Pacific Journal of Mathematics

ON THE PRIME IDEAL DIVISORS OF $(a^n - b^n)$

EDWARD GROSSMAN

Vol. 54, No. 2 June 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 2, 1974

ON THE PRIME IDEAL DIVISORS OF $(a^n - b^n)$

EDWARD H. GROSSMAN

Let a and b denote nonzero elements of the ring of integers O_K of an algebraic number field K, such that ab^{-1} is not a root of unity and the principal ideals (a) and (b) are relatively prime.

DEFINITION 1. A prime ideal $\mathfrak{p}$ is called a *primitive prime divisor* of $(a^n - b^n)$ if $\mathfrak{p} \mid (a^n - b^n)$ and $\mathfrak{p} \nmid (a^k - b^k)$ for $k < n$.

DEFINITION 2. An integer n is called *exceptional for* $\{a, b\}$ if $(a^n - b^n)$ has no primitive prime divisors.

The set of integers exceptional for $\{a, b\}$ is denoted by $E(a, b)$. Using recent deep results of Baker, Schinzel [4] has proved that if $n > n_0(l)$ then $n \notin E(a, b)$, where $l = [K : Q]$ and n_0 is an effectively computable integer. In particular card $E(a, b) \leq n_0$. In this paper, using only elementary methods, upper bounds are obtained for card $\{n \in E(a, b) : n \leq x\}$ which are independent of a and b.

1. Introduction. The prime divisors of the sequence of rational integers $x_n = a^n - b^n$ have been studied by Birkhoff and Vandiver. They showed [**1**, p. 177] that if a and b are positive and relatively prime, then for $n > 6$ there is a prime p which divides $a^n - b^n$ and does not divide $a^k - b^k$ for $k < n$. Postnikova and Schinzel [**3**] have investigated analogues of this result for the ring of integers O_K of an algebraic number field K.

To fix our notation and terminology, a and b will always denote nonzero elements of O_K such that ab^{-1} is not a root of unity, and the principal ideals (a) and (b) are relatively prime. Note then that all the ideals $(a^n - b^n)$ are nonzero.

DEFINITION 1. A prime ideal $\mathfrak{p}$ is called a *primitive prime divisor of* $(a^n - b^n)$ if $\mathfrak{p} \mid (a^n - b^n)$ and $\mathfrak{p} \nmid (a^k - b^k)$ for $k < n$.

DEFINITION 2. An integer n is called *exceptional for* $\{a, b\}$ if $(a^n - b^n)$ has no primitive prime divisors.

The set of integers exceptional for $\{a, b\}$ is denoted by $E(a, b)$. Using a theorem of Gelfond it can be shown [**3**, p. 172] that card $(E(a, b)) < n_0(a, b)$. Recently, using deep methods, Baker [**4**] has improved Gelfond's theorem, and has shown that card $E(a, b) < n_0(l)$, where $l = [K : Q]$. In this paper we obtain by elementary methods upper bounds for card $\{n \in E(a, b) : n \leq x\}$ which are independent of a and b. To state our theorem precisely we introduce the following notation: If $M = 1$ we define $\log_1 x = \log x$ and if $M > 1$ is an integer we define $\log_M x = \log(\log_{M-1} x)$. The main result is

THEOREM 1. *Let K be a finite extension of Q of degree l, a and b elements of O_K such that $(a, b) = O_K$ and a/b is not a root of unity. If $M \geq 1$ is an integer, there is an $x_0 = x_0(M, l)$ such that for $x > x_0$,* card $\{n \in E(a, b) : n \leq x\} \leq \log_M x$.

The proof of Theorem 1 as well as related results will be found in §4. Sections 2 and 3 are preparatory.

2. Preliminary lemmas. Our first lemma provides an algebraic criterion for an integer n to be exceptional for $\{a, b\}$. Let $F_n(x, y)$ denote the nth homogeneous cyclotomic polynomial. We then have

LEMMA 1. *Let $l = [K : Q]$ and suppose $n > 2^l(2^l - 1)$. If the prime ideal $\mathfrak{p} | (a^n - b^n)$ and is not a primitive prime divisor then* $\operatorname{ord}_{\mathfrak{p}}(F_n(a, b)) \leq \operatorname{ord}_{\mathfrak{p}}(n)$. *In particular if $n \in E(a, b)$ then $(F_n(a, b))|(n)$.*

Proof. See [3, p. 172]. We note without proof that the result also holds provided $n > 2l(2^l - 1)$.

From Lemma 1 if n is sufficiently large and $n \in E(a, b)$, then the ideal norm of $F_n(a, b)$ satisfies the inequality $N(F_n(a, b)) \leq n^l$. We will show that this can occur only if some conjugate of a/b is "very close" to a primitive nth root of unity; moreover the set of integers n for which this holds must be spaced very far apart.

We consider K as imbedded in some fixed manner in the field of complex numbers. ζ_n will denote the nth root of unity $e^{2\pi i/n}$. If a and b are any complex numbers such that a/b is not a root of unity, we let $\zeta_n^*(a, b)$ (or simply ζ_n^* if a and b are understood) denote an nth root of unity closest to a/b. For some n and complex numbers a and b, ζ_n^* is a primitive nth root of unity, for others it is not. Moreover, if there is no unique nth root of unity closest to a/b, ζ_n^* will denote a fixed nearest one. Thus

$$|a - b\zeta_n^*| = \min\{|a - b\zeta_n^\nu| : \nu = 1, \dots, n\}.$$

LEMMA 2. *Let $m > n$ and suppose that ζ_n^* and ζ_m^* are primitive nth and mth roots of unity satisfying*

$$|a - b\zeta_n^*| < \max(|a|, |b|) \exp(-n^{1/2})/n$$

and

$$|a - b\zeta_m^*| < \max(|a|, |b|) \exp(-m^{1/2})/m,$$

then $m \geq 2 \exp(n^{1/2})$.

Proof. If $\max(|a|, |b|) = |b|$ then we have $4/mn \leq |\zeta_n^* - \zeta_m^*| \leq \exp(-n^{1/2})/n + \exp(-m^{1/2})/m \leq 2\exp(-n^{1/2})/n$ and so $m \geq 2\exp(n^{1/2})$. If $\max(|a|, |b|) = |a|$ then a similar estimate holds for $|\bar{\zeta}_n^* - \bar{\zeta}_m^*|$.

LEMMA 3. *Let A be a subset of the positive integers such that whenever $n, m \in A$ and $m > n$, then $m > \exp(n^{1/2})$. If M is any positive integer there is an x_M depending only on M such that for $x \geq x_M$,* $\text{card}\{n \in A : n \leq x\} \leq \log_M x$.

Proof. Let $k = \text{card}\{n \in A : n \leq x\}$. If $n_1 < n_2 < \cdots < n_k \leq x$ are the k elements of A less than x, then for any integer $j < k$

$$n_{k-j} \leq (3 \log_j x)^2 \tag{1}$$

if $\log_j x > 2 \log 3$.

We first assume $k > M + 1$. Then taking $j = M + 1$ in (1) and x large enough so that $\log_{M+1} x > 2 \log 3$ we have that $n_{k-M-1} \leq (3 \log_{M+1} x)^2$; in particular $k - M - 1 \leq (3 \log_{M+1} x)^2$ and so $k < (M + 1) + (3 \log_{M+1} x)^2$. Since this inequality also holds when $k \leq M + 1$ and $(M + 1) + (3 \log_{M+1} x)^2 = o(\log_M x)$ the lemma is proven.

Denoting by $E'(a, b)$ the set of n such that ζ_n^* is a primitive nth root of unity and such that $|a - b\zeta_n^*| < \max(|a|, |b|) \exp(-n^{1/2})/n$, Theorem 1 will follow from Lemma 3 if it is shown that if n is sufficiently large and is not in $\cup E'(a^{(v)}, b^{(v)})$, where $a^{(v)}$ and $b^{(v)}$ denote the conjugates of a and b, then $n \notin E(a, b)$.

To perform the analysis we first break up $Z^+ - E'(a, b)$ into two disjoint sets:

$$S_1 = \{n : |a - b\zeta_n^*| > \max(|a|, |b|) \exp(-n^{1/2})/n\}$$

$$S_2 = \{n : |a - b\zeta_n^*| \leq \max(|a|, |b|) \exp(-n^{1/2})/n, \text{ and } \zeta_n^* \text{ not a primitive } n\text{th root of unity}\}.$$

Before continuing we note that if n is an integer for which there is no unique closest nth root of unity to a/b then $n \in S_1$.

It will be convenient to have the following notation. For any ζ_n^* let k

be the divisor of n such that ζ_n^* is a primitive kth root of unity. If $d|n$ define

(2) $$[a^d - b^d] = \begin{cases} a^d - b^d & \text{if } k \nmid d \\ \dfrac{a^d - b^d}{a - b\zeta_n^*} & \text{if } k | d. \end{cases}$$

In terms of this notation we have the following easy but basic lemma.

LEMMA 4. *If ζ_n^* is a primitive kth root of unity and $k < n$ then*

$$F_n(a,b) = \prod_{d|n} [a^d - b^d]^{\mu(n/d)}$$

Proof.

$$\prod_{d|n} [a^d - b^d]^{\mu(n/d)} = \prod_{\substack{d|n \\ k\nmid d}} (a^d - b^d)^{\mu(n/d)} \prod_{\substack{d|n \\ k|d}} \left(\frac{a^d - b^d}{a - b\zeta_n^*}\right)^{\mu(n/d)}$$

$$= F_n(a,b)(a - b\zeta_n^*)^{-L}, \quad \text{where}$$

$L = \sum_{d|n,\, k|d} \mu(n/d)$. Setting $n' = n/k > 1$, $d' = d/k$ we have $L = \sum_{d'|n'} \mu(n'/d') = 0$.

3. Bounds for $|a^d - b^d|$ and $|[a^d - b^d]|$.

The representation of $F_n(a, b)$ given in Lemma 4 as well as the usual product formula

$$F_n(a,b) = \prod_{d|n} (a^d - b^d)^{\mu(n/d)}$$

will be used to provide lower bounds for $N(F_n(a, b))$. In this section we derive the necessary estimates for $|a^d - b^d|$ and $|[a^d - b^d]|$.

LEMMA 5. *For all $d \geq 1$*

(3) $$|a^d - b^d| \leq 2d \max(|a|, |b|)^d$$

(4) $$|[a^d - b^d]| \leq \begin{cases} 2d\max(|a|,|b|)^d & \text{if } k = \text{order } \zeta_n^* \nmid d \\ 2d\max(|a|,|b|)^{d-1} & \text{if } k = \text{order } \zeta_n^* | d \end{cases}$$

Proof. Inequality (3) and (4) in the case $k \nmid d$ follow from $|a^d - b^d| \leq 2\max(|a|, |b|)^d$. If $k|d$ then from (2)

$$|[a^d - b^d]| = \left|\frac{a^d - b^d}{a - b\zeta_n^*}\right| = |a^{d-1} + a^{d-2}(b\zeta_n^*) + \dots + (b\zeta_n^*)^{d-1}|$$
$$\leq d\max(|a|,|b|)^{d-1}.$$

Lower Bound Estimates: We first prove a preliminary lemma.

LEMMA 6. *Let z be a complex number such that $|z| \leq 1$ and $|z - \zeta_n^*(z, 1)| > |\zeta_n - 1| = \lambda_n$. Then $n > 6$ and $1 - |z| > (\sqrt{3}/2)\lambda_n$.*

Proof. Recall that $\zeta_n^*(z, 1)$ is a closest nth root of unity to z. First we show that if $z = re^{i\theta}$, where $1 \geq r \geq \max(0, \cos \pi/n - \sqrt{3}\sin \pi/n)$ and $|\theta| \leq \pi/n$, then $|z - 1| \leq \lambda_n$. We have in fact

$$|z - 1|^2 - \lambda_n^2 \leq (r - (\cos\pi/n - \sqrt{3}\sin\pi/n))(r - (\cos\pi/n + \sqrt{3}\sin\pi/n)) \leq 0.$$

By rotation it now follows that if $1 \geq |z| \geq \max(0, \cos \pi/n - \sqrt{3}\sin \pi/n)$ there is an nth root of unity ζ_n^ν such that $|z - \zeta_n^\nu| \leq \lambda_n$. Finally if $n \leq 6$ we have $\cos \pi/n - \sqrt{3}\sin \pi/n \leq 0$ and so the condition $|z| \leq 1$, $|z - \zeta_n^*| > \lambda_n$ is impossible. If $n > 6$ then $1 - |z| \geq 1 - \cos \pi/n + \sqrt{3}\sin \pi/n > (\sqrt{3}/2)\lambda_n$.

LEMMA 7. *If $n \in S_1$ and $d|n$ then*

(5) $$|a^d - b^d| \geq \max(|a|, |b|)^d \exp(-n^{1/2})/n \qquad \textit{or}$$

(6) $$|a^d - b^d| \geq \max(|a|,|b|)^d\Big(\prod_{\zeta_d^\nu \neq \zeta_d^*(z,1)} |z - \zeta_d^\nu|\Big)\exp(-n^{1/2})/n,$$

in which case $d > 1$ and $|z| \leq 1$ satisfies $|z - \zeta_d^*(z, 1)| \leq \lambda_d$.

Proof. Since $n \in S_1$ we can write

(7) $$|a^d - b^d| = \max(|a|, |b|)^d|z^d - 1|$$

where $z = a/b$ or $z = b/a$ satisfies $|z| \leq 1$ and

(8) $$|z - \zeta_n^*(z, 1)| > \exp(-n^{1/2})/n.$$

If $n \geq 1$ and $d = 1$ then (5) is immediate. If $n > 1$ and $d > 1$ we distinguish two cases accordingly as $|z - \zeta_n^*| > \lambda_n$ or $|z - \zeta_n^*| \leq \lambda_n$. In the former case Lemma 6 gives $1 - |z| > (\sqrt{3}/2)\lambda_n > 2\sqrt{3}/n$; hence $|z^d - 1| \geq 1 - |z| > 2\sqrt{3}/n > \exp(-n^{1/2})/n$, which when combined with (7) gives (5).

If $|z - \zeta_n^*| \leq \lambda_n$ then we must also have $|z - \zeta_d^*| \leq \lambda_d$. Otherwise Lemma 6 gives $(\sqrt{3}/2)\lambda_d < 1 - |z| \leq |z - \zeta_n^*| \leq \lambda_n$ which is impossible since $n/d \geq 2$. Observing now that (6) follows immediately from (7) and (8), the proof is complete.

LEMMA 8. *If $n \in S_2$ and $d|n$, then if* order $\zeta_n^* = k \nmid d$

(9) $$|[a^d - b^d]| \geq \max(|a|, |b|)^d \exp(-n^{1/2})/n \qquad \textit{or}$$

(10) $$|[a^d - b^d]| \geq \max(|a|,|b|)^d \Big(\prod_{\zeta_d^\nu \neq \zeta_d^*(z,1)} |z - \zeta_d^\nu| \Big) \exp(-n^{1/2})/n$$

in which case $d > 1$ and $|z| \leq 1$ satisfies $|z - \zeta_d^*| \leq \lambda_d$. If order $\zeta_n^* = k|d$

(11) $$|[a^d - b^d]| \geq \max(|a|,|b|)^{d-1} \prod_{\zeta_d^\nu \neq \zeta_d^*} |z - \zeta_d^\nu|,$$

where for $d = 1$ the product on the right side of (11) is one and if $d > 1$, $|z| \leq 1$ satisfies $|z - \zeta_d^*| \leq \lambda_d$.

Proof. Since $n \in S_2$, with $z = a/b$ or b/a we have $|z| \leq 1$,

(12) $$|z - \zeta_n^*| \leq \exp(-n^{1/2})/n$$

and order $\zeta_n^* = k < n$.

If $k \nmid d$ we have $n > 1$ and since ζ_n^* is not a dth root of unity (12) implies

$$|z - \zeta_d^*| \geq |\zeta_d^* - \zeta_n^*| - |z - \zeta_n^*| > \exp(-n^{1/2})/n.$$

We can now argue as in the previous lemma.

If $k|d$ then we have

(13) $$|[a^d - b^d]| = \max(|a|,|b|)^{d-1} \left| \frac{z^d - 1}{z - \zeta_n^*(z,1)} \right|$$

where $|z| \leq 1$ satisfies (12). If $d = 1$ then since $k|d$, $\zeta_n^* = 1$ and (13) is precisely (11). For $d > 1$, (11) and the condition $|z - \zeta_d^{\nu}| \leq \lambda_d$ follow from (12) and (13) in view of $\zeta_n^* = \zeta_d^*$.

In order to complete the lower bound estimates we must obtain lower bounds for $\prod_{\zeta_d^{\nu} \neq \zeta_d^*} |z - \zeta_d^{\nu}|$, where $d > 1$ and $|z| \leq 1$ satisfies $|z - \zeta_d^*| \leq \lambda_d$. We first prove

LEMMA 9. *Let $d > 1$ be an integer and r a real number satisfying $0 \leq r \leq 1$ and $|r - 1| \leq \lambda_d$, then*

$$\prod_{\nu=1}^{d-1} |r - \zeta_d^{\nu}| \geq d^{-3\tau+1} \tag{14}$$

where $\tau = \tau_d = [\sqrt{\pi d/2}] + 1$.

Proof. Since r is real we have

$$\prod_{\nu=1}^{d-1} |r - \zeta_d^{\nu}| \geq (1/2) \prod_{\nu=1}^{[d/2]} |r - \zeta_d^{\nu}|^2. \tag{15}$$

We give a lower bound for the latter product. From $|r - 1| \leq \lambda_d$ we obtain

$$|r - \zeta_d^{\nu}| \geq |1 - \zeta_d^{\nu}| - |1 - r| \geq |1 - \zeta_d^{\nu}| - \lambda_d. \tag{16}$$

Let $\tau = \tau_d = [\sqrt{\pi d/2}] + 1$ and suppose first that $[d/2] > \tau_d$ and ν satisfies $[d/2] \geq \nu > \tau_d$. Then

$$\begin{aligned} |1 - \zeta_d^{\nu}| - |\zeta_d^{\nu} - \zeta_d^{\tau}| &= 4 \sin(\pi\tau/2d) \cos \pi(\nu/d - \tau/2d) \\ &\geq (4\tau/d)(d - 2\nu + \tau)/d \geq 4\tau^2/d^2 \geq 2\pi/d \geq \lambda_d. \end{aligned} \tag{17}$$

Thus from (16), $|r - \zeta_d^{\nu}| \geq |\zeta_d^{\nu-\tau} - 1|$ and so

$$\begin{aligned} \prod_{\nu=1}^{[d/2]} |r - \zeta_d^{\nu}| &\geq \prod_{\nu=1}^{\tau} |r - \zeta_d^{\nu}| \prod_{\nu=\tau+1}^{[d/2]} |1 - \zeta_d^{\nu-\tau}| \\ &= \frac{\prod_{\nu=1}^{\tau} |r - \zeta_d^{\nu}| \prod_{\nu=1}^{[d/2]} |1 - \zeta_d^{\nu}|}{\prod_{\nu=[d/2]-\tau+1}^{[d/2]} |1 - \zeta_d^{\nu}|}. \end{aligned} \tag{18}$$

Since $r \geq 0$ we have

$$|r - \zeta_d^{\nu}| \geq |r - \zeta_d| > 2/d \tag{19}$$

if $d \geq 4$ and the same inequality ($|r - \zeta_d^{\nu}| \geq 2/d$) holds for $d < 4$. Observing finally that $|1 - \zeta_d^{\nu}| \leq 2$ and

$$\prod_{\nu=1}^{[d/2]} |1 - \zeta_d^{\nu}|^2 \geq \prod_{\nu=1}^{d-1} |1 - \zeta_d^{\nu}| = d$$

we obtain (14) from (15) and (18).

Now if $\tau_d \geq [d/2]$ we have from (19)

$$\prod_{\nu=1}^{d-1} |r - \zeta_d^{\nu}| \geq (2/d)^{d-1} \geq (2/d)^{2[d/2]} \geq d^{-2\tau}2^{2\tau} \geq d^{-3\tau+1}$$

and so (14) is also proven in this case.

LEMMA 10. *If $d > 1$ and $|z| \leq 1$ satisfies $|z - \zeta_d^*| \leq \lambda_d$ then*

$$\prod_{\zeta_d^{\nu} \neq \zeta_d^*} |z - \zeta_d^{\nu}| \geq d^{-3\tau} \tag{20}$$

where $\tau = \tau_d = [\sqrt{\pi d/2}] + 1$.

Proof. We may assume that $\zeta_d^* = 1$ and $z = re^{i\theta}$, where $0 \leq \theta \leq \pi/d$; thus we must prove the lower bound (20) for $\prod_{\nu=1}^{d-1} |z - \zeta_d^{\nu}|$. Let $z' = re^{2\pi i/d}$. Then if $1 \leq \nu \leq [d/2]$, $|z - \zeta_d^{\nu}| > |z' - \zeta_d^{\nu}|$ and if $[d/2] < \nu \leq d - 1$, $|z - \zeta_d^{\nu}| \geq |r - \zeta_d^{\nu}|$. Combining these results and using $|z - \zeta_d| \geq \lambda_d/2 \geq 2/d$ we obtain

$$\begin{aligned} \prod_{\nu=1}^{d-1} |z - \zeta_d^{\nu}| &\geq \frac{|z - \zeta_d|}{|r - \zeta_d^{[d/2]}|} \prod_{\nu=1}^{d-1} |r - \zeta_d^{\nu}| \\ &\geq d^{-1} \prod_{\nu=1}^{d-1} |r - \zeta_d^{\nu}|. \end{aligned} \tag{21}$$

Since $|r - 1| \leq |z - 1| \leq \lambda_d$, (20) follows from Lemma 9 and (21).

From Lemmas 7, 8 and 10 we arrive at our final lower bound estimates.

LEMMA 11. *If $n \in S_1$ and $d|n$ then*

(22) $$|a^d - b^d| \geq \max(|a|,|b|)^d d^{-3\tau} \exp(-n^{1/2})/n$$

where $\tau = \tau_d = [\sqrt{\pi d/2}] + 1$.
If $n \in S_2$, $d|n$, *then if* order $\zeta_n^*(a, b) = k \nmid d$, (22) *holds for* $|[a^d - b^d]|$. *If* $k|d$ *we have*

(23) $$|[a^d - b^d]| \geq \max(|a|, |b|)^{d-1} d^{-3\tau}.$$

4. Main theorem and related results. The proof of Theorem 1 will follow easily from the following lemma.

LEMMA 12. *There is an integer* n'_0 *such that if* $n > n'_0$ *and* $n \in S_1 \cup S_2$ *then*

(24) $$\log|F_n(a, b)| \geq \varphi(n)\max(\log|a|, \log|b|) - 2^{\nu(n)} n^{5/8}.$$

Proof. If $n \in S_1$ we use Lemmas 5 and 11 and the formula $F_n(a, b) = \prod_{d|n} (a^d - b^d)^{\mu(n/d)}$ to obtain (24).

If $n \in S_2$ then (24) follows from Lemma 4 and the estimates of Lemmas 5 and 11.

Proof of Theorem 1. Recall that $S_1 \cup S_2$ is the complement of the set $E'(a, b) = \{n \in Z^+ : |a - b\zeta_n^*| \leq \max(|a|, |b|) \exp(-n^{1/2})/n$ and ζ_n^* a primitive nth root of unity$\}$. Let $E' = \cup_{\nu=1}^{l} E'(a^{(\nu)}, b^{(\nu)})$, where $l = [K : Q]$ and $a^{(\nu)}$, $b^{(\nu)}$ denote the conjugates of a and b. If $n \notin E'$ then the lower bound (24) is valid for all ν provided $n > n'_0$. Thus

(25) $$\log|N(F_n(a, b))| = \sum_{\nu=1}^{l} \log|F_n(a^{(\nu)}, b^{(\nu)})| \geq A\varphi(n) - l2^{\nu(n)} n^{5/8} \qquad \text{where}$$

(26) $$A = \sum_{\nu=1}^{l} \max(\log|a^{(\nu)}|, \log|b^{(\nu)}|) = \log|N(b)| + \sum_{\nu=1}^{l} \max\left(\log\left|\frac{a^{(\nu)}}{b^{(\nu)}}\right|, 0\right).$$

If $|N(b)| = 1$, then a/b is in O_K and there is a constant c_K, depending only on $[K : Q]$, such that $|a^{(\nu)}/b^{(\nu)}| > c_K$ for some ν. Thus $A \geq \min(\log 2,$

$\log c_K) = C'_K$. Using the well-known [2, p. 114] estimates $\varphi(n) > c_1 n/\log\log n$ and $2^{\nu(n)} < c_2(\epsilon)n^\epsilon$ (with $\epsilon = 1/8$), (25) gives

$$\log|N(F_n(a,b))| \geq \frac{C_K n}{\log\log n} - c_2 l n^{3/4} > l \log n \text{ for}$$

$n > n_0(l)$ and so from Lemma 1, $n \notin E(a, b)$.

Thus $E(a, b) \subset E' \cup \{n \leq n_0\}$ and the density estimate for $E(a, b)$ follows in view of Lemmas 2 and 3.

We can extract additional quantitative information from the above proof. Let us write $(a^n - b^n) = \mathfrak{A}\mathfrak{B}$ where $\mathfrak{A} + \mathfrak{B} = O_K$ and $\mathfrak{P}|\mathfrak{A}$ if and only if $\mathfrak{P}$ is a primitive prime divisor of $(a^n - b^n)$. We call $\mathfrak{A}$ the *primitive part* of $(a^n - b^n)$ and denote it by $P_n(a, b)$. Then we have

LEMMA 13. *If* $n > n_0(K)$ *and* $n \notin E'$ *then*

$$\log|N(P_n(a,b))| = A\varphi(n) + O(n^{3/4}) \tag{27}$$

where A is defined by (26) *and the constant implied by O depends only on K.*

Proof. Lemma 1 implies that for $n > 2^l(2^l - 1)$

$$\log|N(F_n(a,b))| - l\log n \leq \log|N(P_n(a,b))| \leq \log|N(F_n(a,b))|.$$

If $n \in S_1 \cup S_2$ the left side can be bounded from below using (24). Moreover, as in Lemma 12 one shows that for n sufficiently large, $n \in S_1 \cup S_2$

$$\log|F_n(a,b)| \leq \varphi(n)\max(\log|a|, \log|b|) + 2^{\nu(n)} n^{5/8}.$$

Using these estimates we immediately obtain (27).

Lemma 13 and the density estimate for E' enable us to derive both a normal order and average order for $\log|N(P_n(a, b))|$. The proofs are straightforward and are omitted.

THEOREM 2. $\log|N(P_n(a, b))|$ *has* $\varphi(n)A$ *as a normal order, i.e. for any* $\epsilon > 0$ *if*

$$T(\varepsilon,x) = \{n \leq x : |\log|N(P_n(a,b))| - \varphi(n)A| < \varepsilon\varphi(n)A\},$$

then card $T(\epsilon, x)/x \to 1$ *as* $x \to \infty$.

THEOREM 3. $$\sum_{n\leq x}\log|N(P_n(a,b))| = \frac{3A}{\pi^2}x^2 + O(Ax^{7/4})$$

where the constant implied by $O(\ )$ *depends only on K.*

REFERENCES

1. G. D. Birkhoff and H. S. Vandiver, *On the integral divisors of $a^n - b^n$*, Ann. of Math., **5** (1902–03), 173–180.
2. W. J. LeVeque, *Topics in Number Theory,* vol. 1, Addison Wesley, Reading, Mass., 1956.
3. L. P. Postnikova and A. Schinzel, *Primitive divisors of the expression $a^n - b^n$ in algebraic number fields,* Math. U.S.S.R.-Sbornik, **4** (1968), No. 2, 153–159.
4 A. Schinzel, *Primitive divisors of the expression $A^n - B^n$ in algebraic number fields,* J. Reine Angew. Math., **268/269** (1974), 27-33.

Received July 23, 1973.

CITY COLLEGE OF CUNY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the five editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California 90708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 54, No. 2 June, 1974