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INDECOMPOSABLE MODULES FOR DIRECT PRODUCTS OF FINITE GROUPS

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An essentially known result is made explicit and its converse is proved, thereby showing that if K is a field of prime characteristic p, P a finite p-group and H a finite p'-group, then every finitely generated indecomposable $K(P\times H)$-module is a tensor product of an indecomposable KP-module with an indecomposable KH-module if and only if either P is cyclic or K is a splitting field for H.

Let R be a commutative ring with 1, G and H finite groups. If V, W are RG-, RH-modules respectively, then $V \otimes_R W$ is an $R(G \times H)$-module, where $(v \otimes w)gh = vg \otimes wh$ for all $v \in V$, $w \in W$, $g \in G$, $h \in H$. The following false assertion is made in [3]:

If K is a field of prime characteristic p, P a finite p-group, and H a finite p'-group, then every finitely generated indecomposable $K(P \times H)$-module is isomorphic to some $V \otimes W$, where V, W are finitely generated indecomposable KP-, KH-modules respectively.

A correct version, with the additional hypothesis that K is algebraically closed, is proved in [1]. The purpose of this note is to give the exact conditions under which the above conclusion is true.

All rings (and algebras) are assumed to have an identity, all modules (and algebras) are unital and finitely generated, and all groups are finite. $J(R)$ denotes the Jacobson radical of a ring R. Our main result, which is proved after some preliminary steps, is

THEOREM 1. *Let p be a prime, K a field of characteristic p, P a p-group and H a p'-group. Every indecomposable $K(P \times H)$-module is isomorphic to some $V \otimes W$, where V, W are indecomposable KP-, KH-modules respectively, if and only if either P is cyclic or K is a splitting field for H.*

PROPOSITION 2. *Let R be a commutative ring with A. C. C. such that $\bar{R} = R/J(R)$ has D. C. C. Let A and B be R-algebras. Then*

$$A \underset{R}{\otimes} B/J(A \otimes B) \approx ((A/J(A)) \otimes (B/J(B)))/J((A/J(A)) \otimes (B/J(B)))$$

as R-algebras. Furthermore, if either $A/J(A) \approx \bar{R}$ or $\bar{R}$ is a perfect field, then $J((A/J(A)) \otimes (B/J(B))) = (0)$, so that $A \otimes B/J(A \otimes B) \approx (A/J(A)) \otimes (B/J(B))$.

Proof. $J(R)A \subseteqq J(A)$ [4, I. 8.15], so $J(A)/J(R)A = J(A/J(R)A)$. Since $A/J(R)A$ has D. C. C., $J(A)^n \subseteqq J(R)A$ for some positive integer n. Let $\iota(J(A) \otimes B)$ denote the natural image of $J(A) \otimes B$ in $A \otimes B$. Then $(\iota(J(A) \otimes B))^n = \iota(J(A)^n \otimes B) \subseteqq \iota(J(R)A \otimes B) = J(R)(A \otimes B)$. So $\iota(J(A) \otimes B)/J(R)(A \otimes B)$ is a nilpotent ideal in $A \otimes B/J(R)(A \otimes B)$, whence

$$\begin{aligned} \iota(J(A) \otimes B)/J(R)(A \otimes B) &\subseteqq J(A \otimes B/J(R)(A \otimes B)) \\ &= J(A \otimes B)/J(R)(A \otimes B) \,. \end{aligned}$$

Therefore $\iota(J(A) \otimes B) \subseteqq J(A \otimes B)$, and similarly $\iota(A \otimes J(B)) \subseteqq J(A \otimes B)$.

Let $C = \iota(A \otimes J(B)) + \iota(J(A) \otimes B)$. Then $(A/J(A)) \otimes (B/J(B)) \approx (A \otimes B)/C$ as R-algebras. (The obvious homomorphism in each direction is indeed well-defined.) Since we have shown $C \subseteqq J(A \otimes B)$, it follows that

$$\begin{aligned} A \otimes B/J(A \otimes B) &\approx ((A \otimes B)/C)/((J(A \otimes B))/C) \\ &= ((A \otimes B)/C)/J((A \otimes B)/C) \\ &\approx ((A/J(A)) \otimes (B/J(B)))/J((A/J(A)) \otimes (B/J(B))) \,. \end{aligned}$$

Let $A/J(A) = X$, $B/J(B) = Y$. $J(R)X = (0) = J(R)Y$ implies $X \otimes_R Y \approx X \otimes_{\bar{R}} Y$. If $X \approx \bar{R}$, then $X \otimes_{\bar{R}} Y \approx Y = B/J(B)$ implies $J(X \otimes_{\bar{R}} Y) = (0)$.

Suppose $\bar{R}$ is a perfect field. Since X and Y are each the direct sum of a finite number of simple ideals, to prove $J(X \otimes_{\bar{R}} Y) = (0)$ it suffices to assume that X and Y are simple. Let K be the center of X, F the center of Y. K and F are extension fields of $\bar{R}$. By a theorem of Azumaya and Nakayama [6, V. 9.1], the lattice of ideals of $K \otimes_{\bar{R}} F$ is isomorphic to the lattice of ideals of $X \otimes_{\bar{R}} Y$ under the correspondence $I \to I(X \otimes Y)$, where I is an ideal of $K \otimes F$. Since $I(X \otimes Y)$ is nilpotent if and only if I is, this correspondence preserves the radical. So $J(X \otimes Y) = (0)$ if and only if $J(K \otimes F) = (0)$. Since $\bar{R}$ is perfect, F is a separable extension, so that

$$J(K \otimes F) = J(K) \otimes F = (0) \otimes F = (0)$$

[2, (69.10)].

Let R be a complete local domain, as in [4, I. 17] (i.e., R is either a complete discrete valuation ring or a field). Let $\bar{R} = R/J(R)$. Let G and H be groups. Let $E_{RG}(V)$ denote the R-algebra of all RG-linear maps of V into V. Consider the following two properties:

(A) *Every R-free indecomposable $R(G \times H)$-module has the form $V \otimes_R W$ for some R-free indecomposable RG-, RH-modules V, W respectively.*

(B) *For all R-free indecomposable RG-, RH-modules V, W respectively, $V \otimes W$ is an indecomposable $R(G \times H)$-module.*

PROPOSITION 3. (i) (A) *implies* (B).

(ii) *If* $E_{RG}(V)/J(E_{RG}(V)) \approx \bar{R}$ *for each* R*-free indecomposable* RG*-module* V, *then* (B) *holds.*

(iii) *If* $|H|$ *is prime to the characteristic of* $\bar{R}$, *then* (B) *implies* (A).

The proof of (iii) is essentially given in [4, III. 3.7], but we include it here for the sake of completeness. Note that if R is an algebraically closed field of characteristic zero, (ii) and (iii) imply the standard result on the irreducible characters of a direct product.

Proof. (i) Let V, W be R-free indecomposable RG-, RH-modules respectively. Let $V \otimes W = \bigoplus \sum_{i=1}^{n} U_i$ where the U_i are indecomposable $R(G \times H)$-modules (necessarily R-free). Each $U_i \approx V_i \otimes W_i$ for some R-free indecomposable RG-, RH-modules V_i, W_i respectively. Then

$$(V \otimes W)_{RG} = \bigoplus (\mathrm{rank}_R\, W) V \approx \bigoplus \sum_{i=1}^{n} (\mathrm{rank}_R\, W_i) V_i$$

and

$$(V \otimes W)_{RH} = \bigoplus (\mathrm{rank}_R\, V) W \approx \bigoplus \sum_{i=1}^{n} (\mathrm{rank}_R\, V_i) W_i\,.$$

The unique decomposition property [4, I. 11.5] implies $V_i \approx V$ and $W_i \approx W$ for $1 \leqq i \leqq n$. Then $V \otimes W \approx \bigoplus n(V \otimes W)$ implies $n = 1$, hence $V \otimes W$ is indecomposable.

(ii) Let $E = E_{R(G \times H)}(V \otimes W)$. By [4, III. 3.6], $E \approx E_{RG}(V) \otimes_R E_{RH}(W)$ as an R-algebra. Since W is indecomposable, $E_{RH}(W)$ has no idempotents besides the identity. Hence $E_{RH}(W)/J(E_{RH}(W))$ is a division ring [4, I. 12.6, I. 10.1]. By Proposition 2,

$$\begin{aligned} E/J(E) &\approx (E_{RG}(V)/J(E_{RG}(V))) \otimes_R (E_{RH}(W)/J(E_{RH}(W))) \\ &\approx \bar{R} \otimes_R (E_{RH}(W)/J(E_{RH}(W))) \approx \bar{R} \otimes_{\bar{R}} (E_{RH}(W)/J(E_{RH}(W))) \\ &\approx E_{RH}(W)/J(E_{RH}(W))\,. \end{aligned}$$

Thus E is a local ring and has no idempotents other than the identity. It follows that $V \otimes W$ is indecomposable.

(iii) Let U be an indecomposable $R(G \times H)$-module. Since $|G \times H : G|$ is a unit in R, there is an indecomposable RG-module V such that $U \mid V^{G \times H} = V \otimes_{RG} R(G \times H)$ by [4, II. 3] and unique decomposition. $V \otimes_{RG} R(G \times H) \approx V \otimes_{RG} (RG \otimes_R RH) \approx (V \otimes_{RG} RG) \otimes_R RH \approx V \otimes_R RH$, where the isomorphisms are $R(G \times H)$-linear. Let $RH = \bigoplus \sum_{i=1}^{n} W_i$, a sum of indecomposable RH-modules. Then

$U \mid \oplus \sum_{i=1}^{n} V \otimes_R W_i$. Since each $V \otimes W_i$ is indecomposable by (B), $U \approx V \otimes W_j$ for some j by unique decomposition.

PROPOSITION 4. *Let R be a finite field. Then* (B) *holds if and only if for all indecomposable RG-, RH-modules V, W respectively,*

$$([E_{RG}(V)/J(E_{RG}(V)) \colon R]\,,\quad [E_{RH}(W)/J(E_{RH}(W)) \colon R]) = 1\,.$$

Proof. V and W indecomposable imply $E_{RG}(V)/J(E_{RG}(V)) = R'$ and $E_{RH}(W)/J(E_{RH}(W)) = R''$ are division rings, and hence fields by Wedderburn's theorem. Since R is perfect, [4, III. 3.6] and Proposition 2 imply that $V \otimes W$ is indecomposable if and only if $R' \otimes_R R''$ is a field. It is well-known that (for R finite) $R' \otimes_R R''$ is a field if and only if $([R' \colon R], [R'' \colon R]) = 1$.

EXAMPLES. (1) Proposition 3(iii) is not true if the assumption on $|H|$ is dropped. For instance, let $H = G$ be a (non-trivial) cyclic p-group, with $R = \bar{R}$ a field of characteristic p. For all indecomposable RG-modules V_i and V_j (see the proof of Theorem 1 below), $V_i \otimes V_j$ is an indecomposable $R(G \times H)$-module by Proposition 3(ii). But there are only finitely many such modules, while $G \times H$ has infinitely many nonisomorphic indecomposable modules over R [2, (64.1)].

(2) If R is a finite field of characteristic p, and G, H are p'-groups, (A) and (B) may be true with R not a splitting field for either group. Let $R = GF$ (11), G be cyclic of order 7, H cyclic of order 3. Considering the decomposition of RG as a cyclic RG-module, we find there exist two nonlinear irreducible RG-modules, say U_i for $i = 1, 2$ such that $[E_{RG}(U_i) \colon R] = 3$. Similarly, there is one nonlinear irreducible RH-module W with $[E_{RH}(W) \colon R] = 2$. So (B) and (A) hold, by Proposition 4 and Proposition 3(iii).

On the other hand, let $R = GF$ (11), H cyclic of order 3, and G cyclic of order 4. G has one nonlinear irreducible RG-module V with $[E_{RG}(V) \colon R] = 2$. Then Proposition 4 implies (B) and (A) fail. However, if $R = Q$ is the rational field, then there is one nonlinear irreducible QG-module V with $E_{QG}(V) \approx Q(\sqrt{-1})$, and one nonlinear irreducible QH-module W with $E_{QH}(W) \approx Q(\sqrt{-3})$. Since $Q(\sqrt{-1}) \otimes_Q Q(\sqrt{-3})$ is a field, $V \otimes W$ is indecomposable and (A), (B) hold.

Proof of Theorem 1. If K is a splitting field for H, then for all indecomposable (hence irreducible) KH-modules W, $E_{KH}(W) \approx K$. So every indecomposable $K(P \times H)$-module is isomorphic to some $V \otimes W$ by Proposition 3(ii) (applied to H) and (iii).

If P is cyclic of order p^n, then the p^n distinct indecomposable

KP-modules are given by $V_i = KP/J(KP)^i$ for $1 \leqq i \leqq p^n$ [2, (64.2)]. Since V_i is cyclic, $E_{KP}(V_i) \approx KP/J(KP)^i$. Hence $E_{KP}(V_i)/J(E_{KP}(V_i)) \approx KP/J(KP) \approx K$ for $1 \leqq i \leqq p^n$. Again, every indecomposable $K(P \times H)$-module is isomorphic to some $V_i \otimes W$ by Proposition 3(ii) and (iii).

Suppose K is not a splitting field for H and P is not cyclic. There is an irreducible KH-module W with $D = E_{KH}(W)$ a division ring of dimension greater than one as a K-algebra. Pick $\alpha \in D - K$, and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be the irreducible polynomial for α over K. (So each $a_i \in K$ and $n > 1$.) $K \subsetneqq K(\alpha) \subseteqq D$.

Let g_1 and g_2 generate the noncyclic group of order p^2, which is a homomorphic image of P. Let Y be a vector space over K of dimension $2n$, with basis $u_1, u_2, \cdots, u_n, y_1, y_2, \cdots, y_n$.

Let

$$T = (t_{ij}) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots & & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots\cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots\cdots & & -a_{n-1} \end{pmatrix}_{n \times n} .$$

Then Y is an indecomposable KP-module, with $(u_i)(g_1 - 1) = y_i$, $(u_i)(g_2 - 1) = \sum_j t_{ij}y_j$, $(y_i)(g_1 - 1) = 0 = (y_i)(g_2 - 1)$ [5, Proposition 5]. We show that in fact $E_{KP}(Y)/J(E_{KP}(Y)) \approx K(\alpha)$:

$E_{KP}(Y)$ consists of all $2n \times 2n$ matrices over K which commute with both $\begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array}$ and $\begin{array}{c|c} 0 & T \\ \hline 0 & 0 \end{array}$, where the blocks are $n \times n$. A matrix commutes with the first if and only if it has the form $\begin{array}{c|c} A & B \\ \hline 0 & A \end{array}$, and also commutes with the second if and only if $AT = TA$. An n-dimensional K-space $\oplus \sum_{i=1}^n Kv_i$ is a faithful, cyclic $K[T]$-module with $v_iT = \sum_{j=1}^n t_{ij}v_j$. Hence $AT = TA$ implies $A \in K[T]$. Now $f(x)$ is the minimum polynomial for T, so $K[T] \approx K[x]/\langle f(x)\rangle \approx K(\alpha)$. Thus, $J(E_{KP}(Y))$ equals the set of all nonunits in $E_{KP}(Y)$, namely the set of all matrices of the form $\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}$. Then $E_{KP}(Y)/J(E_{KP}(Y)) \approx K[T] \approx K(\alpha)$.

Let $F \supseteqq K$ be a splitting field for H. Then $W \otimes_K F \approx \oplus \sum_{i,j} U_{ij}$ where each U_{ij} is an absolutely irreducible FH-module, and $U_{ij} \approx U_{st}$ if and only if $i = s$. Since $E_{FH}(U_{ij}) \approx F$ for all i, j, we have $D \otimes_K F \approx E_{FH}(W \otimes_K F) \approx \oplus \sum_i F_{n_i}$, a direct sum of full matrix algebras over F. Then D is a separable K-algebra [2, (71.2)]. It follows that $J(K(\alpha) \otimes_K D) = (0)$.

Let $E = E_{K(P \times H)}(Y \otimes_K W)$. By [4, III. 3.6] and Proposition 2,

$$E/J(E) \approx K(\alpha) \otimes D/J(K(\alpha) \otimes D) = K(\alpha) \otimes D .$$

However, $K(\alpha)\otimes D \supseteqq K(\alpha)\otimes K(\alpha) \approx K(\alpha)[x]/\langle f(x)\rangle$ which contains zero divisors, since $f(x)$ is reducible over $K(\alpha)$. Therefore, $E/J(E)$ is not a division ring, whence E contains more than one idempotent. So $Y\otimes W$ is decomposable. Proposition 3(i) implies not every indecomposable $K(P\times H)$-module is of the given form.

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