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SINGULAR PERTURBATION OF A TIME-DEPENDENT CAUCHY PROBLEM IN A HILBERT SPACE

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Let A be a self-adjoint operator, not necessarily bounded, in the Hilbert space H, with resolution of the identity E_λ. Define $h(t, A) = \int_{-\infty}^{\infty} h(t, \lambda) dE_\lambda$. It is shown that as $\varepsilon \to 0+$ the solution of the abstract problem $\varepsilon U_\varepsilon'' + bU_\varepsilon' + h(t, A)U_\varepsilon = 0$, $U_\varepsilon(0) = x_0$, $U_\varepsilon'(0) = x_1$ tends in the norm of H to the solution of $bU_0' + h(t, A)U_0 = 0$, $U_0(0) = x_0$ for data x_0, x_1 in a dense subset of H.

Let A be a (possibly unbounded) self-adjoint operator in a Hilbert space H, and let E_λ be the resolution of the identity for A, so that

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x$$

for $x \in D(A) \subset H$. Let $h(t, \lambda)$ defined on $[0, \infty) \times (-\infty, \infty)$ be a Borel measurable function of λ for fixed t and a continuous function of t for fixed real λ. Then an operator $h(t, A)$ can be defined by

$$h(t, A)x = \int_{-\infty}^{\infty} h(t, \lambda) dE_\lambda x$$

for $x \in D(h(t, A))$, where

$$D(h(t, A)) = \left\{ x \in H: \int_{-\infty}^{\infty} |h(t, \lambda)|^2 d\|E_\lambda x\|^2 < \infty \right\} .$$

We shall be concerned with the behavior as $\varepsilon \to 0+$ of the solution of the problem

$$(1) \qquad \varepsilon U_\varepsilon'' + bU_\varepsilon' + h(t, A)U_\varepsilon = 0,\ U_\varepsilon(0) = x_0,\ U_\varepsilon'(0) = x_1 ,$$

where b is a positive constant. It seems reasonable to expect that $U_\varepsilon \to U_0$ as $\varepsilon \to 0$, where U_0 solves the problem

$$(2) \qquad bU_0' + h(t, A)U_0 = 0,\ U_0(0) = x_0 .$$

We prove this convergence, as well as $U_\varepsilon' \to U_0'$, in the norm of H for data x_0, x_1 restricted to a certain dense subset of H.

Several abstract singular perturbation problems of this nature have been considered before. Kisyński [5] considered the case $h(t, A) \equiv A$ where A is positive as well as self-adjoint; in addition, he considered the inhomogeneous problem. Smoller [9, 10], Latil [6], Friedman [4] and the authors [1] have extended his results to higher-order equations and have removed the restriction that A be positive. The use of the resolution of the identity for A and estimates for the special case $H = L^2(-\infty, \infty)$, A = a real parameter λ is central to all these treatments, as well as to the present study. Singular perturbations in Banach spaces have been studied by Bobisud and Hersh [2], Sova [11], and Schoene [8].

Time-dependent equations of the form

$$\varepsilon p(t) U_\varepsilon'' + q(t) U_\varepsilon' + A U_\varepsilon = 0$$

and higher-order generalizations have been considered by Friedman [4]. The only previous study of a nonfactorable time-variable operator $h(t, A)$ which is known to the authors is that of Nur [7], who considers the case $h(t, \lambda) = e^{\lambda t}$, so $h(t, A)$ is a semigroup with generator A. The result of Nur is contained in the theorems to follow.

As mentioned above, we begin by examining in part 1 the special case $H = L^2(-\infty, \infty)$ and show that $u_\varepsilon(t, \lambda) \to u_0(t, \lambda)$, where

$$\varepsilon u_\varepsilon'' + b u_\varepsilon' + h(t, \lambda) u_\varepsilon = 0,\ u_\varepsilon(0, \lambda) = x_0,\ u_\varepsilon'(0, \lambda) = x_1\ , \tag{3}$$

$$b u_0' + h(t, \lambda) u_0 = 0\ ,\quad u_0(0, \lambda) = x_0\ . \tag{4}$$

We also establish for this case certain estimates to be used in treating the Hilbert space problem in part 2.

1. The problem on the real line.

Since the problems (3), (4) are linear, we may write

$$u_\varepsilon(t, \lambda) = p_\varepsilon(t, \lambda) x_0 + q_\varepsilon(t, \lambda) x_1\ ,$$
$$u_0(t, \lambda) = p_0(t, \lambda) x_0$$

for certain functions p_ε, q_ε, p_0. Regarding a solution of (3) for fixed ε as a solution of the equation $\varepsilon u_\varepsilon'' + b u_\varepsilon' = -h(t, \lambda) u_\varepsilon(t)$, we find that $u_\varepsilon(t, \lambda)$ satisfies the following integral equation:

$$\begin{aligned} u_\varepsilon(t, \lambda) &= x_0 + \frac{\varepsilon}{b}[1 - e^{-(b/\varepsilon)t}] x_1 \\ &\quad - \frac{1}{b}\int_0^t h(s, \lambda)[1 - e^{-(b/\varepsilon)(t-s)}] u_\varepsilon(s, \lambda) ds\ . \end{aligned} \tag{5}$$

Similarly, for u_0 we obtain the integral equation

$$u_0(t, \lambda) = x_0 - \frac{1}{b}\int_0^t h(s, \lambda) u_0(s, \lambda) ds\ ; \tag{6}$$

thus for the difference we have

$$\begin{aligned} u_\varepsilon(t, \lambda) - u_0(t, \lambda) = {} & \frac{\varepsilon}{b}[1 - e^{-(b/\varepsilon)t}]x_1 \\ & - \frac{1}{b}\int_0^t h(s, \lambda)[u_\varepsilon(s, \lambda) - u_0(s, \lambda)]ds \\ & + \frac{1}{b}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}u_\varepsilon(s, \lambda)ds \ . \end{aligned} \tag{7}$$

For convenience we define

$$M_T(\lambda) = 1 + \frac{1}{b}e^{1/b\int_0^T |h(s,\lambda)|ds}\int_0^T |h(s, \lambda)|ds \ .$$

Observe that $M_t(\lambda) \leqq 2\exp\left\{2\int_0^T |h(s, \lambda)|ds\right\}$ since $\alpha e^\alpha \leqq e^{2\alpha}$.

Lemma 1. *For $t \in [0, T]$*

$$\begin{aligned} |p_\varepsilon(t, \lambda)| &\leqq M_T(\lambda) \ , \\ |q_\varepsilon(t, \lambda)| &\leqq \frac{\varepsilon}{b}M_T(\lambda) \ , \\ |p_0(t, \lambda)| &\leqq M_T(\lambda) \ . \end{aligned}$$

Proof. From (5) we obtain

$$|u_\varepsilon(t, \lambda)| \leqq |x_0| + \frac{\varepsilon}{b}|x_1| + \frac{1}{b}\int_0^t |h(s, \lambda)|\,|u_\varepsilon(s, \lambda)|ds \ ;$$

Gronwall's lemma [3, p. 37] then implies that

$$|u_\varepsilon(t, \lambda)| \leqq \left[|x_0| + \frac{\varepsilon}{b}|x_1|\right]M_T(\lambda) \ . \tag{8}$$

The first two statements of the lemma follow on setting $x_0 = 1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. The last statement follows in the same manner from (6).

Theorem 1. *For any $T > 0$ and any fixed λ, $p_\varepsilon(t, \lambda) \to p_0(t, \lambda)$ and $q_\varepsilon(t, \lambda) \to 0$ as $\varepsilon \to 0+$, uniformly in $t \in [0, T]$.*

Proof. That $q_\varepsilon(t, \lambda) \to 0$ is obvious from Lemma 1. Setting $x_1 = 0$, $x_0 = 1$ in (7) yields

$$\begin{aligned} |p_\varepsilon(t, \lambda) - p_0(t, \lambda)| \leqq {} & \frac{1}{b}\int_0^t |h(s, \lambda)|\,|p_\varepsilon(s, \lambda) - p_0(s, \lambda)|ds \\ & + \frac{1}{b}M_T(\lambda)\int_0^t |h(s, \lambda)|e^{-(b/\varepsilon)(t-s)}ds \ . \end{aligned}$$

For any $\delta > 0$ we have

$$\begin{aligned}\int_0^t |h(s, \lambda)| e^{-(b/\varepsilon)(t-s)} ds &\leqq \int_0^{\max(t-\delta,0)} |h(s, \lambda)| e^{-(b/\varepsilon)(t-s)} ds + \int_{\max(t-\delta,0)}^t |h(s, \lambda)| ds \\ &\leqq e^{-(b/\varepsilon)\delta} \int_0^T |h(s, \lambda)| ds + \delta \sup_{0 \leqq t \leqq T} |h(t, \lambda)| \\ &\equiv \delta g(\lambda) + e^{-(b/\varepsilon)\delta} k(\lambda) .\end{aligned}$$

Thus

$$\begin{aligned}|p_\varepsilon(t, \lambda) - p_0(t, \lambda)| \leqq \frac{1}{b} M_T(\lambda)[\delta g(\lambda) + e^{-(b/\varepsilon)\delta} k(\lambda)] \\ + \frac{1}{b}\int_0^t |h(s, \lambda)|\, |p_\varepsilon(s, \lambda) - p_0(s, \lambda)| ds ;\end{aligned}$$

application of Gronwall's lemma yields the inequality

$$|p_\varepsilon(t, \lambda) - p_0(t, \lambda)| \leqq \frac{1}{b}[\delta g(\lambda) + e^{-(b/\varepsilon)\delta} k(\lambda)] M_\lambda(T)^2 .$$

Here the right-hand side can be made arbitrarily small by first choosing $\delta > 0$ small and then requiring ε to be sufficiently small.

LEMMA 2. *For* $t \in [0, T]$,

$$\begin{aligned}|p_\varepsilon'(t, \lambda)| &\leqq \frac{2b}{\varepsilon} e^{2\int_0^T |h(s,\lambda)| ds} , \\ |q_\varepsilon'(t, \lambda)| &\leqq 1 + e^{2\int_0^T |h(s,\lambda)| ds} .\end{aligned}$$

Proof. Differentiating (5), taking absolute values, and using the estimate (8), we get

$$(9) \qquad |u_\varepsilon'(t, \lambda)| \leqq |x_1| + \frac{1}{\varepsilon}[|x_0| + \frac{\varepsilon}{b}|x_1|] M_T(\lambda) \int_0^T |h(s, \lambda)| ds ;$$

setting in turn $x_0 = 1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, and using the inequality $\alpha(1 + \alpha e^\alpha) \leqq 2e^{2\alpha}$ for $\alpha > 0$, yields the result.

LEMMA 3. *For* $t \in [0, T]$,

$$\begin{aligned}|p_\varepsilon''(t, \lambda)| &\leqq \frac{1}{\varepsilon^2}\Big(2b^2 + \varepsilon \max_{[0,T]} |h(t, \lambda)|\Big) e^{2\int_0^T |h(s,\lambda)| ds} , \\ |q_\varepsilon''(t, \lambda)| &\leqq \Big(\frac{2b}{\varepsilon} + \frac{2}{b} \max_{[0,T]} |h(t, \lambda)|\Big) e^{2\int_0^T |h(s,\lambda)| ds} , \\ |p_0'(t, \lambda)| &\leqq \frac{2}{b}\Big(\max_{[0,T]} |h(t, \lambda)|\Big) e^{2\int_0^T |h(s,\lambda)| ds} .\end{aligned}$$

Proof. The proof follows easily by using the differential equations (3), (4) themselves and the estimates contained in (9) and Lemma 1.

THEOREM 2. *For any* $0 < \tau < T$, $p'_\varepsilon(t, \lambda) \to p'_0(t, \lambda)$ *uniformly for* $t \in [0, T]$ *and* $q'_\varepsilon(t, \lambda) \to 0$ *uniformly for* $t \in [\tau, T]$.

Proof. Differentiation of (5) and (6) yields the equations

$$p'_\varepsilon(t, \lambda) = -\frac{1}{\varepsilon}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds ,$$

$$p'_0(t, \lambda) = -\frac{1}{b}h(t, \lambda)p_0(t, \lambda) ,$$

$$q'_\varepsilon(t, \lambda) = e^{-(b/\varepsilon)t} - \frac{1}{\varepsilon}\int_0^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}q_\varepsilon(s, \lambda)ds .$$

From the last of these equations we obtain, using Lemma 1,

$$|q'_\varepsilon(t, \lambda)| \leqq e^{-(b/\varepsilon)t} + \frac{1}{b}M_T(\lambda)\int_0^t |h(s, \lambda)|e^{-(b/\varepsilon)(t-s)}ds ;$$

in the proof of Theorem 1 this integral was shown to approach zero, uniformly in $t \in [0, T]$, as $\varepsilon \to 0$. Since $e^{-(b/\varepsilon)t} \to 0$ uniformly for $t \in [\tau, T]$, the second statement of theorem follows.

We turn to the first statement, writing for any fixed $\delta > 0$

$$\begin{aligned} p'_\varepsilon(t, \lambda) - p'_0(t, \lambda) = & -\frac{1}{\varepsilon}\int_0^{\max(t-\delta,0)} h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds \\ & -\frac{1}{\varepsilon}\int_{\max(t-\delta,0)}^t h(s, \lambda)e^{-(b/\varepsilon)(t-s)}p_\varepsilon(s, \lambda)ds \\ & + \frac{1}{b}h(t, \lambda)p_0(t, \lambda) \equiv J_1 + J_2 + J_3 . \end{aligned}$$

Since $\lambda e^{-\lambda} \leqq e^{-(1/2)\lambda}$, we have the estimate

$$\begin{aligned} |J_1| & \leqq \frac{1}{b\delta}\int_0^{\max(t-\delta,0)} \frac{b}{\varepsilon}(t - s)e^{-(b/\varepsilon)(t-s)}|h(s, \lambda)|\,|p_\varepsilon(s, \lambda)|ds \\ & \leqq \frac{1}{b\delta}M_T(\lambda)e^{-(b/2\varepsilon)\delta}\int_0^T |h(s, \lambda)|ds , \end{aligned}$$

where the final quantity tends to zero with ε. Also, since

$$\int_{t-\delta}^t \frac{b}{\varepsilon}e^{-(b/\varepsilon)(t-s)}ds = 1 - e^{-b\delta/\varepsilon} ,$$

we have that

$$\begin{aligned} |b(J_2 + J_3)| \leqq & \frac{b}{\varepsilon}\int_{t-\delta}^t e^{-(b/\varepsilon)(t-s)}|h(s, \lambda)|\,|p_\varepsilon(s, \lambda) - p_0(s, \lambda)|ds \\ & + e^{-b\delta/\varepsilon}|h(t, \lambda)|\,\,|p_0(t, \lambda)| . \end{aligned}$$

Setting $\mu_\varepsilon(\lambda, T) = \sup_{t \in [0,T]} | p_\varepsilon(t, \lambda) - p_0(t, \lambda) |$, $\nu(\lambda, T) = \sup_{t \in [0,T]} | h(t, \lambda) |$, we get that

$$\begin{aligned} | b(J_2 + J_3) | &\leqq e^{-b\delta/\varepsilon} \nu(\lambda, T) \sup_{t \in [0,T]} | p_0(t, \lambda) | \\ &\qquad + \mu_\varepsilon(\lambda, T) \nu(\lambda, T)(1 - e^{-b\delta/\varepsilon}) ; \end{aligned}$$

the right-hand side tends to zero with ε by Theorem 1.

2. The problem in a Hilbert space. Let H be a Hilbert space, E_λ the resolution of the identity for the self-adjoint operator A, and define operators P_ε, P_0, Q_ε on H by

$$\begin{aligned} P_\varepsilon(t) &= \int_{-\infty}^{\infty} p_\varepsilon(t, \lambda) dE_\lambda , \\ P_0(t) &= \int_{-\infty}^{\infty} p_0(t, \lambda) dE_\lambda , \\ Q_\varepsilon(t) &= \int_{-\infty}^{\infty} q_\varepsilon(t, \lambda) dE_\lambda ; \end{aligned}$$

let $T > 0$ be a fixed number. Let D denote the (dense) domain of the operator

$$(I + A + \max_{[0,T]} | h(t, A) |) \exp \left\{ 2 \int_0^T | h(s, A) | ds \right\}$$

defined as

$$\int_{-\infty}^{\infty} \left(1 + \lambda + \max_{[0,T]} | h(t, \lambda) | \right) e^{2 \int_0^T | h(s,\lambda) | ds} dE_\lambda ;$$

then

$$D = \left\{ x \in H: \int_{-\infty}^{\infty} \left(1 + \lambda + \max_{[0,T]} | h(t, \lambda) | \right)^2 e^{4 \int_0^T | h(s,\lambda) | ds} d(E_\lambda x, x) < \infty \right\} .$$

D is contained in the domains of P_ε, P_0, Q_ε, because for $x \in D$

$$\begin{aligned} \| P_\varepsilon(t) x \|^2 &= \int_{-\infty}^{\infty} | p_\varepsilon(t, \lambda) |^2 d(E_\lambda x, x) \leqq \int_{-\infty}^{\infty} [M_T(\lambda)]^2 d(E_\lambda x, x) \\ &\leqq \int_{\infty}^{\infty} 4 e^{4 \int_0^T | h(s,\lambda) | ds} d(E_\lambda x, x) < \infty ; \end{aligned}$$

similar calculations are valid for P_0, Q_0. Also, if $x \in D$, then $P_\varepsilon x$, $P_0 x$, $Q_\varepsilon x \in D(A)$, as is shown by calculations like the following for $P_\varepsilon x$:

$$\begin{aligned} \int_{-\infty}^{\infty} \lambda^2 d \| E_\lambda P_\varepsilon x \|^2 &= \int_{-\infty}^{\infty} \lambda^2 d \left\| E_\lambda \int_{-\infty}^{\infty} p_\varepsilon(t, \mu) dE_\mu x \right\|^2 \\ &= \int_{-\infty}^{\infty} \lambda^2 d \left\| \int_{-\infty}^{\lambda} p_\varepsilon(t, \mu) dE_\mu x \right\|^2 \leqq \int_{-\infty}^{\infty} \lambda^2 d \int_{-\infty}^{\lambda} 4 e^{4 \int_0^T | h(s,\mu) | ds} d \| E_\mu x \|^2 \\ &= 4 \int_{-\infty}^{\infty} \lambda^2 e^{4 \int_0^T | h(s,\lambda) | ds} d \| E_\lambda x \|^2 < \infty . \end{aligned}$$

Defining $U_\varepsilon(t) = P_\varepsilon(t)x_0 + Q_\varepsilon(t)x_1$ for $\varepsilon > 0$ and $U_0(t) = P_0(t)x_0$, we have

LEMMA 4. *$U_\varepsilon(t)$, $U_0(t)$ solve the problems* (1), (2), *respectively, on* $[0, T]$.

Proof. We shall prove only the statement concerning U_ε; that U_0 solves (2) is proved similarly. In view of the fact that $p_\varepsilon(t, \lambda)$ and $q_\varepsilon(t, \lambda)$ satisfy the differential equation (3), it is enough to show that the first and second derivatives of P_ε, Q_ε can be taken under the integral sign; that is, for $x \in D$,

$$\left(\frac{d}{dt}\right)^{\nu} P_\varepsilon(t)x = \int_{-\infty}^{\infty} \left(\frac{d}{dt}\right)^{\nu} p_\varepsilon(t, \lambda)dE_\lambda x,$$

$$\left(\frac{d}{dt}\right)^{\nu} Q_\varepsilon(t)x \sim = \int_{-\infty}^{\infty} \left(\frac{d}{dt}\right)^{\nu} q_\varepsilon(t, \lambda)dE_\lambda x \quad (\nu = 1, 2) ;$$

we present a proof of the statement for P_ε. By the mean value theorem we have that

$$\frac{1}{h}[P_\varepsilon(t + h)x - P_\varepsilon(t)x] = \int_{-\infty}^{\infty} \frac{d}{dt} p_\varepsilon(t', \lambda)dE_\lambda x$$

for some t' between t and $t + h$. Now

$$\left\| \int_{-\infty}^{\infty} \frac{d}{dt} p_\varepsilon(t', \lambda)dE_\lambda x - \int_{-\infty}^{\infty} \frac{d}{dt} p_\varepsilon(t, \lambda)dE_\lambda x \right\|^2$$
$$\leqq \int_{-\infty}^{\infty} \left| \frac{d}{dt} p_\varepsilon(t', \lambda) - \frac{d}{dt} p_\varepsilon(t, \lambda) \right|^2 d\| E_\lambda x \|^2 ,$$

so the desired result is a consequence of the Lebesgue dominated convergence theorem if we show that the integrand of the last integral is bounded by a function integrable with respect to the measure $\| E_\lambda x \|^2$. To this end restrict h to be small enough that $0 \leqq t + h \leqq T$. Then from Lemma 2

$$\left| \frac{d}{dt} p_\varepsilon(t', \lambda) \right|^2, \ \left| \frac{d}{dt} p_\varepsilon(t, \lambda) \right|^2 \leqq \frac{4b^2}{\varepsilon^2} e^{4\int_0^T |h(s,\lambda)| ds} ,$$

which is integrable for each fixed $\varepsilon > 0$. The proof of the case $\nu = 2$ is similar but uses Lemma 3 instead of Lemma 2.

LEMMA 5. *The solutions of the problems* (1), (2) *are unique.*

Proof. We shall show that $P_\varepsilon(t)x_0$ is the only solution of (1) with $x_1 = 0$; the omitted cases are similar. Let $h_n(t, A)$ for any integer n be the bounded operator

$$h_n(t, A) \equiv \int_{-n}^{n} h(t, \lambda)dE_\lambda = (E_n - E_{-n})h(t, A) .$$

Suppose $R_\varepsilon(t)$ is a solution of (1) with $x_1 = 0$, and set $z_\varepsilon(t) = P_\varepsilon(t)x_0 - R_\varepsilon(t)$. Set $z_{\varepsilon,n}(t) = (E_n - E_{-n})z_\varepsilon(t)$; then

$$\begin{aligned} &\varepsilon z''_{\varepsilon,n} + bz'_{\varepsilon,n} + h_n(t, A)z_{\varepsilon,n} \\ &\quad = (E_n - E_{-n})(\varepsilon z''_\varepsilon + bz'_\varepsilon + h(t, A)z_\varepsilon) = 0 , \end{aligned}$$

and $z_{\varepsilon,n}(0) = 0$, $z'_{\varepsilon,n}(0) = 0$. If we show that $z_{\varepsilon,n} \equiv 0$, we will have $0 = \lim_{n\to\infty} z_{\varepsilon,n}(t) = z_\varepsilon(t)$, as desired. Now $z_{\varepsilon,n}$ satisfies the integral equation

$$z_{\varepsilon,n}(t) = -\frac{1}{b}\int_0^t [1 - e^{-(b/\varepsilon)(t-s)}]h_n(s, A)z_{\varepsilon,n}(s)ds ,$$

whence

$$||z_{\varepsilon,n}(t)|| \leqq \frac{1}{b}\int_0^t ||h_n(s, A)||\, ||z_{\varepsilon,n}(s)||ds ,$$

and $z_{\varepsilon,n} \equiv 0$ follows in standard fashion.

THEOREM 3. *For* $x_0, x_1 \in D$ *we have*

$$\lim_{\varepsilon\to 0+} ||U_\varepsilon(t) - U_0(t)|| = 0$$

uniformly for $t \in [0, t]$, *and for any* $\tau > 0, \tau < T$,

$$\lim_{\varepsilon\to 0+} ||U'_\varepsilon(t) - U'_0(t)|| = 0$$

uniformly for $t \in [\tau, T]$.

Proof. It is necessary to show that

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |p_\varepsilon(t, \lambda) - p_0(t, \lambda)|^2 d||E_\lambda x_0||^2 = 0 ,$$

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |q_\varepsilon(t, \lambda)|^2 d||E_\lambda x_1||^2 = 0 ,$$

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |p'_\varepsilon(t, \lambda) - p'_0(t, \lambda)|^2 d||E_\lambda x_0||^2 = 0 ,$$

$$\lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} |q'_\varepsilon(t, \lambda)|^2 d||E_\lambda x_1||^2 = 0 .$$

Since, by Lemma 1,

$$|p_\varepsilon(t, \lambda) - p_0(t, \lambda)|^2 \leqq 2|p_\varepsilon(t, \lambda)|^2 + 2|p_0(t, \lambda)|^2 \leqq 4M_T(\lambda) ,$$

the first result follows from Theorem 1 and the Lebesgue dominated convergence theorem. The remaining statements follow in a similar manner.

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