = \int_G (hf_x)_{-x} dx \\ &= \int_G h_{-x} f dx = \hat{h}(0) f. \end{aligned}$$

Thus  $f \in L_E^1$  implies  $f \in L_E^q$ , so  $L_E^1 = L_E^q$  and  $E$  is of type  $\Lambda(q)$ .

If  $p > 1$ , then  $L_E^1 = L_E^p$  and the  $L^1$  and  $L^p$  norms are equivalent there. It follows from Theorem 13 of [4] that (2) holds for some  $q > p$ . Thus, as shown above,  $E$  is of type  $\Lambda(q)$ .

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