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**FIXED POINTS AND CHARACTERIZATIONS OF CERTAIN
MAPS**

CHI SONG WONG

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Let T be a self map on a metric space (X, d) such that

$$d(T(x), T(y)) \leq (d(x, T(x)) + d(y, T(y)))/2, \quad x, y \in X.$$

It is proved that: (a) T has a fixed point if T is continuous and X is weakly compact convex subset of a Banach space. (b) All such T which have fixed points can be explicitly determined in terms of d . Related results are obtained.

1. Introduction. In [7], [8], [9], [10], [11], R. Kannan considered the following family of self maps T on a (nonempty) complete metric space (X, d) :

$$(1) \quad d(T(x), T(y)) \leq \frac{1}{2}(d(x, T(x)) + d(y, T(y))), \quad x, y \in X.$$

He obtained a number of results of the following type: T has a (unique) fixed point if X is a weakly compact convex subset of a reflexive Banach space B and for each closed convex subset H of X with $T(H) \subset H$ and $\delta(H) > 0$,

$$(2) \quad \sup \{d(y, T(y)): y \in H\} < \delta(H),$$

where d is the metric induced by the norm $\| \cdot \|$ on B and $\delta(H)$ is the diameter of H . Suppose now that X is a weakly compact convex subset of a Banach space B and T is a self map on X which satisfies (1). P. Soardi [17, Theorems I, II] proved that T has a fixed point if either X has normal structure [3] or T has diminishing orbital diameters [2]. In this paper, the following results are obtained: (a) T has a fixed point if it is continuous (with respect to the strong topology). In fact, T has a fixed point if it is continuous along line segments. It may be worthwhile to mention here that it is a well-known open problem that the same conclusion holds for nonexpansive self maps on X [1, p. 217]. (b) T has a fixed point if for any closed convex subset H of X with more than one point and $T(H) \subset H$,

$$\inf \{d(T(y), y): y \in H\} < \delta(H).$$

It is obvious that this result generalizes the above results of Soardi and the above result of Kannan. (c) Let T be a self map on X such that there exist a_1, a_2, a_3, a_4, a_5 in $[0, 1]$ for which $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ and for all x, y in X ,

$$(3) \quad \begin{aligned} d(T(x), T(y)) \leq & a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) \\ & + a_4 d(y, T(x)) + a_5 d(x, y). \end{aligned}$$

Then T has a fixed point u if for some x_0 in X , the sequence $\{x_n\}$, where $x_{n+1} = (x_n + T(x_n))/2$, converges to u . Note that by calculating $(d(T(x), T(y)) + d(T(y), T(x)))/2$ through (3), one may assume that $a_1 = a_2$ and $a_3 = a_4$. Thus when $a_3 = a_4 = a_5 = 0$, T satisfies (1) if it satisfies (3). Also, (3) is satisfied if T is nonexpansive. Serious consideration of those T which satisfy (3) with $a_1 = a_2 = a_3 = a_4 = 0$ was started by M. Edelstein [5]. In [14], [15], and [16], S. Reich considered those T which satisfy (3) with $a_3 = a_4 = 0$ and thus combined in a natural way, those maps investigated by Edelstein and Kannan. Recently, G. Hardy and T. Rogers [6] considered those maps which satisfy (3) with the possibility that none of the coefficients a_i is zero.

For the interest of the reader, we pose the following statement: Every self map T on a weakly compact convex subset X of a Banach space has a fixed point if

$$\|T(x) - T(y)\| \leq \frac{1}{2}(\|x - T(x)\| + \|y - T(y)\|), \quad x, y \in X.$$

From (a), (b), and (c), a counter example, if exists, would be difficult to construct.

Let T be a self map on a metric space (X, d) which satisfies (1). Once one succeeds in proving that T has a fixed point, it is possible to characterize such T by its fixed point. Because of this observation, the family of all self maps T on a metric space (X, d) which have a fixed point and which satisfy (1) can be found explicitly. As an example, the family of all such self maps T on the unit interval is illustrated. There are exactly 2^c such maps, where c is the cardinal of the real line.

2. Fixed point theorems. Let T be a self map on a convex subset X of a normed linear space. T is *continuous along line segments* if for any x, y in X , $\{T(x_t)\}$ converges to $T(x)$ as t tends to 0, where $x_t = (1-t)x + ty$, $t \in (0, 1)$. Thus T is continuous along line segments if T is continuous.

THEOREM 1. *Let X be a (nonempty) weakly compact convex subset of a normed linear space. Let T be a self map on X which is continuous along line segments and satisfies*

$$\|T(x) - T(y)\| \leq (\|x - T(x)\| + \|y - T(y)\|)/2, \quad x, y \in X.$$

Then T has a unique fixed point in X .

Proof. By Zorn's lemma, there exists a minimal nonempty weakly compact convex subset H of X such that $T(H) \subset H$. Let $x \in H$, $r = \|x - T(x)\|$. It suffices to prove that $r = 0$. Consider

$$W = \{y \in H: \|y - T(y)\| \leq r\}.$$

Since $x \in W$, $W \neq \emptyset$. Since $\|T^2(z) - T(z)\| \leq \|T(z) - z\|$ for each z in X , $T(W) \subset W$. Let V be the closure of the convex hull of $T(W)$. We shall prove that $V \subset W$. Let $v \in V$ and $\varepsilon > 0$. Then there exist y_1, y_2, \dots, y_n in W and t_1, t_2, \dots, t_n in $[0, 1]$ such that $\sum_{i=1}^n t_i = 1$ and

$$\left\|v - \sum_{i=1}^n t_i T(y_i)\right\| < \varepsilon.$$

Thus

$$\begin{aligned} \|v - T(v)\| &\leq \left\|v - \sum_{i=1}^n t_i T(y_i)\right\| + \left\|\sum_{i=1}^n t_i T(y_i) - T(v)\right\| \\ &< \varepsilon + \sum_{i=1}^n t_i \|T(y_i) - T(v)\| \\ &\leq \varepsilon + \left(\sum_{i=1}^n t_i (\|y_i - T(y_i)\| + \|v - T(v)\|)\right)/2 \\ &\leq \varepsilon + (r + \|v - T(v)\|)/2. \end{aligned}$$

So $\|v - T(v)\| \leq 2\varepsilon + r$. Since ε is arbitrarily chosen, $\|v - T(v)\| \leq r$, i.e., $v \in W$. Thus $V \subset W$. So

$$T(V) \subset T(W) \subset V.$$

By minimality of H , $V = H$. So $W = H$. Since x is arbitrarily chosen, it follows from $W = H$ that $\|z - T(z)\| = r$ for all z in H . Now let $y = T(x)$. Then

$$\begin{aligned} r &= \|(1-t)T(x) + tT(y) - T((1-t)T(x) + tT(y))\| \\ &\leq (1-t)\|T(x) - T((1-t)T(x) + tT(y))\| \\ &\quad + t\|T(y) - T((1-t)T(x) + tT(y))\| \\ &\leq \frac{(1-t)}{2}(\|x - T(x)\| + \|(1-t)T(x) + tT(y) \\ &\quad - T((1-t)T(x) + tT(y))\|) \\ &\quad + \frac{t}{2}(\|y - T(y)\| + \|(1-t)T(x) + tT(y) \\ &\quad - T((1-t)T(x) + tT(y))\|) \\ &\leq \frac{1-t}{2}(r+r) + \frac{t}{2}(r+r) = r. \end{aligned}$$

Therefore, all the above inequalities are equalities. So

$$\|T(y) - T((1-t)T(x) + tT(y))\| = r.$$

Since T is continuous along line segments, we have by letting t tend to 0

$$\|T(y) - T^2(x)\| = r.$$

Since $y = T(x)$, $r = 0$.

THEOREM 2. *Let X be a weakly compact convex subset of a Banach space B . Let T be a self map on X such that*

$$\|T(x) - T(y)\| \leq (\|x - T(x)\| + \|y - T(y)\|)/2, \quad x, y \in X.$$

Suppose that for any closed convex subset H of X with $T(H) \subset H$ and $\delta(H) > 0$,

$$(4) \quad \inf \{\|y - T(y)\| : y \in H\} < \delta(H).$$

Then T has a unique fixed point.

Proof. Construct H as in the proof of Theorem 1. Suppose to the contrary that $\delta(H) > 0$. By hypothesis, there exists x in H such that

$$r \equiv \|x - T(x)\| < \delta(H).$$

Construct V as in the proof of Theorem 1. It was proved that $V = W = H$ and $r = \|y - T(y)\|$ for each y in H . Let $u, v \in H$. Then

$$\begin{aligned} \|T(u) - T(v)\| &\leq (\|u - T(u)\| + \|v - T(v)\|)/2 \\ &= (r + r)/2 = r. \end{aligned}$$

So $\delta(T(H)) \leq r$. Thus

$$\delta(H) = \delta(V) = \delta(T(W)) \leq r < \delta(H),$$

a contradiction.

We owe the above argument to Kannan [11] and Soardi [17]. Variants of such argument were used in [4] by R. DeMarr and were later refined by W. A. Kirk [12] for nonexpansive mappings. Then they were used in numerous other articles, e.g., [1], [2], and [13]. We would like to emphasize here that even for reflexive Banach spaces, our results are more general than the corresponding results of Kannan [Theorems 1 and 2, 11].

THEOREM 3. *Let X be a convex subset of a normed linear space B . Let T be a self map on X . Suppose that there exist a_i , $i = 1, 2, 3, 4, 5$, in $[0, 1]$ such that $\sum_{i=1}^5 a_i = 1$ and for all x, y in X ,*

$$\begin{aligned} \|T(x) - T(y)\| \leq & a_1 \|x - T(x)\| + a_2 \|y - T(y)\| + a_3 \|x - T(y)\| \\ & + a_4 \|y - T(x)\| + a_5 \|x - y\|. \end{aligned}$$

Let $x_0 \in X$, $t \in (0, 1)$ and $x_{n+1} = (1 - t)x_n + t T(x_n)$ for each integer $n \geq 0$. Suppose that the sequence $\{x_n\}$ converges to a point u in X . Then u is a fixed point of T .

Proof. Let $n \geq 0$. Then

$$(5) \quad \begin{aligned} \|x_{n+1} - T(u)\| &= \|(1 - t)(x_n - T(u)) + t(T(x_n) - T(u))\| \\ &\leq (1 - t) \|x_n - T(u)\| + t \|T(x_n) - T(u)\|. \end{aligned}$$

By hypothesis,

$$(6) \quad \begin{aligned} \|T(x_n) - T(u)\| &\leq a_1 \|x_n - T(x_n)\| + a_2 \|u - T(u)\| \\ &+ a_3 \|x_n - T(u)\| + a_4 \|u - T(x_n)\| \\ &+ a_5 \|x_n - u\|. \end{aligned}$$

Since $\{x_m\}$ converges to u , and $x_{n+1} - x_n = t(T(x_n) - x_n)$, $\{T(x_m) - x_m\}$ converges to 0. Thus $\{u - T(x_m)\}$ converges to 0. From (5) and (6), we have by letting n tend to ∞ ,

$$(7) \quad \begin{aligned} \|u - T(u)\| &\leq (1 - t) \|u - T(u)\| + t(a_2) \|u - T(u)\| + a_3 \|u - T(u)\| \\ &= ((1 - t) + t(a_2 + a_3)) \|u - T(u)\|. \end{aligned}$$

We may assume that $a_1 = a_2$ and $a_3 = a_4$. Thus $a_2 + a_3 \leq 1/2$. So from (7),

$$\|u - T(u)\| \leq \left(1 - \frac{1}{2}t\right) \|u - T(u)\|.$$

Since $1 - (1/2)t \in (0, 1)$, $T(u) = u$.

3. Examples. Let a be a point in a metric space (X, d) . X_a will denote the set

$$\{(x, y) \in X \times X: d(y, a) \leq \frac{1}{2}d(x, y)\}.$$

THEOREM 4. Let T be a self map on a metric space. Then the following conditions are equivalent:

(a) T has a fixed point and

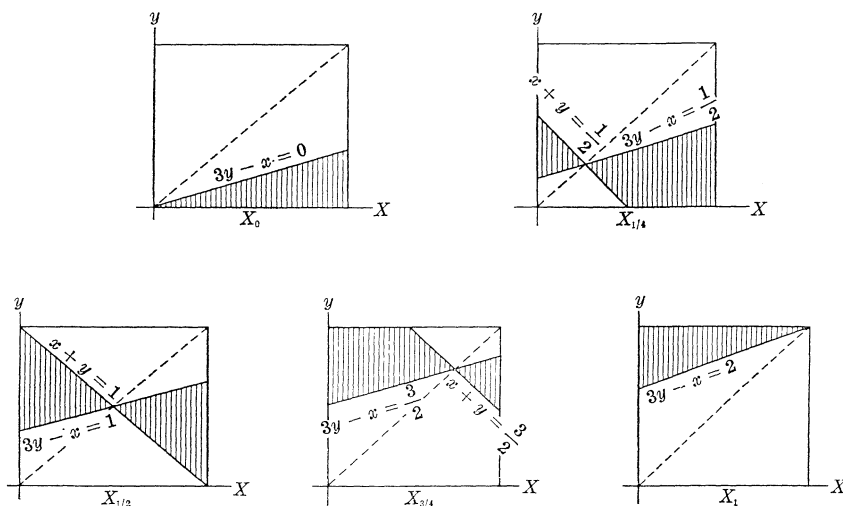
$$d(T(x), T(y)) \leq (d(x, T(x)) + d(y, T(y)))/2$$

for all x, y in X .

(b) *There exists a in X such that $T \subset X_a$, i.e.,*

$$d(T(x), a) \leq d(x, T(x))/2, \quad x \in X.$$

To see the use of Theorems 2 and 4, let T be a self map on the unit interval with the usual metric. Then T satisfies (1) if and only if there exists a in X such that $T \subset X_a$. So in order to find all the maps considered by Kannan, it suffices to find all X_a 's with a in X . For illustration, we list $X_0, X_{1/4}, X_{1/2}, X_{3/4}, X_1$ as follows (the shaded parts): Thus X_a is determined by the lines $x + y = 2a$ and



$3y - x = 2a$: If one cuts $X \times X$ along these lines, then X_a is the closure (in $X \times X$) of the union of those parts which do not contain more than one point of the diagonal of $X \times X$.

Once it was difficult to construct enough examples of those maps considered by Kannan. Now it seems too easy to find a lot. It is then only natural to be more demanding. By looking at the above example where $X = [0, 1]$, we find that it is difficult for a self map on X which satisfies (1) to be surjective. In [18], it was proved that if T is a self map on $[0, 1]$ which satisfies (1) and if $0, 1 \in T[0, 1]$, then $1/2$ is the fixed point of T . This result is an obvious consequence of Theorems 2 and 4. For the interest of the reader, we list two more consequences of Theorems 2 and 4: (a) Let T be a self map on $X = [0, 1]$ such that $0, 1 \in T([0, 1])$ and T is not the identity map on $[0, 1]$. Then the following conditions are equivalent: (i) T satisfies (1). (ii) T satisfies (3). (iii) For any $x, y \in X$,

$$\begin{aligned} & d(T(x), T(y)) \\ & \leq \sup \{d(x, y), (d(x, T(x)) + d(y, T(y)))/2, (d(x, T(y)) + d(y, T(x)))/2\}. \end{aligned}$$

(b) Let X be a regular polygon in the two dimensional Euclidean space. Then there exists a self map T of X onto X which satisfies (1) if and only if X has even number of sides.

Added in proof. Let K be a closed convex subset of a Banach space B . K has a *close-to-normal structure* if for any bounded closed convex subset H of K with $\delta(H) > 0$, there exists x in H such that $\|x - y\| < \delta(H)$ for all y in H . The following related results were announced by the author at the International Congress of Mathematicians in 1974. (a) B has a close-to-normal structure if (1) B is separable, (2) B is strictly convex, or (3) B is reflexive and for any sequence $\{x_n\}$ in B , $\{x_n\}$ converges to x in B whenever $\{x_n\}$ converges weakly to x and $\{\|x_n\|\}$ converges to $\|x\|$. (b) A Hilbert space is isomorphic to a Banach space which has no close-to-normal structure if and only if it is separable. (b) together with Theorem 4 in Chi Song Wong, Proc. Amer. Math. Soc., 46 (1974), to appear, solve the problem we posed in the introduction of this paper. Related to (b), we mentioned here that all theorems for normal structure (=completely normal structure) in L. P. Belluce, W. A. Kirk and E. P. Steiner, Normal structure in Banach spaces, Pacific J. Math., 26 (1968) 433-440, can be modified to results for close-to-normal structure.

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