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Let G be a nondiscrete locally compact abelian group, and M(G) the convolution algebra of bounded regular measures on G. In this paper, the following is proved: Let $\{\lambda_k\}_{k=0}^{\infty}$ be a countable subset of $M_c^+(G)$, $0 \neq \lambda_0 \in M_0(G)$, and $\{C_k\}_{k=0}^{\infty}$ a countable family of σ -compact subsets of G such that $\lambda_k(x + C_k) = 0$ for all $x \in G$ and all $k = 0, 1, 2, \cdots$. Then there exists a nonzero measure $\sigma \in M_0^+(\operatorname{supp} \lambda_0)$ with compact support such that $\lambda_k[x + C_k + G_p(\operatorname{supp} \sigma)] = 0$ for all $x \in G$ and all k = $0, 1, 2, \cdots$. A consequence of this result is the following: Let Y be the closed ideal in M(G) which is generated by $\bigcup \{L^1(\lambda_k): k = 0, 1, 2, \cdots\}$ for some countable subset $\{\lambda_k\}_{k=0}^{\infty}$ of $M_c(G)$. Then there exist "fairly many" symmetric maximal ideals in M(G) which contain $\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G)$ but not $M_0(G)$. Here $L^1(\mu)$ denotes the set of the measures in M(G)which are absolutely continuous with respect to $|\mu|$.

Throughout the paper, let G be a nondiscrete locally compact abelian group, \hat{G} its dual, and M(G) the convolution algebra of bounded regular measures on G. We use the following customary notations:

$$L^{\scriptscriptstyle 1}(G) = M_{\scriptscriptstyle a}(G) \subset M_{\scriptscriptstyle 0}(G) \subset M_{\scriptscriptstyle c}(G) \subset M(G)$$
 .

Here $M_0(G)$ denotes the closed ideal of those measures in M(G) whose Fourier transforms vanish at infinity. For the definitions of $M_a(G)$ and $M_o(G)$, see [3: (19.13)]; for the second inclusion, see [8: 5.6.9] or [4]. Given a measure $\mu \in M(G)$, we denote by $L^1(\mu)$ the set of those measures in M(G) which are absolutely continuous with respect to $|\mu|$. For a set K in G, define

$$(K)_1 = K \cup (-K)$$
 and $(K)_n = (K)_{n-1} + (K)_1$ $(n = 2, 3, \dots)$.

Thus, the union of all $(K)_n$, denoted by $G_p(K)$, is the subgroup of G generated by K.

Our main results are the following.

THEOREM 1. Let $\{\lambda_k\}_{k=0}^{\infty}$ be a countable subset of $M_c^+(G)$, $0 \neq \lambda_0 \in M_0(G)$, and $\{C_k\}_{k=0}^{\infty}$ a countable family of nonempty σ -compact subsets of G such that

(a)
$$\lambda_k(x+C_k) = 0$$
 $(x \in G; k = 0, 1, 2, \cdots)$.

Then there exists a nonzero measure $\sigma \in M_0^+(\operatorname{supp} \lambda_0)$ with compact support such that

(b)
$$\lambda_k[x + C_k + G_p(\operatorname{supp} \sigma)] = 0$$
 $(x \in G; k = 0, 1, 2, \cdots)$.

If, in addition, G is metrizable, such a measure σ can be taken so that $(\text{supp } \sigma) - x_0$ is independent for some $x_0 \in G$.

COROLLARY. Let $\{\lambda_k\}_{k=0}^{\infty}$ be a countable subset of $M_c(G)$, $0 \neq \lambda_0 \in M_0(G)$, and Y the closed ideal in M(G) which is generated by $\bigcup \{L^1(\lambda_k): k = 0, 1, 2, \cdots\}$. Then there exists a nonzero measure $\sigma \in M_0^+(\operatorname{supp} \lambda_0)$ with compact support such that

$$|\mu| [x + G_p(\operatorname{supp} \sigma)] = 0 \qquad (x \in G; \mu \in Y) .$$

If, in addition, G is metrizable, such a measure σ can be taken so that $(\text{supp } \sigma) - x_0$ is independent for some $x_0 \in G$.

THEOREM 2. Let $\{\lambda_k\}_{k=0}^{\infty}$ and Y be as in the Corollary. Then there exists a symmetric maximal ideal Θ in M(G) such that

$$\bigcup \{L^{\scriptscriptstyle 1}(\mu) \colon \mu \in Y\} \cup M_{\scriptscriptstyle a}(G) \subset \Theta \qquad but \ M_{\scriptscriptstyle 0}(G) \not\subset \Theta \ .$$

Furthermore, the set of all Θ 's with these properties has cardinal number larger than or equal to max $\{2^{\omega}, \operatorname{Card} \hat{G}\}$. Here ω denotes the smallest uncountable cardinal.

Theorem 1 improves the main result in [6] and its Corollary generalizes a theorem of Rudin [7] (see N. Th. Varopoulos [9] in this connection). The idea of our proof is due to T. W. Körner [5: Ch. XIII]. Although the arguments needed are similar to those in [6], we give a detailed proof of Theorem 1.

We need some lemmas.

LEMMA 1. Let λ be a measure in $M^+_{\varepsilon}(G)$, and D a compact subset of G such that $\lambda(x + D) = 0$ for all $x \in G$. Then, for each finite set F in G, $n \in N$ (the natural numbers) and $\varepsilon > 0$, there exists a neighborhood V of $0 \in G$ such that

$$\lambda [x + D + (F + V)_n] < \varepsilon$$
 $(x \in G)$.

Proof. Let F, n, and ε be as above. Take a compact set K in G so that $\lambda(G\setminus K) < \varepsilon$, and fix any neighborhood V_0 of 0 with compact closure. Since F is finite, $\lambda[x + D + (F)_n] = 0$ for all $x \in G$ by hypothesis. Thus, for each $x \in G$, we can find a neighborhood W_x of 0 so that

$$\lambda[x + D + (F)_n + (W_x)_{2n}] < \varepsilon$$
.

(Note that $D + (F)_n$ is compact.) It follows from compactness of the

set $K - [D + (F)_n + (\overline{V}_0)_n]$ that there exist finitely many points x_1 , x_2 , \cdots , $x_m \in G$ such that

$$K - [D + (F)_n + (V_0)_n] \subset \bigcup_{j=1}^m (x_j + W_{x_j})$$
.

Put $V = \bigcap_{j=1}^{m} W_{x_j} \cap V_0$. If $x \in K - [D + (F)_n + (V_0)_n]$, then $x \in x_j + W_{x_j}$ for some j = j(x), and so

$$egin{aligned} \lambda[x+D+(F+V)_n] &\leq \lambda[x_j+W_{x_j}+D+(F)_n+(V)_n] \ &\leq \lambda[x_j+D+(F)_n+(W_{x_j})_{2n}] < arepsilon \ . \end{aligned}$$

If $x \notin K - [D + (F)_n + (V_0)_n]$, then

$$[x+D+(F+V)_n]\cap K\!\subset\! [x+D+(F)_n+(V_0)_n]\cap K=arnothing$$
 ,

and so $\lambda[x + D + (F + V)_n] \leq \lambda(G \setminus K) < \varepsilon$. This completes the proof.

LEMMA 2. Suppose that G is metrizable and λ_0 a nonzero measure in $M^+_e(G)$. Then there exists a point $x_0 \in G$ and a nonempty, totally disconnected, compact, perfect subset K_0 of supp λ_0 with the following three properties.

(a) Every nonempty (relatively) open subset of K_0 has positive λ_0 -measure.

(b) The elements of $K_0 - x_0$ have the same order, say q_0 .

(c) If V_1, V_2, \dots, V_m are *m* disjoint, nonempty, open subsets of K_0 , there exist *m* points $x_j \in V_j$ such that $x_1 - x_0, x_2 - x_0, \dots, x_m - x_0$ are independent.

Proof. Since G is metrizable and λ_0 is continuous, we may assume that λ_0 is carried by a totally disconnected compact set.

Suppose first that there exist a natural number q and an element $y \in G$ such that

$$E(q, y) = \{x \in \text{supp } \lambda_0 \colon qx = y\}$$

has positive λ_0 -measure. Let q_0 be the smallest natural number such that $\lambda_0[E(q_0, y_0)] > 0$ for some $y_0 \in G$. Fix any element $x_0 \in E(q_0, y_0)$. Then $\lambda_0[E(q, qx_0)] = 0$ for all $q \in N$ with $1 \leq q < q_0$, so that there exists a compact subset K_0 of $E(q_0, y_0) \setminus \{\bigcup_{q=1}^{q_0-1} E(q, qx_0)\}$ with $\lambda_0(K_0) > 0$. Replacing K_0 by the support of $\lambda_0 \mid K_0$, we may assume that K_0 is perfect and satisfies (a). Evidently (b) holds. Suppose now that (c) holds for some $m \in N$ (note that (c) is trivial for m = 1). Let V_1, \dots, V_m, V_{m+1} be m + 1 disjoint, nonempty, open subsets of K_0 . There are m points $x_1 \in V_1, \dots, x_m \in V_m$ such that $x_1 - x_0, \dots, x_m - x_0$ are independent. Let H be the subgroup of G which is generated by x_0, x_1, \dots, x_m . By minimality of q_0 , we have $\lambda_0[E(q, y)] = 0$ for all $q \in N$ with $1 \leq q < q_0$ and all $y \in H$. Since $\lambda_0(V_{m+1}) > 0$ by (a) and H is at most countable, we can find an element $x_{m+1} \in V_{m+1}$ so that

$$x_{m+1}
otin E(q, y) \qquad (q=1, 2, \cdots, q_0-1; y \in H) \;.$$

It is now easy to prove that the elements $x_1 - x_0, \dots, x_m - x_0, x_{m+1} - x_0$ are independent. By induction on m, we obtain (c).

Suppose next that $\lambda_0[E(q, y)] = 0$ for all $q \in N$ and all $y \in G$. Then $F = \{x \in \operatorname{supp} \lambda_0 : \operatorname{ord} x < \infty\}$ has λ_0 -measure zero, so that there exists a nonempty compact, perfect subset K_0 of $(\operatorname{supp} \lambda_0) \setminus F$ which satisfies (a). It is now easy to prove that (b) and (c) hold for $x_0 = 0$.

LEMMA 3. Let K be a totally disconnected, compact subset of G, and σ a nonzero measure in $M_c^+(G)$ with supp $\sigma = K$. Then, for each compact subset \hat{F} of \hat{G} and $\varepsilon > 0$, there exists a finite partition $\{K_j\}_{j=1}^n$ of K into disjoint clopen subsets such that:

(i) $0 < \sigma(K_j) < \varepsilon$ $(j = 1, 2, \dots, n);$

 $\begin{array}{c|c} (\text{ ii }) & \left|\sum\limits_{j=1}^n \sigma(K_j) \hat{\nu}_j(\chi) - \hat{\sigma}(\chi)\right| < \varepsilon \quad (\chi \in \hat{F}) \\ \text{whenever } \nu_j \in M^+(K_j) \text{ and } ||\nu_j||_{\scriptscriptstyle M} = 1 \text{ for all } j = 1, 2, \cdots, n. \end{array}$

Proof. Since \hat{F} is compact while K is totally disconnected and compact, there is a finite partition $\{K_j\}_{j=1}^n$ of K into disjoint clopen subsets which satisfies (i) and

$$\sup \left\{ \mid \chi(x) - \chi(y) \mid : x, \ y \in K_{j}
ight\} < (3 \mid \mid \sigma \mid \mid_{M})^{-1} arepsilon$$

for all $\chi \in \hat{F}$ and all $j = 1, 2, \dots, n$. If $\nu_j \in M^+(K_j)$ and $||\nu_j||_M = 1$ for $j = 1, 2, \dots, n$, then we have

$$\|\sigma(K_j) \hat{
u}_j(\chi) - \widehat{\sigma} \| \widetilde{K}_j(\chi) \| < \|\sigma\|_{_M}^{-1} \sigma(K_j) arepsilon \qquad (\chi \in \widehat{F}) \;.$$

To see this, take any $x_j \in K_j$. Then $\chi \in \hat{F}$ implies

$$egin{aligned} &|\, \sigma(K_j) \widehat{m{
u}}_j(\chi) - \widehat{\sigma \mid K_j}(\chi) \,| \ &= \left| \sigma(K_j) \int_{\kappa_j} \overline{\chi} dm{
u}_j - \int_{\kappa_j} \overline{\chi} d\sigma
ight| \ &\leq \sigma(K_j) \int_{\kappa_j} |\, \overline{\chi} - \overline{\chi}(x_j) \,| \, dm{
u}_j + \int_{\kappa_j} |\, \overline{\chi} - \overline{\chi}(x_j) \,| \, d\sigma \ &\leq 2\sigma(K_j) (3 \,||\, \sigma \,||_{\mathfrak{U}})^{-1} arepsilon < ||\, \sigma \,||_{\mathfrak{U}}^{-1} \sigma(K_j) arepsilon \;. \end{aligned}$$

Adding these inequalities for all j's, we obtain (ii).

To prove the following lemma, we need a definition. Let K be a subset of G whose elements have the same order $q_0(2 \leq q_0 \leq \infty)$. Let also L_1, L_2, \dots, L_n be finitely many subsets of K, and M any natural number. We say that L_1, L_2, \dots, L_n are *M*-independent if and only if $\sum_{j=1}^{n} m_j x_j \neq 0$ whenever $m_j \in Z$ (the integers), $|m_j| < q_0$,

 $x_j \in L_j \ (j = 1, 2, \dots, n) \text{ and } 0 \neq \sum_{j=1}^k |m_j| < M.$

LEMMA 4. Suppose that G is metrizable, and that $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ are as in Theorem 1. Let also $x_0 \in G$ and $K_0 \subset \operatorname{supp} \lambda_0$ be as in Lemma 2. Then there exists a nonzero measure $\sigma \in M_0^+(K_0)$ such that $(\operatorname{supp} \sigma) - x_0$ is independent and

$$(\mathbf{P}_1) \qquad \lambda_k [x + C_k + (\operatorname{supp} \sigma)_1] = 0 \qquad (x \in G; \, k = 0, \, 1, \, 2, \, \cdots) \,.$$

Proof. Write

$$\widehat{G} = \bigcup_{n=1}^{\infty} \widehat{E}_n$$
 and $C_k = \bigcup_{n=1}^{\infty} C_{kn}$ $(k = 0, 1, 2, \cdots)$,

where the \hat{E}_n are compact subsets of \hat{G} while the C_{kn} are compact subsets of G such that $C_{kn} \subset C_{k(n+1)}$ for all k and n. (It is well-known that G is metrizable if and only if \hat{G} is σ -compact. See, for example, [4].) Let λ be the measure in M(G) defined by $\lambda(E) = \lambda_0[(E + x_0) \cap K_0]$ for all Borel subsets E of G. Then, $0 \neq \lambda \in M_0^+(G)$ and the elements in supp $\lambda = K_0 - x_0$ have the same order q_0 .

We shall now construct a sequence $(n_p)_{p=1}^{\infty}$ of natural numbers, a sequence $(\mathscr{I}_n)_{n=1}^{\infty}$ of finite collections of disjoint clopen subsets of $K_0 - x_0$, a sequence $(\sigma_n)_{n=1}^{\infty}$ of probability measures in $L^1(\lambda)$, and a sequence $(\widehat{F}_n)_{n=1}^{\infty}$ of compact subsets of \widehat{G} . They will satisfy the following three conditions. Every σ_n has the form

(i)
$$\sigma_n = \sum_{I \in \mathcal{F}_n} a_I \lambda_I$$

where each a_I is a positive real number, $\lambda_I = \lambda | I$ the restriction of λ to I, and

$$\|\sigma_n\|_M = \sum_{I \in \mathcal{I}_n} a_I \lambda(I) = 1.$$

$$(\text{ii}) \qquad \sup \{ | \, \widehat{\sigma_n} \, | \, \widehat{I}(\chi) \, | \colon \chi \in \widehat{G} \backslash \widehat{F}_n \} < 2^{-n} \sigma_n(I) \qquad \forall \, I \in \mathscr{I}_n \; .$$

(iii)
$$\widehat{E}_n \subset \widehat{F}_n$$
 .

For n = 1, such \mathscr{I}_i , σ_i , and \hat{F}_i may be quite arbitrary. We set $n_i = 1$, and suppose that n_p , \mathscr{I}_{n_p} , σ_{n_p} , and \hat{F}_{n_p} have been constructed for some $p \in N$. Let $l_p = \text{Card } \mathscr{I}_{n_p}$, and write

$$\mathscr{I}_{n_p} = \{I_i\}_{i=1}^{l_p} = \{I_i^p\}_{i=1}^{l_p}$$

Let M_p be the largest natural number such that

(1)
$$\max \left\{ \sigma_{n_p}(I) \colon I \in \mathscr{I}_{n_p} \right\} \leq M_p^{-2} ,$$

and set

(2)
$$T_p = \{A \subset \mathscr{I}_{n_p} : 1 \leq \text{Card } A \leq M_p\} = \{A_r\}_{r=1}^{s_p}.$$

We may assume

(3)
$$A_r = \{I_r\} = \{I_r\}$$
 $(r = 1, 2, \dots, l_p)$.

We shall inductively construct the \mathscr{I}_n , σ_n , and \hat{F}_n for all $n \in N$ with $n_p < n \leq n_p + s_p$ as follows. Suppose that \mathscr{I}_n , σ_n and \hat{F}_n have been constructed for some $n = n_p + r - 1$ $(r = 1, 2, \dots, s_p)$, and put

(4)
$$\mathscr{K}_n = \{I \in \mathscr{I}_n \colon I \subset J \text{ for some } J \in A_r\}$$
.

We can find (finite) collections $\{b_j^\kappa\}_j$ of real numbers and collections $\{L_j^\kappa\}_j$ of disjoint clopen subsets of $K \in \mathscr{K}_n$ which satisfy the following six conditions:

$$(5) 0 < b_j^{\scriptscriptstyle K} \sigma_n(L_j^{\scriptscriptstyle K}) < 2^{-1} \sigma_n(K) \forall K \in \mathscr{K}_n \text{ and } \forall_j;$$

(6)
$$\sum_{j} b_{j}^{\kappa} \sigma_{n}(L_{j}^{\kappa}) = \sigma_{n}(K) \quad \forall K \in \mathscr{K}_{n};$$

(7)
$$\left|\sum_{j} b_{j}^{K} \widehat{\sigma_{n} \mid L_{j}^{K}}(\chi) - \widehat{\sigma_{n} \mid K}(\chi)\right| < 2^{-n} \sigma_{n}(K) \quad \forall K \in \mathscr{K}_{n} \text{ and } \forall \chi \in \widehat{F}_{n};$$

(8)
$$\sum_{j} \operatorname{dia} \left(L_{j}^{\scriptscriptstyle K} \right) < n^{-1}$$
 $\forall K \in \mathscr{K}_{n};$

(9) The sets
$$\{L_j^{\kappa}\}_{\kappa,j}$$
 are M_p -independent;

(10)
$$\sup_{x \in G} \lambda_k \left[x + C_{kn} + \left(\bigcup_j L_j^K \right)_1 \right] < (nl_p)^{-1} \\ \forall K \in \mathscr{K}_n \text{ and } \forall k = 0, 1, \dots, n.$$

The above conditions are met as follows: For each $K \in \mathscr{K}_n$, apply Lemma 3 to $\sigma = \sigma_n | K, \varepsilon = 2^{-n} \sigma_n(K)$ and $\hat{F} = \hat{F}_n$. Let $\{K_j\}_j$ be a finite partition of K as in Lemma 3. Using property (c) in Lemma 2, we can find $x_j^{\kappa} \in K_j$ so that $\bigcup \{\{x_j^{\kappa}\}_j: K \in \mathscr{K}_n\}$ is independent. If we choose $L_j^{\kappa} \subset K_j$ so that $x_j^{\kappa} \in L_j^{\kappa}$ and the diameter of each L_j^{κ} is "sufficiently small", then (8) and (9) hold and so does (10) by Lemma 1. Finally, it suffices to set $b_j^{\kappa} = \sigma_n(K_j)/\sigma_n(L_j^{\kappa})$.

We now define

(11)
$$\mathscr{L}_{n} = \mathscr{I}_{n} \setminus \mathscr{K}_{n}, \ \mathscr{I}_{n+1} = \mathscr{L}_{n} \cup \left(\bigcup_{K \in \mathscr{K}_{n}} \{L_{j}^{K}\}_{j} \right);$$

(12)
$$\theta_{n+1} = \sum_{I \in \mathscr{L}_n} a_I \lambda_I + \sum_{K \in \mathscr{K}_n} \sum_j b_j^K \sigma_n \mid L_j^K ,$$

and take a compact subset \hat{F}_{n+1} of \hat{G} , with $\hat{F}_{n+1} \supset \hat{E}_{n+1} \cup \hat{F}_n$, so that (ii) holds with n replaced by n + 1.

We repeat the above process with n_p replaced by $n_{p+1} = n_p + s_p$, which completes our induction. Let σ_{∞} be a weak-* cluster point of

 $(\sigma_n)_{n=1}^{\infty}$ in M(G), and σ the measure in M(G) defined by $\sigma(E) = \sigma_{\infty}(E - x_0)$ for all Borel sets E in G. We claim that σ has the required properties.

First note that

$$\bigcup_{I\in\mathscr{S}_{n+1}}I\subsetigcup_{I\in\mathscr{S}_n}I=\mathrm{supp}\ \sigma_{n}\subset K_{\scriptscriptstyle 0}-x_{\scriptscriptstyle 0}$$
 ,

and so we have

(13)
$$\sigma_{\infty} \geq 0$$
, $\sigma_{\infty}(G) = 1$ and $\operatorname{supp} \sigma_{\infty} \subset \bigcap_{n=1}^{\infty} \left(\bigcup_{I \in \mathcal{F}_n} I\right)$.

Let $p \in N$ be given. It is easily seen from (3), (4), and (11) that

$$\mathscr{I}_{n_p+l_p} = \{L_j^{I_1}\}_j \cup \{L_j^{I_2}\}_j \cup \cdots \cup \{L_j^{I_l}\}_j \;,\;\; ext{ where } l = l_p \;.$$

This, combined with (10) and (13), shows

$$egin{aligned} \lambda_k [x + C_{kn_p} + (\operatorname{supp} \sigma_{\circ\circ})_1] \ &\leq \sum_{i=1}^{l_p} \lambda_k igg[x + C_{kn_p} + \left(igcup_j L_j^{{\scriptscriptstyle I}_i}
ight)_1 igg] \leq l_p \cdot (n_p l_p)^{-1} = n_p^{-1} \end{aligned}$$

for all $x \in G$ and all $k = 0, 1, \dots, n_p$. (Note that $C_{kn} \subset C_{k(n+1)}$ for all k and n.) Thus, fixing $x \in G$ and $k \in \{0, 1, 2, \dots\}$, and letting $p \to \infty$, we have

$$\lambda_k[x+C_k+(\mathrm{supp}\,\sigma_{\scriptscriptstyle\infty})_{\scriptscriptstyle 1}]=0\qquad (x\in G;\,k=0,\,1,\,2,\,\cdots)\;.$$

But evidently supp $\sigma = (\operatorname{supp} \sigma_{\infty}) + x_0$, and so (P₁) holds.

It remains to show that $\hat{\sigma}$ vanishes at infinity and that $(\text{supp } \sigma) - x_0$ is independent. Although these are proved in [5:Ch. XIII, 151-153 and 155-156], we give their proofs to make the paper self-contained.

Suppose $n_p \leq n < n_{p+1}$ $(p, n \in N)$, and write $n = n_p + r - 1$ $(r = 1, 2, \dots, s_p)$. Then we have

(14)
$$\sum_{K \in \mathscr{K}_n} \sigma_n(K) = \sum_{J \in A_r} \sigma_{n_p}(J) \leq (\operatorname{Card} A_r) \cdot \max_{J \in A_r} \sigma_{n_p}(J) \leq M_p^{-1}.$$

Here the equality follows from (4), (6) and (12) while the last inequality follows from (1) and (2). If $\chi \in \hat{F}_n$, then

$$egin{aligned} &| \ \hat{\sigma}_{n+1}(\chi) - \ \hat{\sigma}_n(\chi) \ | &\leq \sum\limits_{K \ \in \ \mathscr{K}_n} \left| \sum\limits_j b_j^{\kappa} \widehat{\sigma_n} \ | \ L_j^{\kappa}(\chi) - \widehat{\sigma_n} \ | \ K(\chi)
ight| \ &< \sum\limits_{K \ \in \ \mathscr{K}_n} 2^{-n} \sigma_n(K) \leq 2^{-n} \end{aligned}$$

by (i), (12) and (7). It follows that

 $|\, \hat{\sigma}_{\scriptscriptstyle\infty}(\chi) \, - \, \hat{\sigma}_{\scriptscriptstyle n+1}(\chi) \, | < 2^{\scriptscriptstyle -n} \qquad orall \, \chi \in \hat{F}_{\scriptscriptstyle n+1}$,

since $\hat{F}_n \subset \hat{F}_{n+1} \subset \cdots$ by construction. For $\chi \in \hat{G} \setminus \hat{F}_n$, we have

$$egin{aligned} &|\, \hat{\sigma}_{n+1}(\chi)\,| &\leq \sum\limits_{I\,\in\,\mathscr{L}_n} \,|\, a_I \widehat{\lambda}_I(\chi)\,| + \sum\limits_{K\,\in\,\mathscr{K}_n} \sum\limits_{j}\, b_j^K \sigma_n(L_j^K) \ &\leq \sum\limits_{I\,\in\,\mathscr{L}_n} \,|\, \widehat{\sigma_n}\,|\, I(\chi)\,| + \sum\limits_{K\,\in\,\mathscr{K}_n} \sigma_n(K) \ &\leq 2^{-n} \sum\limits_{I\,\in\,\mathscr{L}_n} \sigma_n(I) + M_p^{-1} = 2^{-n} + M_p^{-1} \end{aligned}$$

by (12), (i), (6), (ii) and (14). Hence

$$| \, {\widehat \sigma}_{\scriptscriptstyle\infty}(\chi) \, | \leq 2^{-n} \, + \, 2^{-n} \, + \, M_{\scriptscriptstyle \mathcal{P}}^{-1} \qquad orall \, \chi \in {\widehat F}_{n+1} ackslash {\widehat F}_n$$
 .

But $\hat{G} = \bigcup_{n=1}^{\infty} \hat{F}_n$ by (iii) and $\lim_p M_p = \infty$ by construction. Thus the above inequality shows that $\hat{\sigma}_{\infty} \in C_0(\hat{G})$, or equivalently, that $\hat{\sigma} \in C_0(\hat{G})$.

Finally we prove that $(\operatorname{supp} \sigma) - x_0 = \operatorname{supp} \sigma_{\infty}$ is independent. Let x_1, x_2, \dots, x_t be distinct elements of $\operatorname{supp} \sigma_{\infty}$. It is easy to see that

 $\max_{I \in \mathscr{I}_n} \operatorname{dia} (I) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$

Therefore, there is an $n_0 \in N$ such that x_1, x_2, \dots, x_t belong to distinct sets in \mathscr{I}_n whenever $n \ge n_0$. Take any $p \in N$ so that $n_p \ge n_0$ and $M_p > t$, and let

$$A = \{I \in \mathscr{I}_{n_n} : I \text{ contains some } x_n\}$$
.

Then $1 \leq \text{Card } A = t < M_p$; hence $A = A_r$ for some $r = 1, 2, \dots, s_p$. Thus x_1, x_2, \dots, x_t belong to distinct sets in $\bigcup \{\{L_j^K\}_j : K \in \mathscr{K}_n\}$, where $n = n_p + r - 1$. It follows from (9) that x_1, x_2, \dots, x_t are M_p -independent. Since p can be taken as large as one pleases, we conclude that x_1, x_2, \dots, x_t are independent.

This establishes Lemma 4.

LEMMA 5. Let G, $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ be as in Lemma 4. Let also $\{K_j\}_{j=1}^{m}$ be finitely many, disjoint, compact subsets of G such that $M_0(K_j) \neq 0$ for all $j = 1, 2, \dots, m$. Then, for each $n \in N$, there exist m nonzero measures $\mu_j \in M_0^+(K_j)$ such that

$$(\mathbf{P}_n) \qquad \lambda_k \bigg[x + C_k + \Big(\bigcup_{j=1}^m \operatorname{supp} \mu_j \Big)_n \bigg] = 0 \qquad (x \in G; \, k = 0, \, 1, \, 2, \, \cdots) \;.$$

Proof. For each $j = 1, 2, \dots, m$, choose and fix a measure $\tau_j \in M_0^+(K_j)$ with $||\tau_j||_M = m^{-1}$ whose support is totally disconnected. In the proof of Lemma 4, replace λ and $K_0 - x_0$ by $\lambda' = \sum_{j=1}^m \tau_j$ and $\bigcup_{j=1}^m \operatorname{supp} \tau_j$, respectively; take off condition (9); and let σ_{∞} be any measure constructed as there with $\sigma_1 = \lambda'$ and $\mathscr{I}_1 = \{\operatorname{supp} \tau_j\}_{j=1}^m$. Then

$$0
eq \mu_j=\sigma_{_\infty}\mid K_j\in M^+_{\scriptscriptstyle 0}(K_j)\qquad (j=1,\,2,\,\cdots,\,m)$$
 ,

and $\{\mu_j\}_{j=1}^m$ satisfy (P₁). We repeat the same argument replacing

 $\{C_k\}_{k=0}^{\infty}$ and $\{K_j\}_{j=1}^m$ by $\{C_k + (\bigcup_{j=1}^m \operatorname{supp} \mu_j)_l\}_{k=0}^{\infty}$ and $\{\operatorname{supp} \mu_j\}_{j=1}^m$, respectively; and continue this process. At the *n*th step, we will obtain *m* nonzero measures satisfying the required condition. This completes the proof.

Proof of Theorem 1 for metrizable groups. Suppose that G is a given metrizable group. Let $\{\lambda_k\}_{k=0}^{\infty}$, $\{C_k\}_{k=0}^{\infty}$, x_0 , and K_0 be as in Lemma 4. Let also $\hat{G} = \bigcup_{n=1}^{\infty} \hat{E}_n$ be as in the proof of Lemma 4. We construct a sequence $(\mathscr{I}_n)_{n=1}^{\infty}$ of finite collections of disjoint compact subsets of K_0 , a sequence $(\sigma_n)_{n=1}^{\infty}$ of probability measures in $M_0(K_0)$, and a consequence $(\hat{F}_n)_{n=1}^{\infty}$ of compact subsets of \hat{G} . They satisfy the following four conditions. Every σ_n has the form

(i)
$$\sigma_n = \sum_{I \in \mathcal{F}_n} a_I \mu_I,$$

where each a_I is a real positive number, μ_I a probability measure in $M_0(I)$ with supp $\mu_I = I$, and

$$||\sigma_n||_M = \sum_{I \in \mathcal{F}_n} a_I = 1$$
 .

$$({\rm ii}) \qquad \qquad {\rm sup} \ \{ \mid \hat{\mu}_I(\chi) \mid : \chi \in \hat{G} \backslash \hat{F}_n \} < 2^{-n} a_I \qquad \forall \ I \in \mathscr{I}_n$$

(iii)
$$\widehat{E}_n \subset \widehat{F}_n$$

(iv)
$$\lambda_k [x + C_k + (K_1)_1 + (K_2)_2 + \cdots + (K_n)_n] = 0$$

 $(x \in G; k = 0, 1, 2, \cdots),$

where $K_1 = \bigcup \{I: I \in \mathscr{I}_1\}$ and $K_n = \bigcup \{I \in \mathscr{I}_n: I \notin \mathscr{I}_{n-1}\}$ for $n=2, 3, \cdots$.

For n = 1, we apply Lemma 4 to obtain a probability measure $\sigma_{_1} \in M_{_0}(K_{_0})$ such that $(\operatorname{supp} \sigma_{_1}) - x_{_0}$ is independent and

$$\lambda_k[x + C_k + (\operatorname{supp} \sigma_1)_1] = 0$$
 $(x \in G; k = 0, 1, 2, \cdots)$.

Set $\mathscr{I}_1 = \{I = \operatorname{supp} \sigma_i\}, \ \mu_I = \sigma_i, \ a_I = 1, \text{ and take any compact subset } \hat{F}_1 \text{ of } \hat{G} \text{ satisfying (ii) and (iii) for } n = 1.$

Suppose that $(\mathscr{I}_i)_{i=1}^n$, $(\sigma_i)_{i=1}^n$ and $(\hat{F}_i)_{i=1}^n$ have been constructed for some $n \in \mathbb{N}$. Choose and fix any $I_n \in \mathscr{I}_n$ with

(1)
$$a_{I_n} = \sup \{a_I \colon I \in \mathscr{I}_n\}.$$

Applying Lemmas 3 and 5, we can find a (finite) collection $\{b_{nj}\}_j$ of real numbers, a collection $\{L_{nj}\}_j$ of disjoint compact subsets of I_n , and a collection $\{\mu_{nj}\}_j$ of probability measures in $M_0(I_n)$ with $\operatorname{supp} \mu_{nj} = L_{nj}$ which satisfy the following four conditions.

$$(2) 0 < b_{nj} < n^{-1} \cdot \min_{I \in \mathscr{I}_n} a_I \forall j.$$

$$(3) \qquad \qquad \sum_{j} b_{nj} = a_{I}$$

$$(4) \qquad \left|\sum_{j} b_{nj} \hat{\mu}_{nj}(\chi) - a_{I_n} \hat{\mu}_{I_n}(\chi)\right| < 2^{-n} \quad \forall \, \chi \in \hat{F}_n \, .$$

$$(5) \qquad \lambda_k \left[x + C_k + (K_1)_1 + \cdots + (K_n)_n + \left(\bigcup_j L_{nj} \right)_{n+1} \right] = 0 \qquad \forall x \in G$$

for all $k = 0, 1, 2, \cdots$. Put $\mathscr{I}_{n+1} = (\mathscr{I}_n \setminus \{I_n\}) \cup \{L_{nj}\}_j$, and $a_I = b_{nj}$, $\mu_I = \mu_{nj}$ for $I = L_{nj} \forall j$. Define σ_{n+1} by the right-hand side of (i) with *n* replaced by n + 1. Finally, we take any compact subset \hat{F}_{n+1} of \hat{G} , with $\hat{F}_{n+1} \supset \hat{E}_{n+1} \cup \hat{F}_n$, so that (ii) holds with *n* replaced by n + 1.

This completes the induction. Let σ be a weak-* cluster point of $(\sigma_n)_{n=1}^{\infty}$ in M(G). Then it is easy to prove that σ has all the required properties (see [5: Ch. XIII, 151-153]). This establishes Theorem 1 for metrizable groups.

To prove the general case, we need one more lemma.

LEMMA 6. Let $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ be as in Theorem 1. Then, given a σ -compact subset \hat{F} of \hat{G} , we can find a σ -compact, noncompact, open subgroup Γ of \hat{G} so that $\hat{F} \subset \Gamma$ and

(i)
$$\lambda_k[x + C_k + H_r] = 0$$
 $(x \in G; k = 0, 1, 2, \cdots)$,

where H_{Γ} denotes the annihilator of Γ .

Proof. Let C_{kn} be as in the proof of Lemma 4, and let \mathscr{F} be the family of all σ -compact, noncompact, open subgroups of \hat{G} which contain \hat{F} . Since every C_{kn} is compact, we have

(1)
$$C_{kn} = \bigcap \{C_{kn} + H_{\Gamma} \colon \Gamma \in \mathscr{F}\}$$
 $(k = 0, 1, 2, \dots; n = 1, 2, \dots)$.

Applying Lemma 1, we can find neighborhoods V_{kn} of 0 so that

$$(2) \quad \lambda_k[x+C_{kn}+V_{kn}] < n^{-1} \qquad (k=0,\,1,\,2,\,\cdots;\,n=1,\,2,\,\cdots:x\in G)\;.$$

By (1), there exist subgroups Γ_{kn} in \mathscr{F} such that

$$(3)$$
 $C_{kn} + H_{kn} \subset C_{kn} + V_{kn}$ $(k = 0, 1, 2, \dots; n = 1, 2, \dots)$,

where H_{kn} is the annihilator of Γ_{kn} . Let Γ be any subgroup in \mathscr{F} which contains all Γ_{kn} . Then, it follows from (2) and (3) that (i) holds. This completes the proof.

Proof of Theorem 1 for general groups. Let G be an arbitrary nondiscrete LCA group, and let $\{\lambda_k\}_{k=0}^{\infty}$ and $\{C_k\}_{k=0}^{\infty}$ be as in Theorem 1. For $\hat{F} = \{\chi \in \hat{G} : \hat{\lambda}_0(\chi) \neq 0\}$, take a $\Gamma \subset G$ as in Lemma 6. Setting

 $H = H_{\Gamma}$, we denote by π and m_{H} the natural mapping of G onto $G_{0} = G/H$ and the Haar measure of H with $m_{H}(H) = 1$, respectively. For each $\mu \in \mathcal{M}(G)$, define a measure $\mu' \in \mathcal{M}(G_{0})$ by setting

(1)
$$\int_{G_0} f d\mu' = \int_G f \circ \pi d\mu \quad \forall f \in C_0(G_0)$$
.

Identifying Γ with \hat{G}_0 in the usual way, we see $\hat{\mu}' = \hat{\mu} | \Gamma$ for all $\mu \in M(G)$, so that $0 \neq \lambda'_0 \in M_0(G_0)$. On the other hand, we have

(2)
$$\lambda_k'[x'+C_k']=0$$
 $(x'\in G_0; k=0, 1, 2, \cdots)$

by (i) in Lemma 6 and (1), where $C'_k = \pi(C_k)$. Therefore $\{\lambda'_k\}_{k=0}^{\infty} \subset M_c^+(G)$, and we can apply our result for metrizable groups to find a nonzero measure $\sigma' \in M_0^+(\operatorname{supp} \lambda'_0)$ with compact support such that

$$(3) \quad \lambda'_k[x' + C'_k + G_p(\operatorname{supp} \sigma')] = 0 \qquad (x' \in G_0; k = 0, 1, 2, \cdots).$$

Now define a measure $\sigma \in M(G)$ by setting

$$(4) \qquad \int_{G} f d\sigma = \int_{G_0} \left\{ \int_{H} f(x+t) dm_H(t) \right\} d\sigma'(x') \qquad \forall f \in C_0(G) \ .$$

As is easily seen, we then have

$$(\,5\,)\qquad \quad \mathrm{supp}\,\sigma=\pi^{\scriptscriptstyle -1}[\mathrm{supp}\,\sigma']\quad \mathrm{and}\quad \mathrm{supp}\,\lambda_{\scriptscriptstyle 0}=\pi^{\scriptscriptstyle -1}[\mathrm{supp}\,\lambda_{\scriptscriptstyle 0}']$$

(note that $\sigma * m_{II} = \sigma$ and $\lambda_0 * m_{II} = \lambda_0$). It is also easy to check that $0 \neq \sigma \in M_0^+(G)$, that supp σ is a compact subset of supp λ_0 , and that

$$\lambda_k[x + C_k + G_p(\operatorname{supp} \sigma)] = \lambda'_k[x' + C'_k + G_p(\operatorname{supp} \sigma')] = 0$$

for all $x \in G$ and all $k = 0, 1, 2, \cdots$.

This establishes Theorem 1.

Proof of Corollary. Let Y be as in the present Corollary. Setting $C_k = \{0\}$ for all k and applying Theorem 1 to $\{|\lambda_k|\}_{k=0}^{\infty}$, we obtain a nonzero measure $\sigma \in M_0^+(\operatorname{supp} \lambda_0)$ with compact support such that

$$(1) \qquad \qquad |\tau| [x + G_p(\operatorname{supp} \sigma)] = 0 \qquad (x \in G)$$

holds for all $\tau \in \bigcup_{k=0}^{\infty} L^1(\lambda_k)$. But then we have

$$egin{aligned} &|oldsymbol{
u}* au \mid [x+G_p(ext{supp }\sigma)] &\leq (|oldsymbol{
u}|*| au |)[x+G_p(ext{supp }\sigma)] \ &\leq \int_{G} |oldsymbol{
u}\mid [x-y+G_p(ext{supp }\sigma)]d \,|oldsymbol{
u}| \,(y) = 0 \end{aligned}$$

for all $x \in G$ whenever $\nu \in M(G)$ and $\tau \in \bigcup_{k=0}^{\infty} L^1(\lambda_k)$. Since the ideal Y is generated by $\bigcup_{k=0}^{\infty} L^1(\lambda_k)$, this implies that (1) holds for all $\tau \in Y$.

The last statement in the Corollary is now trivial and the proof is complete.

To prove Theorem 2, we need some notation. Let \mathscr{T} be a nonempty family of (locally) Borel measurable subgroups of G such that for any countable subfamily \mathscr{T}_0 of \mathscr{T} there exists a subgroup $H \in \mathscr{T}$ which contains all $L \in \mathscr{T}_0$. Define

$$I(\mathscr{T}) = \{ \mu \in M(G) \colon | \mu| (x + H) = 0 \ \forall x \in G \text{ and } \forall H \in \mathscr{T} \}$$

and

$$\mathscr{R}(\mathscr{T}) = \{ \nu \in M(G) \colon |\nu| (G \setminus (D + H)) = 0 \text{ for some countable} \\ D \subset G \text{ and some } H \in \mathscr{T} \}.$$

Then it is easy to prove the following (cf. [1]):

(a) I(𝒴) is a closed ideal in M(G) such that I(𝒴)* = I(𝒴).
(b) 𝔅(𝒴) is a closed subalgebra of M(G) such that 𝔅(𝒴)* =

(b) $\mathscr{R}(\mathscr{G})$ is a closed subalgebra of M(G) such that $\mathscr{R}(\mathscr{G})^* = \mathscr{R}(\mathscr{F})$.

(c) $M(G) = I(\mathcal{F}) + \mathscr{R}(\mathcal{F})$ and $I(\mathcal{F}) \cap \mathscr{R}(\mathcal{F}) = \{0\}$. We denote by $\Phi_{\mathcal{F}}$ the projection of M(G) onto $\mathscr{R}(\mathcal{F})$ which is induced by the direct sum decomposition $M(G) = I(\mathcal{F}) + \mathscr{R}(\mathcal{F})$. Note that $\Phi_{\mathcal{F}}$ is a *-homomorphism of M(G) onto $\mathscr{R}(\mathcal{F})$.

Proof of Theorem 2. Let $\{\lambda_k\}_{k=0}^{\infty}$ and Y be as in Corollary; without loss of generality, we may assume that $\lambda_k \geq 0$ for all k. By Lemma 6, there is a σ -compact, noncompact, open subgroup Γ of \hat{G} such that

(1)
$$\lambda_k(x + H) = 0$$
 $(x \in G; k = 0, 1, 2, \cdots),$

where *H* is the annihilator of *Γ*. Let $G_0 = G/H$, and let $\mu \to \mu'$ be the mapping of M(G) onto $M(G_0)$ defined in the proof of Theorem 1 for general groups. Note that $\mu \to \mu'$ is a *-homomorphism. Since (1) implies $\lambda'_k \in M_c(G)$ for all *k*, Theorem 1 assures that there exists a nonzero measure $\sigma' \in M_0^-(G_0)$ with compact support such that K' =supp σ' is independent and

(2)
$$\lambda'_k[x' + G_p(K')] = 0$$
 $(x' \in G_0; k = 0, 1, 2, \cdots)$.

Let ω_1 be the first countable ordinal and let $W = \{1, 2, \dots\}$ be the well-ordered set consisting of all ordinals smaller than ω_1 . We now construct a family $\{L'_{\alpha} : \alpha \in W\}$ of disjoint compact subsets of K'such that

$$(3)$$
 $M_{\scriptscriptstyle 0}(L'_{\scriptscriptstyle lpha})
eq \{0\} \hspace{0.1in} ext{and} \hspace{0.1in} \sigma'(L'_{\scriptscriptstyle lpha}) = 0$

for all $\alpha \in W$. First, by Theorem 1, there exists a compact subset L'_1 of K' having property (3). Let $\beta \in W$, $\beta \geq 2$, and suppose that L'_{α} has been constructed for all $\alpha \in W$ with $\alpha < \beta$. Then $E'_{\beta} = \bigcup \{L'_{\alpha} : \alpha < \beta\}$ is σ -compact, and by (3), has σ' -measure zero. Therefore, there exists a compact subset F'_{β} of $K' \setminus E'_{\beta}$ having positive σ' -

measure. Applying Theorem 1 again, we can find a compact subset L'_{β} of F'_{β} so that (3) holds for $\alpha = \beta$. By transfinite induction, we obtain a family $\{L'_{\alpha}: \alpha \in W\}$ of disjoint compact subsets of K' satisfying (3).

Let $\mathscr{P}(W)$ be the family of all nonempty subsets of W; hence Card $\mathscr{P}(W) = 2^{\omega}$, where ω denotes the smallest uncountable cardinal. For each $A \in \mathscr{P}(W)$ and $\chi \in \Gamma$, we construct a complex homomorphism $\Psi_{A\chi}$ of M(G) as follows. Let $\mathscr{T} = \mathscr{T}_A$ be the family of subgroups of G each of which is generated by $\bigcup \{L'_{\alpha} : \alpha \in B\}$ for some countable subset B of A. We define $\Psi_{A\chi}$ by setting

(4)
$$\Psi_{A\chi}(\mu) = \widehat{\Phi_{\mathscr{I}}}((\chi \mu)')(1) \quad (\mu \in M(G)).$$

It is easy to see that $\Psi_{A\chi}$ is a symmetric complex homomorphism of M(G). Also $\Psi_{A\chi} \neq 0$ because

(5)
$$\Psi_{A\chi}(\delta_x) = \chi(x) \neq 0 \qquad (x \in G)$$

where δ_x denotes the unit mass at x.

Fixing an $A \in \mathscr{P}(W)$ and $\chi \in \Gamma$, we now prove

(6)
$$\bigcup \{L^1(\mu): \mu \in Y\} \cup M_a(G) \subset \operatorname{Ker} \Psi_{A\chi} \text{ but } M_0(G) \not\subset \operatorname{Ker} \Psi_{A\chi}.$$

First note that $\nu \in M^+(G)$ and $\mu \in L^1(\nu)$ imply $\mu' \in L^1(\nu')$. In fact, if w is a bounded Borel function on G, we have

$$igg| \int_{\sigma_0} f d(w oldsymbol{
u})' igg| = igg| \int_{\sigma} (f \circ \pi) w doldsymbol{
u} igg| \leq \int_{\sigma} (|f| \circ \pi) |w| doldsymbol{
u} \ \leq ||w||_{\infty} \int_{\sigma} (|f| \circ \pi) doldsymbol{
u} = ||w||_{\infty} \cdot ||f||_{L^1(
u')}$$

for all $f \in C_0(G_0)$, so that $(w\nu)' \in L^1(\nu')$. Since the mapping $\tau \to \tau'$ is norm-decreasing, we see

$$\inf\left\{\parallel \mu' - au' \parallel_{\scriptscriptstyle M}: au' \in L^{\imath}(oldsymbol{
u}')
ight\} \leq \parallel \mu' - (w oldsymbol{
u})' \parallel_{\scriptscriptstyle M} \leq \parallel \mu - w oldsymbol{
u} \parallel_{\scriptscriptstyle M}.$$

Since w was arbitrary and $\mu \in L^1(\nu)$, this implies $\mu' \in L^1(\nu')$. Suppose now that $\nu \in M(G)$ and $\lambda \in L^1(\lambda_k)$ for some k. Then, the above observation and (2) show

$$egin{aligned} &|(m{
u}*m{\lambda})'|(x'+T') = |m{
u}'*m{\lambda}'|(x'+T') \leqq (|m{
u}'|*|m{\lambda}'|)(x'+T') \ &\leq (|m{
u}'|*|m{\lambda}'|)[x'+G_p(K')] = 0 \end{aligned}$$

for all $x' \in G_0$ and all $T' \in \mathcal{T} = \mathcal{T}_A$. Since the linear span of the sets $M(G) * L^1(\lambda_k)$, $k = 0, 1, 2, \cdots$, is dense in Y, it follows that

$$| \tau' | (x' + T') = 0$$
 $(x' \in G_0; T' \in \mathscr{T})$

holds for all $\tau \in Y$, and so for all $\tau \in \bigcup \{L^{\iota}(\mu): \mu \in Y\}$. Therefore we have

$$\bigcup \{L^{1}(\mu): \mu \in Y\} \subset \operatorname{Ker} \Psi_{A\chi} .$$

Note now that $\lambda'_0 \neq 0$, and so $G_p(\operatorname{supp} \sigma')$ has no interior point by (2); hence the Haar measure of $G_p(\operatorname{supp} \sigma')$ is zero. Since $M_a(G)' = M_a(G_0)$, it follows that $M_a(G) \subset \operatorname{Ker} \Psi_{A\chi}$. To prove that $M_0(G) \not\subset \operatorname{Ker} \Psi_{A\chi}$, take any $\alpha \in A$. Then $L'_{\alpha} \subset G_p(L'_{\alpha}) \in \mathscr{T}$, and so $\Phi_{\mathcal{T}}[M_0(L'_{\alpha})] = M_0(L'_{\alpha}) \neq \{0\}$. This establishes (6) because $M_0(G)' = M_0(G_0)$.

Finally, take any $A, B \in \mathscr{P}(W)$ and any $\chi, \gamma \in \Gamma$. If $\chi \neq \gamma$, (5) implies that $\Psi_{A\chi} \neq \Psi_{B\gamma}$. If $A \neq B$ (say $A \not\supset B$), take any $\beta \in B \setminus A$; we claim

$$(7) M_c(L'_{\beta}) \subset \operatorname{Ker} \, \varPhi_{\mathscr{T}} \text{ where } \, \mathscr{T} = \mathscr{T}_A \, .$$

In fact, let T' be an arbitrary subgroup in \mathscr{T} ; there exists a countable subset A_0 of A such that $T' = G_p(\bigcup \{L'_{\alpha}: \alpha \in A_0\})$. Since K' is independent and since L'_{β} and $\bigcup \{L'_{\alpha}: \alpha \in A_0\}$ are disjoint subsets of K', it follows that $L'_{\beta} \cap (x' + T')$ contains at most one point for each $x' \in G_0$. In particular, if $\mu' \in M_c(L'_{\beta})$, then $|\mu'|(x' + T') = 0$ for all $x' \in G_0$. Since $T' \in \mathscr{T}$ was arbitrary, we see that (7) holds. On the other hand, we have $\Phi_{\ll}(M(L'_{\beta})) = M(L'_{\beta})$ for $\mathscr{U} = \mathscr{T}_B$. Thus $\Psi_{AZ} \neq$ Ψ_{BT} , as is easily seen. This clearly establishes Theorem 2.

REMARKS. (i) If G is a metrizable I-group, then the element x_0 in Theorem 1 (and Corollary) can be chosen $x_0 = 0$. In fact, take any nonzero $\lambda_0 \in M_0^+(G)$, and assume that $E_{qy} = \{x \in G : qx = y\}$ has positive λ_0 -measure for some $q \in N$ and some $y \in G$. Let μ_0 be the restriction of λ_0 to E_{qy} , so that $0 \neq \mu_0 = M_0^-(G)$. It is trivial that E_{qy} is a coset of some closed subgroup H of G which is of bounded order. If Γ is the annihilator of H, we see $|\hat{\mu}_0| = \text{const} \neq 0$ on Γ . Since $\hat{\mu}_0$ vanishes at infinity, it follows that Γ is compact, or, equivalently, that H is an open subgroup of G. This is a contradiction because G is an I-group while H is of bounded order. Thus, our assertion follows from the last paragraph of the proof of Lemma 2 and the proof of Theorem 1 for metrizable groups.

(ii) Let \widehat{G}^- denote the closure of \widehat{G} in the maximal ideal space Δ_G of M(G), and let Y be as in Theorem 2. Then, for some $\tau \in M_0^+(G)$, the set E_{τ} of all symmetric $\Theta \in \Delta_G$ such that

$$igcup_{a} \{L^{\imath}(\mu) \colon \mu \in Y\} \cup M_{a}(G) \subset \Theta \quad ext{and} \quad \widehat{ au}(\Theta) = 1$$

has cardinal number $\geq 2^{\omega}$, where $\hat{\tau}$ denotes the Gelfand transform of τ . Note that E_{τ} is a closed subset of \varDelta_G disjoint from \hat{G}^- . To see this, redefine $\mathscr{P}(W)$ in the proof of Theorem 2 to be the family of all subsets of W containing $1 \in W$, and fix any probability measure $\tau \in M_0^+(G)$ such that $\tau' \in M(L'_1)$. Then we have

$$\varPsi_{_{A1}}(au)=\widehat{ au}'(1)=\widehat{ au}(1)=1 \qquad (A\in\mathscr{P}(W)) \;.$$

(iii) Let Y be as in Theorem 2. Then there exist a measure $\tau \in M_0^+(G)$, a nondiscrete LCA group G_0 , and an independent compact subset K' thereof, with $M_0(K') \neq \{0\}$, having the following property: the set of all asymmetric $\Theta \in \Delta_G$ such that

$$\bigcup \left\{ L^{\scriptscriptstyle 1}(\mu) \colon \mu \in Y \right\} \cup M_a(G) \subset \Theta \quad \text{and} \quad \widehat{\tau}(\Theta) = 1$$

has cardinal number \geq Card $M_c(K')^*$, where $M_c(K')^*$ denotes the conjugate space of $M_c(K')$. This can be proved using the proof of Theorem 2 and a theorem of Hewitt and Kakutani [2]. We omit the details.

(iv) Some analogs to our results hold for non-abelian groups. For example, we have the following: Let G be a nondiscrete locally compact group, $\{\lambda_k\}_{k=0}^{\infty} \subset M_c^+(G), \ \lambda_0 \neq 0$, and let $\{C_k\}_{k=0}^{\infty}$ be a countable family of σ -compact subsets of G such that

$$\lambda_k(xC_k) = 0$$
 $(x \in G; k = 0, 1, 2, \cdots)$.

Then there exists a nonzero measure $\sigma \in M^+_e(\operatorname{supp} \lambda_0)$ with compact support such that

$$\lambda_k[xG_p(\operatorname{supp} \sigma)C_k] = 0 \qquad (x \in G; \, k = 0, \, 1, \, 2, \, \cdots) \; .$$

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