FIXED POINT THEOREMS FOR MULTIVALUED NONCOMPACT ACYCLIC MAPPINGS

Patrick Michael Fitzpatrick and Walter Volodymyr Petryshyn
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P. M. Fitzpatrick and W. V. Petryshyn

Let $X$ be a Frechet space, $D$ a closed convex subset of $X$, and $T: D \to 2^X$ an upper semicontinuous multivalued acyclic mapping. Using the Eilenberg-Montgomery Theorem and the earlier results of the authors, it is first shown that if $W \supseteq T(D)$ and $f: W \to D$ is a single-valued continuous mapping such that $fT: D \to 2^X$ is $\Phi$-condensing, then $fT$ has a fixed point. This result is then used to obtain various fixed point theorems for acyclic $\Phi$-condensing mappings $T: D \to 2^X$ under the Leray-Schauder boundary conditions in case $D = \text{Int}(D)$ and under the outward and/or inward type conditions in case $\text{Int}(D) = \emptyset$.

Introduction. Let $X$ be a Frechet space and $D$ an open or a closed convex subset of $X$. It is our object in this paper to establish fixed point theorems for not necessarily compact (e.g. condensing) multivalued acyclic mappings $T: D \to 2^X$ which need not satisfy the condition “$T(D) \subseteq D$” but instead are required to satisfy weaker conditions of the Leray-Schauder type. Our results are based upon the Eilenberg-Montgomery Theorem [4] and upon our Lemma 1 in [16]. The fixed point theorems presented in this paper for multivalued maps in infinite dimensional spaces strengthen and extend certain fixed point theorems of Górniewicz-Granas [7] and Powers [17] for acyclic compact maps, the results for star-shaped-valued maps of Halpern [8] for compact maps and our own [16] for condensing maps, and a number of fixed point theorems for convex-valued compact and noncompact maps (see Ky Fan [5], Browder [1], Reich [18], Ma [12], Walt [20], and [20, 8, 15] for related results and further references).

1. Let $X$ be a Frechet space. If $D \subseteq X$, then we will denote by $\bar{D}$ and $\partial D$ the closure and boundary of $D$, respectively.

Definition 1. If $C$ is a lattice with a minimal element, which we will denote by $0$, then a mapping $\Phi: 2^X \to C$ is called a measure of noncompactness provided that the following conditions hold for any $A, B$ in $2^X$:

1. $\Phi(A) = 0$ if and only if $A$ is precompact.
2. $\Phi(\text{co}A) = \Phi(A)$, where $\text{co}A$ denotes the convex closure of $A$.
3. $\Phi(A \cup B) = \max \{\Phi(A), \Phi(B)\}$. 


It follows that if \( A \subseteq B \), then \( \Phi(A) \leq \Phi(B) \). The above notation has been used in \([16, 19]\) and is a generalization of the set-measure \([11]\) and the ball-measure of noncompactness \([6]\) defined either in terms of a family of seminorms or of a single norm when \( X \) is a Banach space. Specifically, if \( \{P_\alpha | \alpha \in \mathcal{A}\} \) is a family of seminorms which determines the topology on \( X \), then for each \( \alpha \in \mathcal{A} \) and \( \Omega \subseteq X \) we define \( \gamma_\alpha(\Omega) = \inf\{d > 0 | \Omega \) can be covered by a finite number of sets each of which has \( P_\alpha \)-diameter less than \( d \}\), and \( \chi_\alpha(\Omega) = \inf\{r > 0 | \Omega \) can be covered by a finite number of \( P_\alpha \)-balls each of which has \( P_\alpha \)-radius less than \( r \}\).

Then letting \( C = \{f: \mathcal{A} \to [0, \infty]\} \), with \( C \) ordered pointwise, we define the set-measure of noncompactness \( \gamma: 2^X \to C \) by \( (\gamma(\Omega))(\alpha) = \gamma_\alpha(\Omega) \) for each \( \alpha \in \mathcal{A} \) and the ball-measure of noncompactness \( \chi(\Omega) \) by \( (\chi(\Omega))(\alpha) = \chi_\alpha(\Omega) \) for each \( \alpha \in \mathcal{A} \) (see\([15]\) for more details and properties of \( \gamma \) and \( \chi \)).

The class of mappings considered here is given by the following.

**Definition 2.** If \( \Phi \) is a measure of noncompactness of \( X \) and \( D \subseteq X \), an upper semicontinuous (u.s.c.) mapping \( T: D \to 2^X \) is called \( \Phi \)-condensing provided that if \( \Omega \subseteq D \) and \( \Phi(T(\Omega)) \geq \Phi(\Omega) \), then \( \Omega \) is relatively compact.

It follows immediately that a compact mapping is \( \Phi \)-condensing with respect to any measure of noncompactness \( \Phi \). Classes of \( \Phi \)-condensing mappings which are not compact have been considered in \([19, 13, 14, 18]\). In particular, if \( X \) is a Banach space, \( D \subseteq X \) is closed, \( C: D \to 2^X \) is compact, and \( S: X \to 2^X \) is such that \( S(x) \) is compact for each \( x \in X \), and \( d^*(S(x), S(y)) \leq kd(x, y) \) for all \( x, y \in X \) and some \( k \in (0, 1) \), where \( d^* \) denotes the Hausdorff metric on the compact subsets of \( 2^X \) generated by the norm \( d \), then \( S + C: D \to 2^X \) is \( \gamma \)-condensing.

By homology we mean Čech homology with rational coefficients, and call a compact metric space \( Y \) acyclic if it has the same homology as a one point space. In particular, any contractable space is acyclic and thus any convex or star-shaped subset of \( X \) is acyclic. A mapping \( T: D \to 2^X \) is called acyclic if \( T(x) \) is compact and acyclic for each \( x \in D \).

The following theorem of Eilenberg and Montgomery \([4]\) together with the succeeding result from \([16]\) will form the basis from which we will deduce our results.

**Theorem A.** \([4]\) Let \( M \) be an acyclic absolute neighborhood retract (ANR), \( N \) a compact metric space, \( r: N \to M \) a continuous single-valued mapping and \( T: M \to 2^N \) a u.s.c. acyclic mapping. Then the mapping
$rT: M \to 2^M$ has a fixed point, i.e., there exist $x \in M$ such that $x \in r(T(x))$.

**Lemma A.** [16] Let $D \subset X$ be closed and convex and $T: D \to 2^X$. Then for each $\Omega \subset D$ there exists a closed convex set $K$, depending on $T$, $D$, and $\Omega$, with $\Omega \subset K$ and $\overline{\text{co}}\{T(D \cap K) \cup \Omega\} = K$.

Our first result is the following fixed point theorem.

**Theorem 1.** Let $X$ be a Frechet space with $D \subset X$ closed and convex. Suppose $T: D \to 2^X$ is u.s.c. and acyclic and $f: W \to D$ is single-valued and continuous, where $W \supset T(D)$. If $fT: D \to 2^X$ is $\Phi$-condensing, then $fT$ has a fixed point. In particular, if $T(D) \subset D$ and $T$ is $\Phi$-condensing, then $T$ has a fixed point.

**Proof.** Choose $x_0 \in D$. By Lemma A, we obtain a closed convex set $K$ such that $x_0 \in K$ and $\overline{\text{co}}\{f(T(K \cap D)) \cup \{x_0\}\} = K$. Since $f(T(D)) \subset D$, we see that $K \cap D = K$ and so $\overline{\text{co}}\{f(T(K)) \cup \{x_0\}\} = K$. By the defining properties of the measure of noncompactness $\Phi$, and, since $fT$ is $\Phi$-condensing, $K$ must be compact. In view of the results in [3, 10], every compact convex subset of a Frechet space is an ANR, and is acyclic. Consequently, letting $M = K, N = T(K)$, and $f = r$ we may invoke Theorem A to conclude that $fT$ has a fixed point. The last part of the theorem follows by letting $f =$ identity.

**Remark 1.** Using the above result, it is clear that a theorem analogous to Theorem 3.4 in [15] is valid for acyclic 1-set and 1-ball contractive mappings.

The second part of Theorem 1 has been obtained in [7, 17] for the case when $T$ is compact and $X$ is a Banach space.

**Theorem 2.** Let $X$ be a Frechet space and $D \subset X$ open and convex with $0 \in D$. If $T: \bar{D} \to 2^X$ is a $\Phi$-condensing and acyclic mapping such that

\[(4) \quad T(x) \cap \{\lambda x| \lambda > 1\} = \emptyset \quad \text{for} \quad x \in \partial D,\]

then $T$ has a fixed point. In particular, if $T(\partial D) \subset \overline{D}$, $T$ has a fixed point.

**Proof.** Let $\rho: X \to \bar{D}$ be the single-valued mapping defined by: $\rho(x) = x$ if $x \in \bar{D}$, and $\rho(x) = x/p(x)$ if $x \in X \setminus \bar{D}$, where $p$ is the support function of $\bar{D}$. Since $0 \in D$, it follows that $\rho$ is continuous. Furthermore, for each $A \subset X, \rho(A) \subset \overline{\text{co}}\{A \cup \{0\}\}$, so that, by the defining properties of $\Phi$,
Φ(ρ(A)) ≤ Φ(A). Hence, ρT is a Φ-condensing mapping of \( D \) into \( \mathcal{D} \) because if \( Ω \subset \mathcal{D} \) and \( Φ(ρ(T(A))) ≥ Φ(Ω) \), \( Ω \) must be relatively compact. Thus, by Theorem 1, we may choose \( x \in \mathcal{D} \), with \( x = ρ(z) \) and \( z \in T(x) \), i.e., \( x \in ρT(x) \). It follows from (4) that \( x \in T(x) \). Indeed, if \( z \in \mathcal{D} \), then \( ρ(z) = z = x \) and so \( x \in T(x) \), and if \( z \notin \mathcal{D} \), then \( ρ(z) = βz \) for some \( β < 1 \) and so \( (1/β)x \in T(x) \), in contradiction to (4). The last assertion follows from the fact that, for each \( y \in \partial D \) and \( β < 1 \), \( βy \in D \) and so \( T(\partial D) \subset \mathcal{D} \) implies (4).

In case \( T(x) \) is convex for each \( x \in \mathcal{D} \), the above result has been obtained in [15] by use of a topological degree argument, without the assumption that \( D \) is convex.

1. In case \( X \) is a Banach space, whose norm has certain additional properties, we will now prove some results for acyclic mappings \( T: D \rightarrow 2^X \), where \( D \) is closed and convex, without the assumption that \( T(D) \subset D \). In particular, we strengthen the results of [8, 16] for mappings satisfying the so-called "nowhere normal outward" condition and without the assumptions (as in [8, 16]) that \( D \) contains a set with a nonempty core and that \( X \) is equipped with a collection of approximation maps (see [8] for definitions of these concepts).

We recall that a Banach space \( X \) is said to have Property (H) if \( X \) is strictly convex and whenever \( \langle x_n \rangle \subset X \) is such that \( \langle \|x_n\| \rangle \rightarrow \|x\| \) and \( \langle x_n \rangle \) converges weakly to \( x \), then \( \langle x_n \rangle \rightarrow x \). Every locally uniformly convex Banach space has this property. We will use the following lemma concerning such spaces, and use the notation \( \langle x_n \rangle \rightarrow x \) to denote the weak convergence of the sequence \( \langle x_n \rangle \) to \( x \).

**Lemma 1.** Let \( X \) be a reflexive Banach space with Property (H), and suppose \( D \subset X \) is closed and convex. Then to each \( x \in X \) there exists a unique point \( N(x) \) in \( D \) such that \( \|x - N(x)\| = \inf_{y \in D} \|y - x\| \). Furthermore, the mapping \( x \rightarrow N(x) \) is continuous.

**Proof.** Let \( x \in X \) and let \( d = \inf_{y \in D} \|y - x\| \). Choose \( \langle u_n \rangle \subset D \) such that \( \langle \|u_n - x\| \rangle \rightarrow d \). Then \( \langle u_n \rangle \) is a bounded subset of \( D \) and since \( X \) is reflexive and \( D \) is weakly complete we may choose a subsequence \( \langle u_{n_k} \rangle \) of \( \langle u_n \rangle \) with \( \langle u_{n_k} \rangle \rightarrow z \in D \). Since \( \langle u_{n_k} - x \rangle \rightarrow z - x \),

\[
d = \lim_k \|u_{n_k} - x\| = \lim_k \inf \|u_{n_k} - x\| \geq \|z - x\|.
\]

But \( \|z - x\| \geq d \), and so \( \langle \|u_{n_k} - x\| \rangle \rightarrow \|z - x\| \). Since \( X \) has Property (H) we must have \( \langle u_{n_k} \rangle \rightarrow z \). The point \( z \) with \( z \in D \) and \( \|z - x\| = d \) is unique
because $X$ is strictly convex, and since, by the above argument, any subsequence of $\langle u_n \rangle$ will in turn have a subsequence which converges to $z$, we see that $\langle u_n \rangle \to z = N(x)$.

We now show that $N$ is continuous. Let $y \in X$ with $\langle y_n \rangle \subset X$ such that $\langle y_n \rangle \to y$. For each $n$ we have $||y_n - N(y_n)|| \leq ||y_n - N(y)||$, so that $\limsup ||y_n - N(y_n)|| \leq ||y - N(y)||$. Since $\langle N(y_n) \rangle$ is a bounded subset of $D$ we may choose $\langle N(y_{nk}) \rangle$ such that $\langle N(y_{nk}) \rangle \to z \in D$. Then

$$||y - N(y)|| \leq ||y - z|| \leq \liminf ||y_{nk} - N(y_{nk})|| \leq \limsup ||y_{nk} - N(y_{nk})|| \leq ||y - N(y)||.$$  

Consequently, $\lim ||y_{nk} - N(y_{nk})|| = ||y - N(y)||$, and so by the first part of the proof, $\langle N(y_{nk}) \rangle \to N(y)$. This argument shows that any subsequence of $\langle N(y_n) \rangle$ in turn has a subsequence which converges to $N(y)$, so that $\langle N(y_n) \rangle \to N(y)$.

We point out that any uniformly convex Banach space is reflexive and has Property (H).

Following Halpern [8], for a subset $D$ of a Banach space $X$, we define the outward set of a point $x \in D$, denoted by $n_D(x)$, to be

$$n_D(x) = \{y \in X| y \neq x, ||y - x|| \leq ||y - z|| \text{ for all } z \in D\}.$$  

We add in passing that, as was shown in [9], if $I_D(x)$ is the inward set of $x \in X$, i.e., $I_D(x) = \{y \in X|\lambda x + (1 - \lambda)y \in D \text{ for some } \lambda \in [0, 1)\}$, then $n_D(x) \cap \overline{I_D(x)} = \emptyset$.

**Theorem 3.** Let $X$ be a Banach space with $D \subset X$ closed and convex. Suppose that $T: D \to 2^X$ is acyclic and "nowhere normal outward," i.e.,

$$(5) \quad T(x) \cap n_D(x) = \emptyset \quad \text{for } x \in D.$$  

Furthermore, suppose that one of the following conditions holds:

(i) $X$ is strictly convex and $D$ is compact.

(ii) $X$ is reflexive, satisfies condition (H), and $T(D)$ is compact.

Then $T$ has a fixed point.

**Proof.** (i) Since $X$ is strictly convex and $D$ is compact, the mapping $N: X \to D$ defined by the inequality $||N(x) - x|| \leq ||y - x||$ for all $y \in D$, is well defined and continuous [8]. Since $D$ is an acyclic ANR, we use
Theorem A to conclude that $NT$ has a fixed point in $D$. Since $T$ satisfies (5), the fixed point of $NT$ must also be a fixed point of $T$.

(ii) By Lemma 1, the above mapping $N$ is continuous. Since $T(D)$ is relatively compact, $NT$ is condensing, and so $NT$ has a fixed point by Theorem 1. Again, using (1), this fixed point must also be a fixed point of $T$.

**COROLLARY 1.** Theorem 3 holds with the hypothesis "$T$ is nowhere normal outward" replaced by either of the stronger conditions, "$T(x) \subset \overline{I_D(x)}$ for all $x \in D$" or "$T(x) \subset I_D(x)$ for all $x \in D.$"

In case $T(x)$ is star-shaped for each $x \in \partial D$, Theorem 3 has been proved in [8, Theorem 20] under the additional condition that $X$ is equipped with a collection of approximation maps and that the core $(D) \neq \Phi$.

**THEOREM 4.** Let $X$ be a Banach space with $D \subset X$ closed and convex. Suppose $T: D \to 2^X$ is acyclic and $\Phi$-condensing. Furthermore, assume that one of the following conditions holds:

(i) $X$ is strictly convex and $T(x) \subset I_D(x)$ for $x$ in $D$.

(ii) $X$ is a Hilbert space, $T(x) \cap n_D(x) = \Phi$ for each $x \in D$, and $\Phi$ is either the ball-measure or the set-measure of noncompactness defined in §1. Then $T$ has a fixed point.

**Proof.** (i) Let $x_0 \in D$. By Lemma A, we may choose a closed convex set $K$ which contains $x_0$ and such that $\overline{\co(T(D) \cap K)} \cup \{x_0\} = K$. By previously used arguments, $K$ must be compact. Let $x \in K \cap D$ with $z \in T(x)$. Then $z \in I_D(x)$, so that for some $\lambda \in [0, 1)$, $\lambda x + (1 - \lambda)z \in D \cap K$. This shows that $T(x) \subset I_D \cap K(x)$ for each $x \in D \cap K$. Hence, by Corollary 1, $T$ has a fixed point.

(ii) Let $N: X \to D$ be defined by $\|N(x) - x\| = \inf \{ \|z - x\| \text{ for each } x \in D \}$. Now, $X$ is a Hilbert space, and Cheney and Goldstein [2] have shown that $\|N(x) - N(y)\| \leq \|x - y\|$ for each $x$ and $y$ in $X$. It is not hard to show that this implies that for each $A \subset X$, $\Phi(N(A)) \leq \Phi(A)$. Consequently, $NT: D \to 2^D$ is $\Phi$-condensing, and hence, by Theorem 1, $NT$ has a fixed point. Since $T(x) \cap n_D(x) = \Phi$, this fixed point must also be a fixed point of $T$.

Under hypothesis (i) the above result strengthens Theorem 3 in [16] and, in particular, Theorem 24 in [8].

**REMARK 2.** If $X$ is a Hilbert space and $D = B(0, 1)$, then for $x \in \partial D$, $n_D(x) = \{\lambda x | \lambda > 1\}$. Hence for a mapping $T: D \to 2^X$ the Leray-Schauder
condition (4) of Theorem 2 coincides with the requirement that $T(x) \cap n_D(x) = \emptyset$ for all $x \in D$.

References


Received May 16, 1973. Supported in part by the NSF Grant GP-20228.

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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California 90708.

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