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ON THE RADICALS OF LATTICE-ORDERED RINGS

H. J. SHYR AND T. M. VISWANATHAN

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In this note, it is shown that for several classes of lattice-ordered rings, the l -radical $L(A)$ and the prime radical $P(A)$ coincide and that A modulo the l -radical is an f -ring. In particular, this is true for the class of positive square rings satisfying the identity $a_+a_- = 0$.

The most well-behaved lattice-ordered rings are the f -rings satisfying the identities $xa_+ \wedge a_- = 0$ where x is an arbitrary positive element and a an arbitrary element of the l -ring A . All other rings are then studied by dissecting the ring into parts — one part called the radical where the idiosyncracies of the ring play a role and the other is the ring modulo the radical where the ring is expected to behave more like an f -ring. The radicals are themselves varied: There is the l -radical $L(A)$ of Birkhoff and Pierce which is the union of nilpotent l -ideals of A and the P -radical $\mathcal{P}(A)$, being the intersection of all the prime l -ideals of A . It is known that $L(A) \subseteq P(A)$. The object of this note is to show that equality holds and that the radicals behave well for several classes of l -rings.

2. Square-archimedean rings. A *square-archimedean* ring A is an l -ring satisfying the following: Given x, y in the positive cone A_+ , there exists a positive integer $n = n(x, y)$ such that $xy + yx \leq n(x^2 + y^2)$. The positive square l -rings, having square elements positive or zero are indeed square-archimedean. The following is an example of a commutative l -ring with identity which is square-archimedean but not positive square: The ring A has the additive group of two copies of the ordered group Z of integers with multiplication defined by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_1 + a_1b_2)$ and order provided by (a_1, a_2) in A^+ if $a_2 \geq a_1 \geq 0$ in Z . Notice also that the bound $n(x, y)$ may not be uniform.

It is appropriate at this point to introduce the upper l -radical $U(A)$ which is the union of all nil l -ideals of A . $U(A)$ is an l -ideal whereas the set $H(A)$ of all absolutely nilpotent elements need not be an ideal. We have the containment relation $L(A) \subseteq P(A) \subseteq U(A) \subseteq H(A)$. Throughout the remaining part of this section A is assumed to be a square-archimedean ring.

PROPOSITION 1. *If x and y are elements of A^+ and m a positive integer, then there exist positive integers λ_m and μ_m such that $(x + y)^{2^m} \leq \lambda_m(x^{2^m} + y^{2^m})$ and $(xy)^{2^m} \leq \mu_m(x^{2^{m+1}} + y^{2^{m+1}})$.*

Proof. Use induction on m . For the second inequality, $xy \leq xy + yx \leq n(x^2 + y^2)$ and so $(xy)^{2^m} \leq n^{2^m}(x^2 + y^2)^{2^m}$ and now use the first.

PROPOSITION 2. *The set $H(A)$ is a sublattice subring of A which is also square-archimedean.*

Proof. This is a consequence of Proposition 1 and the following identity in A : $a + b = (a \vee b) + (a \wedge b)$.

THEOREM 1. *If A is a square-archimedean ring, then $L(A) = P(A) = U(A)$. In particular, the three radicals coincide for positive square l -rings.*

Proof. We shall obtain a reduction to the case when A itself will be a nil ring. For this, $U(A)$ is an l -ideal of A and so by (2.18) of [2], the l -radical of $U(A)$ is equal to $L(A)$. Since $U(A)$ is a nil ring, the theorem will be proved if we show that the l -radical of a nil ring is the whole ring. This is the next lemma.

LEMMA 1. *For every integer $m \geq 1$, let $p(m) = 2^m$. If A is a nil ring then the set $I_m = \{x \in A : |x|^{p(m)} = 0\}$ is a nil potent l -ideal. Hence $L(A) = A$.*

Proof. It is enough to prove the result for $m = 1$, since the general case would then follow by induction by passing to the quotient say A/I_{m-1} . For $m = 1$, we already know from Proposition 1 that I_1 is a sublattice subring of A . Given $x \geq 0$ in I_1 and a in A^+ , we have $xax = xax + ax^2 \leq n(ax)^2$ for some positive integer n and by iteration, $xax \leq n^s a^s xax$ for every $s \geq 2$ and so $xax = 0$, making the square of both ax and xa vanish. Thus I_1 is a nilpotent l -ideal of index 2.

REMARK 1. The question naturally arises whether there exists a positive square l -ring for which $U(A) \neq H(A)$. This is another form of a question of Diem. (See p. 79 of [2].)

3. Rings with well-behaved radicals. We shall now complete the work of Diem by showing that for several classes of rings satisfying specific l -ring identities, the l -radical equals the set N of nilpotents so that all the radicals coincide. A basic tool is the notion of an \underline{f} -ideal, which is an l -ideal I such that A/I is an \underline{f} -ring. Thus an l -ideal I is ad \underline{f} -ideal if and only if it contains all elements of the form $xa^+ \wedge a^-$ and $a^+x \wedge a^-$ for all $x \geq 0$ and for all a in A . We observe that if the l -ring A has a nilpotent \underline{f} -ideal, then $L(A) = N$, making all the radicals coincide and in this case the l -radical indeed behaves well since $A/L(A)$ is an \underline{f} -ring without nilpotent elements.

THEOREM 2. *Let A be an l -ring which satisfies one of the following identities:*

- (i) $xa^+ \wedge xa^- = 0$ and $a^+x \wedge a^-x = 0$ for all $x \geq 0$ and a in A .
- (ii) $xa^+x \wedge xa^-x = 0$ for all $x \geq 0$ and a in A .
- (iii) $a^+xa^- = 0$ for all $x \geq 0$ and a in A .
- (iv) $xa^+xa^- = 0$ for all $x \geq 0$ and a in A .
- (v) $a^+a^- = 0$ for all a in A . Then $L(A) = N$.

Proof. We shall produce a nilpotent \underline{f} -ideal in all cases except (v).

(i) and (ii). Let $I = \{x \in A : Ax A = 0\}$. Let us show that I is an \underline{f} -ideal in the case of (ii). A similar proof works for (i). If c, d , and $x \geq 0$ in A and a an element of A , then $c(xa^+ \wedge a^-)d \leq cxa^+d \wedge ca^-d \leq ea^+e \wedge ea^-e$ where e is any upper bound of c, cx , and d and this last element is 0. Since any element is the difference of two positive elements, this shows that $xa^+ \wedge a^-$ belongs in I . Similarly $a^+x \wedge a^-$ belongs in I . Clearly I is a nilpotent \underline{l} -ideal.

(iii) and (iv). It is clearly enough to prove (iv). Notice that for every $x \geq 0$ and a in A , the element $(xa)^2x \geq 0$. Using this, it is easy to show that the set $J = \{a \in A : (x|a|)^2x = 0 \forall x \in A^+\}$ is a nilpotent \underline{f} -ideal.

(v) Since A in this case is a positive square ring, by Theorem 1, $L(A) = P(A)$ and by Corollary 4.6 of [2], $P(A) = N$.

COROLLARY. *Let A be an l -ring. Suppose the upper radical is square-archimedean or satisfies one of the identities above, then $L(A) = P(A) = U(A)$.*

REMARK 2. The l -ring satisfying the identity $a^+a^- = 0$ also has a nilpotent \underline{f} -ideal. The proof however requires that $H(A)$ be an \underline{l} -ideal, a consequence of Corollary 3.8 of [2]. Since the existence of a nilpotent \underline{f} -ideal implies that only a part of the \underline{l} -radical behaves undesirably, it may be useful to describe this \underline{f} -ideal.

From Lemma 1, if a and s are elements of A^+ and if $a^2 = 0$ and s nilpotent, then $asa = 0$. Now if $r \in A^+$ and $a \in A^+$ an element such that $a^2 = 0$, then rar is nilpotent, since $H(A)$ is an \underline{l} -ideal. Hence for every r in A^+ we have $arara = 0$.

Now if $a \in A$ and $r \in A^+$ then $(ra^+ \wedge a^-)^2 \leq ra^+a^- = 0$. Hence $(ra^+ \wedge a^-)^2 = 0$. Similarly $(a^+r \wedge a^-)^2 = 0$.

Let $Z_1(A) = \{a \in A : (x|a|)^2x = 0 \forall x \in A^+\}$. Since A is a positive square ring, $Z_1(A)$ is a nilpotent \underline{l} -ideal. Since it may not contain $ra^+ \wedge a^-$, we construct $Z_2(A)$ as the inverse image of $Z_1(A/Z_1(A))$, using the natural epimorphism $A \rightarrow A/Z_1(A)$. $Z_2(A)$ is a nilpotent \underline{f} -ideal of A .

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