ON $\Lambda(p)$ SETS

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In this note it is shown that if \(1 \leq p < 2\) and \(E\) is a set of type \(\Lambda(p)\) in the dual of a compact abelian group, then \(E\) is of type \(\Lambda(p + \varepsilon)\) for some \(\varepsilon > 0\).

Introduction. Let \(G\) be a compact abelian group with dual group \(\Gamma\). For \(0 < p < \infty\), we denote by \(L^p(G)\) the set of complex-valued measurable functions \(f\) on \(G\) such that

\[
\|f\|_p = \left( \int_G |f(x)|^p \, dx \right)^{1/p}
\]

is finite, where \(dx\) denotes normalized Haar measure on \(G\). For \(f \in L^1(G)\), the Fourier transform is defined by

\[
\hat{f}(\gamma) = \int_G f(x)(x, \gamma) \, dx, \quad \gamma \in \Gamma.
\]

As in [5], we call a subset \(E \subseteq \Gamma\) a set of type \(\Lambda(p)\) if there exists a \(q < p\) and a constant \(K_q\) such that

\[
(1) \quad \|P\|_p \leq K_q \|P\|_q
\]

for all trigonometric polynomials \(P\) such that \(\hat{P} = 0\) outside \(E\).

As shown in [5], if (1) holds for some \(q\), \(0 < q < p\), then it holds for all such \(q\). Also, if \(p > 1\), then the definition of \(\Lambda(p)\) set is equivalent to the statement that \(L^q_{\#} = L^p_{\#}\) for some \(q, 1 \leq q < p\), where \(L^q_{\#} = \{f \in L^q : \hat{f} = 0\}\) outside \(E\). For further details on \(\Lambda(p)\) sets, the reader is referred to [1] or [5].

In this note we apply results of [4] to show the following:

**Theorem.** Let \(1 \leq p < 2\). If \(E\) is of type \(\Lambda(p)\), then \(E\) is of type \(\Lambda(p + \varepsilon)\) for some \(\varepsilon > 0\).

This result is in contrast to the situation when \(p\) is an even integer, \(p \geq 4\). In that case there are known to exist sets of type \(\Lambda(p)\) which are not of type \(\Lambda(p + \varepsilon)\) when \(G\) is the circle group [5], and also for a large class of compact abelian groups [2].

The Main Result. We shall proceed to the proof of the theorem after establishing two lemmas; these lemmas were communicated to the authors by Haskell Rosenthal.

**Lemma 1.** Suppose \(X\) is a nonreflexive subspace of \(L'(\mu)\), where
μ is a probability measure on some measure space. Then given δ > 0 and M > 0 there exists f ∈ X with ∥f∥₁ = 1 and
\[ \int_S |f(x)| \, dμ(x) > 1 - δ, \]
where \( S = \{x : |f(x)| \geq M\} \).

Proof. Suppose there exists \( M > 0 \) and \( δ > 0 \) so that if \( f \in X \) and ∥f∥₁ = 1 then
\[ \int_S |f(x)| \, dμ(x) \leq 1 - δ. \]

Choose ε > 0 so that \( Mε < δ/2 \). Since X is nonreflexive, it follows from Lemmas 6 and 7 of [4] that there exists \( f \in X \) and a measurable set \( F \) with ∥f∥₁ = 1, \( μ(F) < ε \) and
\[ \int_F |f(x)| \, dμ(x) > 1 - δ/2. \]

We have
\[
1 - δ/2 < \int_F |f(x)| \, dμ(x) = \int_{F \cap S} |f(x)| \, dμ(x) + \int_{F \setminus S} |f(x)| \, dμ(x) \\
\leq \int_S |f(x)| \, dμ(x) + \int_F Mdμ(x) \leq 1 - δ + Mε \\
< 1 - δ + δ/2 = 1 - δ/2,
\]
a contradiction.

Lemma 2. If \( E \) is of type \( A(1) \), then \( L^1_E \) is reflexive.

Proof. Suppose \( L^1_E \) is nonreflexive. Let \( M, δ > 0 \) and let \( f \in L^1_E \) be as given by Lemma 1.

If \( 0 < p < 1 \), then
\[
1 \geq \int_S |f(x)| \, dx = \int_S |f(x)|^p |f(x)|^{1-p} \, dx \geq \left( \int_S |f(x)|^p \, dx \right)^{1-p} M^{1-p},
\]
so
\[ \int_S |f(x)|^p \, dx \leq 1/M^{1-p}. \]

But
\[
\left( \int_{S^c} |f(x)|^p \, dx \right)^{1/p} \leq \int_{S^c} |f(x)| \, dx < δ,
\]
so
Now this last quantity can be made arbitrarily small, so it follows from (1) that $E$ is not of type $A(1)$.

**Proof of Theorem.** First suppose that $p = 1$. By Lemma 2, $L^1_E$ is reflexive. It follows from Theorem 1 and Lemma 6 of [4] that there exists $q > 1$ and a nonnegative function $\phi \in L^1$ such that $0 \neq ||\phi||_1 \leq 1$ and

\[
\left( \int_G |f(x)|^q \phi^{1-q}(x) dx \right)^{1/q} \leq K \int_G |f(x)| dx \quad , \quad f \in L^1_E .
\]

Letting $f$ be some element of $E$, we see that $\phi^{1-q} \in L^q$. Let $h = \phi^{1-q-1}$. Then $h^q = \phi^{1-q} \in L^1$, so $h \in L^q \subset L^1$ and $h(0) > 0$.

For $f \in L^1_E$, let

\[ T(f)(x) = f(x)h(x) . \]

It follows from (2) that $Tf \in L^q$ and

\[ ||Tf||_q \leq K ||f||_1 . \]

If $f \in L^1_E$ and $x \in G$ then $f_x \in L^1_E$, where $f_x(y) = f(x + y)$, since $L^1_E$ is a translation-invariant subspace of $L^1$.

The map $x \rightarrow (T(f_x))_{-x}$ is continuous from $G$ into $L^q$. Thus we may define $T$ from $L^1_E$ to $L^q$ by the following vector-valued integral:

\[ \tilde{T}(f) = \int_G (T(f_x))_{-x} dx \quad , \quad f \in L^1_E , \]

(cf. [3], p. 154). Then

\[ ||\tilde{T}(f)||_q \leq ||T(f)||_q \leq K ||f||_1 \quad , \quad f \in L^1_E , \]

so $\tilde{T}$ is a bounded linear operator from $L^1_E$ to $L^q$. Now

\[ \tilde{T}(f) = \int_G (T(f_x))_{-x} dx = \int_G (hf_x)_{-x} dx \]

\[ = \int_G h_{-x}f dx = \tilde{h}(0)f . \]

Thus $f \in L^1_E$ implies $f \in L^q_E$, so $L^1_E = L^q_E$ and $E$ is of type $A(q)$.

If $p > 1$, then $L^1_E = L^p_E$ and the $L^1$ and $L^p$ norms are equivalent there. It follows from Theorem 13 of [4] that (2) holds for some $q > p$. Thus, as shown above, $E$ is of type $A(q)$. 
REFERENCES


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