ON THE RADICALS OF LATTICE-ORDERED RINGS

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In this note, it is shown that for several classes of lattice-ordered rings, the $\delta$-radical $L(A)$ and the prime radical $P(A)$ coincide and that $A$ modulo the $\delta$-radical is an $f$-ring. In particular, this is true for the class of positive square rings satisfying the identity $a_+a_- = 0$.

The most well-behaved lattice-ordered rings are the $f$-rings satisfying the identities $xa_+ \wedge a_- = 0$ where $x$ is an arbitrary positive element and $a$ an arbitrary element of the $\delta$-ring $A$. All other rings are then studied by dissecting the ring into parts—one part called the radical where the idiosyncracies of the ring play a role and the other is the ring modulo the radical where the ring is expected to behave more like an $f$-ring. The radicals are themselves varied: There is the $\delta$-radical $L(A)$ of Birkhoff and Pierce which is the union of nilpotent $\delta$-ideals of $A$ and the $P$-radical $P(A)$, being the intersection of all the prime $\delta$-ideals of $A$. It is known that $L(A) \subseteq P(A)$. The object of this note is to show that equality holds and that the radicals behave well for several classes of $\delta$-rings.

2. Square-archimedean rings. A square-archimedean ring $A$ is an $\delta$-ring satisfying the following: Given $x, y$ in the positive cone $A_+$, there exists a positive integer $n = n(x, y)$ such that $xy + yx \leq n(x^2 + y^2)$. The positive square $\delta$-rings, having square elements positive or zero are indeed square-archimedean. The following is an example of a commutative $\delta$-ring with identity which is square-archimedean but not positive square: The ring $A$ has the additive group of two copies of the ordered group $Z$ of integers with multiplication defined by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_1 + a_1b_2)$ and order provided by $(a_1, a_2) \in A^+$ if $a_2 \geq a_1 \geq 0$ in $Z$. Notice also that the bound $n(x, y)$ may not be uniform.

It is appropriate at this point to introduce the upper $\delta$-radical $U(A)$ which is the union of all nil $\delta$-ideals of $A$. $U(A)$ is an $\delta$-ideal whereas the set $H(A)$ of all absolutely nilpotent elements need not be an ideal. We have the containment relation $L(A) \subseteq P(A) \subseteq U(A) \subseteq H(A)$. Throughout the remaining part of this section $A$ is assumed to be a square-archimedean ring.

**Proposition 1.** If $x$ and $y$ are elements of $A^+$ and $m$ a positive integer, then there exist positive integers $\lambda_m$ and $\mu_m$ such that $(x + y)^m \leq \lambda_m(x^m + y^m)$ and $(xy)^m \leq \mu_m(x^{m+1} + y^{m+1})$. 

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Proof. Use induction on $m$. For the second inequality, $xy \leq xy + yx \leq n(x^2 + y^2)$ and so $(xy)^m \leq n^m(x^2 + y^2)^m$ and now use the first.

**Proposition 2.** The set $H(A)$ is a sublattice subring of $A$ which is also square-archimedean.

**Proof.** This is a consequence of Proposition 1 and the following identity in $A$: $a + b = (a \lor b) + (a \land b)$.

**Theorem 1.** If $A$ is a square-archimedean ring, then $L(A) = P(A) = U(A)$. In particular, the three radicals coincide for positive square l-rings.

**Proof.** We shall obtain a reduction to the case when $A$ itself will be a nil ring. For this, $U(A)$ is an $l$-ideal of $A$ and so by (2.18) of [2], the $l$-radical of $U(A)$ is equal to $L(A)$. Since $U(A)$ is a nil ring, the theorem will be proved if we show that the $l$-radical of a nil ring is the whole ring. This is the next lemma.

**Lemma 1.** For every integer $m \geq 1$, let $p(m) = 2^m$. If $A$ is a nil ring then the set $I_m = \{x \in A : |x|^{p(m)} = 0 \}$ is a nilpotent $l$-ideal. Hence $L(A) = A$.

**Proof.** It is enough to prove the result for $m = 1$, since the general case would then follow by induction by passing to the quotient say $A/I_m$. For $m = 1$, we already know from Proposition 1 that $I_1$ is a sublattice subring of $A$. Given $x \geq 0$ in $I_1$ and $a$ in $A^+$, we have $xax = xax + ax^2 \leq n(ax)^2$ for some positive integer $n$ and by iteration, $xax \leq n^a xax$ for every $s \geq 2$ and so $xax = 0$, making the square of both $ax$ and $xa$ vanish. Thus $I_1$ is a nilpotent $l$-ideal of index 2.

**Remark 1.** The question naturally arises whether there exists a positive square $l$-ring for which $U(A) \neq H(A)$. This is another form of a question of Diem. (See p. 79 of [2].)

3. **Rings with well-behaved radicals.** We shall now complete the work of Diem by showing that for several classes of rings satisfying specific $l$-ring identities, the $l$-radical equals the set $N$ of nilpotents so that all the radicals coincide. A basic tool is the notion of an $f$-ideal, which is an $l$-ideal $I$ such that $A/I$ is an $f$-ring. Thus an $l$-ideal $I$ is ad $f$-ideal if and only if it contains all elements of the form $xa^+ \land a^-$ and $a^+x \land a^-$ for all $x \geq 0$ and for all $a$ in $A$. We observe that if the $l$-ring $A$ has a nilpotent $f$-ideal, then $L(A) = N$, making all the radicals coincide and in this case the $l$-radical indeed behaves well since $A/L(A)$ is an $f$-ring without nilpotent elements.
**Theorem 2.** Let $A$ be an $l$-ring which satisfies one of the following identities:

(i) $xa^+ \wedge xa^- = 0$ and $a^+x \wedge a^-x = 0$ for all $x \geq 0$ and $a$ in $A$.
(ii) $xa^+x \wedge xa^-x = 0$ for all $x \geq 0$ and $a$ in $A$.
(iii) $a^+xa^- = 0$ for all $x \geq 0$ and $a$ in $A$.
(iv) $xa^+xa^-x = 0$ for all $x \geq 0$ and $a$ in $A$.
(v) $a^+a^- = 0$ for all $a$ in $A$. Then $L(A) = N$.

**Proof.** We shall produce a nilpotent $f$-ideal in all cases except (v).

(i) and (ii). Let $I = \{x \in A : Ax = 0\}$. Let us show that $I$ is an $f$-ideal in the case of (ii). A similar proof works for (i). If $c, d,$ and $x \geq 0$ in $A$ and $a$ an element of $A$, then $c(xa^+ \wedge a^-)d \leq cxa^+d \wedge ca^-d \leq ea^+e \wedge ea^-e$ where $e$ is any upper bound of $c, cx,$ and $d$ and this last element is 0. Since any element is the difference of two positive elements, this shows that $xa^+ \wedge a^-$ belongs in $I$. Similarly $a^+x \wedge a^-$ belongs in $I$. Clearly $I$ is a nilpotent $l$-ideal.

(iii) and (iv). It is clearly enough to prove (iv). Notice that for every $x \geq 0$ and $a$ in $A$, the element $(xa)^2x \geq 0$. Using this, it is easy to show that the set $J = \{a \in A : (x \mid a \mid)^2x = 0 \forall x \in A^+\}$ is a nilpotent $f$-ideal.

(v) Since $A$ in this case is a positive square ring, by Theorem 1, $L(A) = P(A)$ and by Corollary 4.6 of [2], $P(A) = N$.

**Corollary.** Let $A$ be an $l$-ring. Suppose the upper radical is square-archimedean or satisfies one of the identities above, then $L(A) = P(A) = U(A)$.

**Remark 2.** The $l$-ring satisfying the identity $a^+a^- = 0$ also has a nilpotent $f$-ideal. The proof however requires that $H(A)$ be an $l$-ideal, a consequence of Corollary 3.8 of [2]. Since the existence of a nilpotent $f$-ideal implies that only a part of the $l$-radical behaves undesirably, it may be useful to describe this $f$-ideal.

From Lemma 1, if $a$ and $s$ are elements of $A^+$ and if $a^2 = 0$ and $s$ nilpotent, then $asa = 0$. Now if $r \in A^+$ and $a \in A^+$ an element such that $a^2 = 0$, then $rar$ is nilpotent, since $H(A)$ is an $l$-ideal. Hence for every $r \in A^+$ we have $arara = 0$.

Now if $a \in A$ and $r \in A^+$ then $(ra^+ \wedge a^-)^2 \leq ra^+a^- = 0$. Hence $(ra^+ \wedge a^-)^2 = 0$. Similarly $(a^+r \wedge a^-)^2 = 0$.

Let $Z_1(A) = \{a \in A : (x \mid a \mid)^2x = 0 \forall x \in A^+\}$. Since $A$ is a positive square ring, $Z_1(A)$ is a nilpotent $l$-ideal. Since it may not contain $ra^+ \wedge a^-$, we construct $Z_2(A)$ as the inverse image of $Z_1(A)/Z_1(A)$, using the natural epimorphism $A \to A/Z_1(A)$. $Z_2(A)$ is a nilpotent $f$-ideal of $A$. 


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