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In this note it is shown that if $1 \leqq p < 2$ and E is a set of type $\Lambda(p)$ in the dual of a compact abelian group, then E is of type $\Lambda(p+\varepsilon)$ for some $\varepsilon > 0$.

Introduction. Let G be a compact abelian group with dual group Γ. For $0 < p < \infty$, we denote by $L^p(G)$ the set of complex-valued measurable functions f on G such that

$$\| f \|_p = \left(\int_G | f(x) |^p \, dx \right)^{1/p}$$

is finite, where dx denotes normalized Haar measure on G. For $f \in L^1(G)$, the Fourier transform is defined by

$$\hat{f}(\gamma) = \int_G f(x) \overline{(x, \gamma)} dx , \qquad \gamma \in \Gamma .$$

As in [5], we call a subset $E \subset \Gamma$ *a set of type* $\Lambda(p)$ if there exists a $q < p$ and a constant K_q such that

$$\| P \|_p \leqq K_q \| P \|_q \tag{1}$$

for all trigonometric polynomials P such that $\hat{P} = 0$ outside E.

As shown in [5], if (1) holds for some q, $0 < q < p$, then it holds for all such q. Also, if $p > 1$, then the definition of $\Lambda(p)$ set is equivalent to the statement that $L_E^q = L_E^p$ for some $q, 1 \leqq q < p$, where $L_E^q = \{ f \in L^q : \hat{f} = 0 \text{ outside } E \}$. For further details on $\Lambda(p)$ sets, the reader is referred to [1] or [5].

In this note we apply results of [4] to show the following:

THEOREM. *Let $1 \leqq p < 2$. If E is of type $\Lambda(p)$, then E is of type $\Lambda(p+\varepsilon)$ for some $\varepsilon > 0$.*

This result is in contrast to the situation when p is an even integer, $p \geqq 4$. In that case there are known to exist sets of type $\Lambda(p)$ which are not of type $\Lambda(p+\varepsilon)$ when G is the circle group [5], and also for a large class of compact abelian groups [2].

The Main Result. We shall proceed to the proof of the theorem after establishing two lemmas; these lemmas were communicated to the authors by Haskell Rosenthal.

LEMMA 1. *Suppose X is a nonreflexive subspace of $L^1(\mu)$, where*

μ is a probability measure on some measure space. Then given $\delta > 0$ and $M > 0$ there exists $f \in X$ with $\|f\|_1 = 1$ and

$$\int_S |f(x)|\, d\mu(x) > 1 - \delta ,$$

where $S = \{x: |f(x)| \geqq M\}$.

Proof. Suppose there exists $M > 0$ and $\delta > 0$ so that if $f \in X$ and $\|f\|_1 = 1$ then

$$\int_S |f(x)|\, d\mu(x) \leqq 1 - \delta .$$

Choose $\varepsilon > 0$ so that $M\varepsilon < \delta/2$. Since X is nonreflexive, it follows from Lemmas 6 and 7 of [4] that there exists $f \in X$ and a measurable set F with $\|f\|_1 = 1$, $\mu(F) < \varepsilon$ and

$$\int_F |f(x)|\, d\mu(x) > 1 - \delta/2 .$$

We have

$$\begin{aligned} 1 - \delta/2 &< \int_F |f(x)|\, d\mu(x) = \int_{F\cap S} |f(x)|\, d\mu(x) + \int_{F\cap S^\sim} |f(x)|\, d\mu(x) \\ &\leqq \int_S |f(x)|\, d\mu(x) + \int_F M d\mu(x) \leqq 1 - \delta + M\varepsilon \\ &< 1 - \delta + \delta/2 = 1 - \delta/2 , \end{aligned}$$

a contradiction.

LEMMA 2. *If E is of type $\Lambda(1)$, then L^1_E is reflexive.*

Proof. Suppose L^1_E is nonreflexive. Let $M, \delta > 0$ and let $f \in L^1_E$ be as given by Lemma 1.

If $0 < p < 1$, then

$$1 \geqq \int_S |f(x)|\, dx = \int_S |f(x)|^p |f(x)|^{1-p} dx \geqq \left(\int_S |f(x)|^p\, dx\right) M^{1-p} ,$$

so

$$\int_S |f(x)|^p dx \leqq 1/M^{1-p} .$$

But

$$\left(\int_{S^\sim} |f(x)|^p\, dx\right)^{1/p} \leqq \int_{S^\sim} |f(x)|\, dx < \delta ,$$

so

$$\|f\|_p = \left(\int_S |f(x)|^p\,dx + \int_{S^\sim} |f(x)|^p\,dx\right)^{1/p}$$
$$\leqq (1/M^{1-p} + \delta^p)^{1/p} .$$

Now this last quantity can be made arbitrarily small, so it follows from (1) that E is not of type $\Lambda(1)$.

Proof of Theorem. First suppose that $p = 1$. By Lemma 2, L_E^1 is reflexive. It follows from Theorem 1 and Lemma 6 of [4] that there exists $q > 1$ and a nonnegative function $\phi \in L^1$ such that $0 \neq \|\phi\|_1 \leqq 1$ and

$$(2) \qquad \left(\int_G |f(x)|^q \phi^{1-q}(x)dx\right)^{1/q} \leqq K\int_G |f(x)|dx , \qquad f \in L_E^1 .$$

Letting f be some element of E, we see that $\phi^{1-q} \in L^1$. Let $h = \phi^{1/q-1}$. Then $h^q = \phi^{1-q} \in L^1$, so $h \in L^q \subset L^1$ and $\hat{h}(0) > 0$.

For $f \in L_E^1$, let

$$Tf(x) = f(x)h(x) .$$

It follows from (2) that $Tf \in L^q$ and

$$\| Tf \|_q \leqq K\|f\|_1 .$$

If $f \in L_E^1$ and $x \in G$ then $f_x \in L_E^1$, where $f_x(y) = f(x + y)$, since L_E^1 is a translation-invariant subspace of L^1.

The map $x \longrightarrow (T(f_x))_{-x}$ is continuous from G into L^q. Thus we may define $\tilde{T}$ from L_E^1 to L^q by the following vector-valued integral:

$$\tilde{T}(f) = \int_G (T(f_x))_{-x}dx , \qquad f \in L_E^1 ,$$

(cf. [3], p. 154). Then

$$\| \tilde{T}(f) \|_q \leqq \| T(f) \|_q \leqq K\| f \|_1 , \qquad f \in L_E^1 ,$$

so $\tilde{T}$ is a bounded linear operator from L_E^1 to L^q. Now

$$\tilde{T}(f) = \int_G (T(f_x))_{-x}dx = \int_G (hf_x)_{-x}dx$$
$$= \int_G h_{-x}f\,dx = \hat{h}(0)f .$$

Thus $f \in L_E^1$ implies $f \in L_E^q$, so $L_E^1 = L_E^q$ and E is of type $\Lambda(q)$.

If $p > 1$, then $L_E^1 = L_E^p$ and the L^1 and L^p norms are equivalent there. It follows from Theorem 13 of [4] that (2) holds for some $q > p$. Thus, as shown above, E is of type $\Lambda(q)$.

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