Pacific Journal of Mathematics

RIGHT SIMPLE CONGRUENCES ON A SEMIGROUP

FRANCIS MASAT

Vol. 54, No. 2 June 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 54, No. 2, 1974

RIGHT SIMPLE CONGRUENCES ON A SEMIGROUP

FRANCIS E. MASAT

The purpose of this paper is to investigate necessary and sufficient conditions on an algebraic semigroup in order that it have non-trivial right simple homomorphic images. Relative to this, the relation between the structure of S and the structure of its right simple homomorphs is characterized.

The main questions considered are:

(1) What characterizes a right simple (right group) congruence on a semigroup?

(2) Can the conditions found for question (1) be made minimal in order that a maximum right simple homomorph occurs?

In §2, question (1) is answered in terms of right neat subsets of a semigroup. The concepts of minimal right neat subsets and minimal right ideals are used in §3 to obtain some sufficient conditions in order that question (2) may be answered in the affirmative. Relative to the stated sufficient conditions, a structure theorem is given. Also in §3, left cancellative congruences are used to generate right group homomorphs of a semigroup.

The right-left duals of all results established will be taken for granted without further comment. For basic concepts, definitions, and terminology, the reader is referred to Clifford and Preston (1). Also, $S\setminus I$ denotes set difference, $|S|$ denotes the cardinality of the set S.

2. Right simple congruences. A semigroup S is called *right* [*left*] *simple* if it contains no proper right [left] ideal. A group, then, is just a semigroup that is both left and right simple, [**1**, p. 39]. A semigroup S is called a *right group* if it is right simple and left cancellative. This means that for any a and b in S there exists a unique x in S such that $ax = b$.

A semigroup S is said to be *regular* if for each a in S, a is also in aSa. When S is regular, the set of idempotents of S is nonempty and will be denoted by E_S. If there is no danger of ambiguity, E will be used instead of E_S.

The following results are stated for later application.

LEMMA 2.1. [**1**, Lemma 1.26, p. 37]. *Every idempotent element of a right simple semigroup S is a left identity element of S.*

LEMMA 2.2 [1]. *The following assertions concerning a semigroup S are equivalent:*

(1) *S is a right group.*

(2) *S is right simple, and contains an idempotent.*

(3) *S is a direct product $G \times E$ of a group G and a right zero semigroup E.*

(4) *S is a union of isomorphic disjoint groups such that the set of identity elements of the groups is a right zero subsemigroup of S.*

(5) *S is regular and left cancellative.*

REMARK 2.3. When a congruence ρ is such that S/ρ is the maximal homomorphic image of S of type C, as in [2, p. 275] and [2, Theorem 11. 25(A), p. 276], then ρ will be called the *minimum congruence* on S of type C and S/ρ will be called the *maximum homomorphic image* of S of type C.

In other words, S/ρ is the maximum C-image if and only if ρ is of type C and $\rho \subseteq \sigma$ for each congruence σ which is of type C. Moreover, the phrases "right group congruence" and "group congruence" will be denoted by RGC and GC respectively. When such a congruence is minimum, it will be denoted by $MRGC$ and MGC respectively.

The preceding proposition, (2.3), would seem to indicate that the minimum right simple [right group, group] congruence on a semigroup S could be found by considering the intersection of all the right simple [right group, group] congruences on S. However, the intersection of the right simple congruences on a semigroup S need not be a right simple congruence on S.

EXAMPLE 2.4. Let S be the additive semigroup of positive integers. The group images of S are exactly all the finite multiplicative cyclic groups and there is no maximal group among these. The intersection of the induced congruences on S is the identity congruence ι, but ι is not a right simple congruence since S is not right simple. Hence ι is neither a right group congruence nor a group congruence.

PROPOSITION. *Every homomorphic image of a right simple semigroup is right simple.*

Alternately, a semigroup is right simple if and only if its minimum right simple congruence is the identity congruence.

A subset X of a semigroup S is said to be a *right* [*left*] *neat* subset of S if for each a in S there exists s in S such that $as \in X$ [$sa \in X$]. Right [left] neat subsets may also be viewed as those subsets of S which meet (have non-empty intersection with) each right [left] ideal of the form $aS[Sa]$ where a is in S. A set U is a *cross-section* of a collection $\{X_a : a \in A\}$ of sets if and only if $|U \cap X_a| = 1$ for all a in A.

If a semigroup S has proper right ideals, define the subset U of S to be any cross-section of the set of all (proper) right ideals of S. The set U meets each principal right ideal of S and hence U is a proper right neat subset of S. The converse is false since each singleton subset of a right simple semigroup is right neat.

The next theorem shows the close relationship between the right neat subsets of a semigroup S and the right simple congruences on S.

THEOREM. *A homomorph $S\theta$ of a semigroup S is right simple if and only if each congruence class of S modulo $\theta \circ \theta^{-1}$ is right neat.*

The following theorem characterizes those right simple congruences on a semigroup which are right group congruences.

THEOREM 2.5. *A necessary and sufficient condition that a right simple congruence on a semigroup S be a right group congruence is that some congruence class contain a subsemigroup of S.*

The next theorem generalizes a result due to Fontaine [4, Thm. 12, p. 784].

THEOREM 2.6. *Suppose a semigroup S has a right neat subset K. Let ρ be a congruence on S such that for each b in S and for each k in K, there exists an x in S such that $(kx)\rho b$. Then S/ρ is a right simple semigroup.*

Proof. For a in S, there exists y in S such that $ay \in K$. Hence *for b in S there exists x in S such that $(ayx)\rho b$. Thus $a\rho \cdot (yx)\rho = b\rho$, so S/ρ is a right simple semigroup.*

COROLLARY [4, Thm. 12, p. 784]. *Let ρ be a congruence on a semigroup S containing a right neat subset K. If $(kb)\rho b$ for each b in S and for each k in K, then ρ is a right group congruence on S.*

The next result shows what happens to the right ideals of a semigroup under a right simple homomorphism.

THEOREM 2.7. *If θ is a homomorphism of a semigroup S onto a right simple semigroup S', then θ maps every right ideal of S onto S'.*

Proof. If I is a right ideal of S, then $(I)\theta$ is a right ideal of S', and therefore $(I)\theta = S'$.

COROLLARY 2.8. *If a semigroup S has a left zero, then the minimum right simple congruence on S is the universal congruence ω.*

Semigroups with a right zero are considered in §3. A second corollary is the following theorem of Lefebrve [**6**, Lemma 1, p. 2277].

COROLLARY 2.9. *If θ is a homomorphism of a semigroup S onto a group S^*, then θ maps every ideal of S onto S^*.*

The following result is a partial converse for Theorem 2.7.

THEOREM. *If θ is a homomorphism of a right ideal R of a semigroup S onto a right group S^*, and if R has a right identity, then θ can be extended to a homomorphism of S onto S^*.*

Proof. Define a mapping g from S to S^* by $(s)g = (is)\theta$, where i is the right identity of R. It is evident that g is well-defined, maps S homomorphically onto S^*, and equals θ when restricted to R.

3. Minimum right simple congruences. If X is a right [left] neat subset of a semigroup S and if x is in X, then the set of all s in S such that $xs \in X$ [$sx \in X$] is denoted by $x^{[-1]}X$ [$Xx^{[-1]}$]. In the following development the set $xS \cap X$[$Sx \cap X$] will be denoted by $x(x^{[-1]}X)$ [$(Xx^{[-1]})x$].

The following theorem of Dubreil [**3**, Thm. 1, p. 34] characterizes those right neat subsets of a semigroup which are minimal.

THEOREM 3.1. *In order that a right neat subset K of a semigroup S be minimal, it is necessary and sufficient that for each k in K, $k(k^{[-1]}K) = \{k\}$.*

Additionally we need the following results from Lefebvre concerning semigroups possessing MRN (minimal right neat) subsets and MRI's (minimal right ideals).

THEOREM 3.2. *Every minimal right neat subset K of S has the following properties:*

(i) *For each $k \in K$, there exists $x \in S$ such that $kx = k$.*

(ii) *For each $a \in S$ and for each $k \in K$, there exists $x \in S$ such that $kax = k$.*

(iii) *For each* $k_1, k_2 \in K$ *and for each* $x_1, x_2 \in S$, *if* $k_1 x_1 = k_2 x_2$ *then* $k_1 = k_2$.

(iv) *For each* $a \in S$, *Ka is a minimal right neat subset of S.*

Proof. [5, Properties 1, 2, 3 and Theorem 2, p. 394].

THEOREM 3.3. *Let S be a semigroup with minimal right neat subsets. Then:*

(i) *Every right neat subset of S contains a minimal right neat subset of S.*

(ii) *S contains a minimal right ideal.*

(iii) *Any two minimal right neat subsets of S correspond bijectively.*

Proof. [5, Thm. 4, Cor. 2, and Thm. 3, pp. 394–395].

THEOREM 3.4. *If a semigroup S contains a minimal right ideal, it contains a minimal right neat subset. Moreover, the union of the minimal right neat subsets of S coincides with the union of the minimal right ideals of S.*

Proof. [5, Theorem 7, p. 395]. It should be noted that the MRN subsets of S are formed by taking cross-sections of the MRI's of S. Denote the union of the *MRN* subsets of S by N, and the union of the MRI's of S by R. That $R = N$ follows from [5, Corollary 3, p. 395].

The next result shows the cardinality and structure relationship between minimal right ideals of S and minimal right neat subsets of S.

COROLLARY 3.5. *Let S have a minimal right ideal. Then:*

(i) *Every minimal right neat subset of S is a cross-section of the set of minimal right ideals of S.*

(ii) *The number of elements in any minimal right neat subset of S equals the number of minimal right ideals of S.*

(iii) *The following are equivalent:*

(A) *The minimal right neat subsets of S are mutually disjoint.*

(B) *The minimal right neat subsets of S are singletons.*

(C) *S contains exactly one minimal right ideal.*

Proof. Denote the set of *MRI*'s of S by $\{R_a : a \in A\}$ for an index set

A, and denote the set of *MRN* subsets of S by $\{K_i : i \in I\}$ for an index set I.

(i) By Theorem 3.4, any cross-section of $\{R_a : a \in A\}$ is a *MRN* subset of S. Conversely, for i in I, $K_i \cap R_a \neq \square$ for each a in A. If a in A and $h, k \in K_i \cap R_a$, then $kR_a = hR_a = R_a$. It follows from Theorem 3.2 (iii) that $k = h$.

(ii) By Theorem 3.3 (iii), the number of elements in any K_i is independent of i in I, and hence, by (i), is equal to $|A|$.

(iii) Proceeding indirectly, assume some (and hence every) K_i contains more than one element, and consider the set K defined by $K = (K_i \cap R_a) \cup (K_j \setminus R_a)$, for a fixed i, j in I, a in A. By part (i), K is a MRN subset of S, but $K \cap K_i \neq \square$. Thus (A) implies (B).

If the K_i's are all singletons, then by (ii), (B) implies (C).

If $|A| = 1$, then by (i), a cross-section of the MRI is precisely a MRN singleton subset and therefore (C) implies (A).

The following theorem relates the MRN subsets of a semigroup S to the right simple congruences on S.

THEOREM 3.6. *Let S be a semigroup with minimal right ideals. Then:*

(i) *S is a disjoint union of right neat subsets.*

(ii) *The decomposition of S in* (i) *induces a right congruence ρ on S.*

(iii) *ρ can be characterized as follows. For some minimal right neat subset K of S and for a fixed x in K,*

$$\rho = \{(s, t) \in S \times S : xs = xt\}.$$

(Accordingly, we write $\rho = \rho_x$.)

(iv) *ρ_x is right simple in that for each a, b in S there exists u in S such that $(au)\rho_x = b\rho_x$.*

Proof. (i) Let x be fixed in some MRN subset K. For each s in S, xs in Ks implies that $s \in x^{[-1]}Ks$. For convenience, denote $x^{[-1]}Ks$ by $[s]$. Thus

$$S = \cup \{[s] : s \in S\}. \tag{3.7}$$

If $[s] \cap [r] \neq \square$ for some r and s in S, then there exists b in S such that $xb \in Ks \cap Kr$. Hence there exists h, k in K such that $xb = hs$ and $xb = kr$. But then by Theorem 3.2 (iii), $h = x = k$ and so $xr = xb = xs$. Hence if c

$\in [s]$ then $xc = xs = xr$, so $c \in [r]$. Therefore $[s] \subseteq [r]$, and similarly $[r] \subseteq [s]$. Hence $[s] = [r]$ and the decomposition (3.7) is a partition of S.

Let Y denote a set of representatives for $\{[s] : s \in S\}$. Thus

$$S = \cup \{[y] : y \in Y\}, \tag{3.8}$$

and this union is disjoint. For a in S and y in Y, the set Ky is right neat and so there exists z in S such that $(xa)z \in Ky$, i.e., $az \in [y]$ and therefore $[y]$ is right neat.

(ii) Let ρ_x denote the equivalence relation induced on S by the partition (3.8). If $(s, t) \in \rho_x$ and $c \in S$, then there exists y in Y such that $s, t \in [y]$ and hence $xs = xt = xy$. Thus $xsc = xtc = xyc$, so $sc, tc \in [yc]$. Assuming $[yc]$ meets $[y']$ for $y' \in Y$, implies by part (i) that ρ_x is right compatible.

(iii) If $(s, t) \in \rho_x$ then $xs = xt = xy$ for some y in Y. Conversely, suppose $xs = xt$. For some y in Y, $s \in [y]$ and thus $xs = xt = xy$, i.e., $(s, t) \in \rho_x$.

(iv) Since $x \in K$, a MRN subset of S, x lies in a MRI R of S; thus $xS = R$. For any a, b in S, there exists u in S such that $xau = xb$ since $xS = xaS = R$. Hence by the definition of ρ_x, $au\rho_x b$.

The following definitions will be used frequently. Let S be a semigroup and let X be a subset of S.

(3.9) For x in S, let $\rho_x = \{(a, b) \in S \times S : xa = xb\}$;

(3.10) $\rho_X = \{(a, b) \in S \times S : xa = xb \text{ for all } x \text{ in } X\}$.

REMARK 3.11. Note that $\rho_X = \cap \{\rho_x : x \in X\}$, and that for each x in X, ρ_x is a right congruence on S. Hence, ρ_X is also a right congruence on S.

The above definitions are motivated by the observation that a right simple congruence is a right group congruence if and only if it is left cancellative. The following remark easily generalizes to the left [right] cancellative case.

REMARK 3.12. [2, Lemma 9.49, p. 164]. Let $\{\sigma_i : i \in I\}$ be the family of all cancellative congruences on a semigroup S. Then $\sigma = \cap \{\sigma_i : i \in I\}$ is a cancellative congruence on S.

Since ω is a cancellative congruence on any semigroup S, the foregoing remark shows that there exists a (unique) minimum left cancellative congruence on S.

Define λ by

(3.13) $\lambda = \{(a, b) \in S \times S : xa = xb \text{ for some } x \text{ in } S\}.$

It is evident that λ is a reflexive, symmetric, and left cancellative relation on S. If σ is any left cancellative relation on S and $(a, b) \in \lambda$, then $xa = xb$ for some x in S, so $(xa, xb) \in \sigma$. By the left cancellativity of σ, $(a, b) \in \sigma$; therefore $\lambda \subseteq \sigma$. Thus if λ is a congruence on S, then it is the minimum left cancellative congruence on S. A major difficulty with λ is in its being transitive for an arbitrary semigroup S.

In the following development, the approach used to overcome the transitivity problem of λ is to fix the element x in the definition of λ. Thus for a fixed x in S, $\lambda = \rho_x$. Relabeling λ as ρ_x, ρ_x (and hence ρ_X) becomes a right congruence on S. Moreover, $\rho_X \subseteq \rho_x \subseteq \lambda$ for all x in X. It remains then to find conditions on S such that ρ_x is left compatible and right simple. A result in this direction is the following theorem.

THEOREM 3.14. *Let A be a subset of the semigroup S and let σ be a congruence on S. Define σ_A by*

$$\sigma_A = \cap \{\sigma_x : x \in A\}$$

where, for each x in A

$$\sigma_x = \{(a, b) \in S \times S : (xa, xb) \in \sigma\}.$$

(1) *If A is a right ideal of S, then σ_A is a congruence on S.*

(2) *If σ_A is left cancellative, then $\sigma_A = \sigma_x$ for each x in A.*

(3) *If A is a minimal right ideal of S, then σ_A is left cancellative if and only if it is a right group congruence on S.*

Proof. (1) The relation σ_x is a right congruence on S for each x in A. Thus it suffices to show that σ_A is left compatible since the intersection of right congruences is a right congruence. If $(a, b) \in \sigma_A$, then $(a, b) \in \sigma_x$ for all x in A. If $c \in S$, then $xc \in AS \subseteq A$, so that $a\sigma_{xc}b$, that is, $xca\sigma xcb$. But then by the definition of σ_x, $ca\sigma_x cb$, i.e., $(ca, cb) \in \sigma_x$ for each x in A. Hence $(ca, cb) \in \sigma_A$ and σ_A is a congruence on S.

(2) Suppose σ_A is left cancellative. Clearly $\sigma_A \subseteq \sigma_x$ for all x in A. Conversely let $(a, b) \in \sigma_x$ and $y \in A$. Since $xa\sigma xb$ and σ is a congruence, then $(yxa, yxb) \in \sigma$ and therefore $(xa, xb) \in \sigma_y$ for every y in A, i.e., $(xa, xb) \in \sigma_A$. Since σ_A is left cancellative, then $(a, b) \in \sigma_A$. Thus for each x in A, $\sigma_x \subseteq \sigma_A \subseteq \sigma_x$, i.e., $\sigma_A = \sigma_x$.

(3) If A is a MRI of S then $xA = A$ for each x in A. Suppose σ_A is left cancellative. If $a, b \in S$ then for any x in A, xa and xb are in A and therefore there exists y in A such that $xay = xb$, that is, $ay\sigma_x b$. By (2) above, $ay\sigma_A b$. This shows that S/σ_A is a right simple semigroup.

If x is in A, then there exists y in A such that $xy = x$. Therefore $xy^2 = xy$ which implies that $y^2\sigma_x y$ and hence that $y^2\sigma_A Y$. Thus, $y\sigma_A$ is an idempotent in S/σ_A and therefore by Lemma 2.2 (2), σ_A is a RGC.

Conversely, if σ_A is a RGC on S then, by definition, it is a left cancellative congruence on S.

REMARK. The converse of Theorem 3.14 (1) is false. Let S be the additive semigroup of positive integers, and let $\sigma = \iota$ and $A = \{1\}$. Then for $a, b \in S$, $(a, b) \in \sigma_A$ if and only if $(a, b) \in \sigma_1$, that is, if and only if $1 + a = 1 + b$. But this last equation is true if and only if $a = b$ and hence $\sigma_A = \sigma_1 = \sigma = \iota$. So σ_A is a congruence but A is not a right ideal of S.

The converse of Theorem 3.14 (2) is also false. Consider for example, the multiplicative semigroup of the integers modulo four, i.e., $S = \{0', 1', 2', 3'\}$. Choose the subset $\{2'\}$ of S to be A and set $\sigma = \iota$. Then $\sigma_A = \sigma_{2'} = \iota \cup \{(0, 2), (2, 0), (1, 3), (3, 1)\}$. Now $(0' \cdot 2', 0' \cdot 3') \in \sigma_A$ but $(2', 3') \notin \sigma_A$, i.e., σ_A is not left cancellative.

The following special case of Theorem 3.14 is worth noting.

COROLLARY 3.15. *Let A be a minimal right ideal of a semigroup S, let σ be the identity relation on S and define ρ_A by equation* (3.10). *Then: $\sigma_A = \rho_A$. Moreover, ρ_A is left cancellative if and only if ρ_A is the minimum right group congruence on S.*

Proof. Certainly σ_A is equal to ρ_A. Now suppose ρ_A is left cancellative. Then by part (3) of Theorem 3.14, ρ_A is a RGC on S.

If σ is RGC on S, then by Theorem 2.7, σ maps A onto S/σ. Hence there exists y in A such that $y\sigma \in E_{S/\sigma}$. For $a, b \in S$ such that $a\rho_A b$, $xa = xb$ for all x in A. In particular then, $ya = yb$. Thus under σ, $ya\sigma = yb\sigma$. Since $y\sigma$ is a left identity for S/σ, $a\sigma = b\sigma$, that is, $\rho_A \subseteq \sigma$. Hence ρ_A is the MRGC on S.

The converse is evident.

It is now logical to consider necessary and sufficient conditions on S such that ρ_x of equation (3.9) is a congruence. The next theorem characterizes those ρ_x which are left compatible.

THEOREM 3.16. *Let S be a semigroup containing minimal right ideals, let K be a minimal right neat subset of S and let $x \in K$. Let Y denote a set of representatives of the set of disjoint right neat subsets of S. Then the relation ρ_x, as defined by equation* (3.9), *is a congruence on S if and only if the following condition is satisfied:*

(C) *For each c in S and for each y in Y, there exists u in Y such that* $x^{[-1]}Ky = (xc)^{[-1]}Ku$.

Proof. By Remark 3.11, ρ_x is a right congruence on S. Thus to prove the theorem it suffices to show that condition (C) is necessary and sufficient for ρ_x to be left compatible. Assume that ρ_x is a left congruence on S and let $s, t \in [y]$ where for convenience $x^{[-1]}Ky$ is denoted by $[y]$. Thus $xs = xy = xt$ and therefore $(s, t) \in \rho_x$. For any c in S, ρ_x being left compatible implies that $(cs, ct) \in \rho_x$. Hence there exists y' in Y such that $cs, ct \in [y']$, that is, $xcs = xct = xy'$. Thus $s, t \in (xc)^{[-1]}Ky'$.

Conversely, let $(s, t) \in \rho_x$, $c \in S$, and assume condition (C) holds. Since $(s, t) \in \rho_x$, there exists y in Y such that $xs = xt = xy$. By hypothesis then, there exists y' in Y such that $xcs = xct = xy'$. This last equation implies that $cs, ct \in [y']$, that is, $(cs, ct) \in \rho_x$.

THEOREM 3.17. *Let S be a semigroup containing minimal right ideals and denote their union by R. Then the relation* ρ_R *defined by*

$$\rho_R = \{(a, b) \in S \times S : ra = rb \text{ for all } r \text{ in } R\},$$

is a congruence on S. Moreover, the following are equivalent:

(1) ρ_x *is a congruence for some x in R.*
(2) ρ_x *is a congruence for every x in R.*
(3) $\rho_x = \rho_R$ *for some x in R.*
(4) $\rho_x = \rho_R$ *for every x in R.*
(5) $\rho_x = \rho_y$ *for all x, y in R.*
(6) ρ_R *is the minimum right group congruence on S.*

Proof. By Remark 3.11, ρ_R is a right congruence on S with $X = R$. To show that ρ_R is left compatible, let $(a, b) \in \rho_R$ and $c \in S$. For any $r \in R$, rc is in R. Hence $rca = rcb$, and therefore $(ca, cb) \in \rho_r$ for all r in R.

(1) *implies* (3). Assume that for some $x \in R$, ρ_x is a congruence on S. For $r \in R$ there exists $u \in S$ such that $rxu = r$ since rxS is the MRI of S containing r. If $(a, b) \in \rho_x$, then $(ua, ub) \in \rho_x$ and therefore $xua = xub$. Multiplication by r yields $rxua = rxub$, i.e., $ra = rb$. Hence $(a, b) \in \rho_r$ and therefore $\rho_x \subseteq \rho_r$ for all r in R. Thus $\rho_R = \rho_x$.

(3) *implies* (6). There exists $u \in S$ such that $xu = x$ and therefore $xu^2 = xu$. Thus $(u^2, u) \in \rho_R$ and E_{S/ρ_R} is nonempty. By part (iv) of Theorem 3.6, ρ_x is right simple and hence S/ρ_R is a right group by Lemma 2.2 (2).

Next let σ be any RGC on S. By Theorem 2.7, $\sigma^\natural$ maps xS onto S/σ and hence there exists $e \in xS$ such that $e\sigma \natural = e\sigma \in E_{S/\sigma}$, say $e = xs$. If $(a, b) \in \rho_R$, then since ρ_R is a congruence and equal to ρ_x, $(sa, sb) \in \rho_x$. Hence $xsa = xsb$ and applying $\sigma \natural$ yields $(xsa)\sigma = (xsb)\sigma$, that is, $(ea)\sigma(eb)$. But by Lemma 2.1, $e\sigma$ is a left identity for S/σ and therefore $a\sigma b$. Hence $\rho_R \subseteq \sigma$ so ρ_R is the MRGC on S.

(6) *implies* (5). Let $(a, b) \in \rho_k$ for some k in R. Then $ka = kb$ and therefore $(ka)\rho_R = (kb)\rho_R$. Since ρ_R is a RGC it is left cancellative and therefore $a\rho_R = b\rho_R$, i.e., $(a, b) \in \rho_r$ for all r in R.

(5) *implies* (1). Since $\rho_x = \rho_r$ for all x, r in R, then $\rho_R = \rho_x$ for some x in R. By the first portion of the theorem, ρ_x is therefore a congruence on S.

(5) *implies* (2). Since ρ_R is a congruence and $\rho_x = \rho_y$ for all x, y in R, then $\rho_R = \rho_x$ for each x in R, i.e., ρ_x is a congruence for each x in R.

It is evident that (2) implies (4) and that (4) implies (5).

COROLLARY 3.18. *If S is a commutative semigroup with kernel R, then for any x in R, ρ_x is the minimum group congruence on S and S/ρ_x is isomorphic to R.*

Proof. For every x in S, ρ_x is a right congruence on S and thus a congruence on S. In particular, taking x in R, Theorem 3.17 and its left-right dual imply that $\rho = \rho_x$ is the minimum right simple and left simple congruence on S, i.e., ρ is the MGC on S.

The kernel R of S is also a group and if x denotes the identity of R, then $xr = xt$ for r, t in R if and only if $r = t$. Thus $\rho^\natural$ is one-to-one on R, so, by Theorem 2.7,

$$S/\rho_x = S\rho_x{}^\natural = R\rho_x{}^\natural \cong R.$$

REMARK. The converse to the first conclusion of Theorem 3.17 is false. For example, let $S(\cdot)$ be the semigroup defined by the following table:

$\cdot$	c	d	e	f
c	c	d	e	f
d	d	d	d	f
e	e	d	d	f
f	f	d	d	f

The unique MRI of S is $dS = fS = \{d, f\}$, so $R = \{d, f\}$. Neither ρ_f nor ρ_d are congruences on S. However, ρ_R is a congruence, but not a right simple congruence on S.

Another condition which is sufficient to insure the left compatibility of ρ_x is given by the following theorem.

THEOREM 3.19. *Let S be a semigroup and assume there exists an x in S and a subset A of S such that $Ax = xS$. Then ρ_x is a congruence on S, and moreover, if A is a subsemigroup of S, then S/ρ_x is a homomorphic image of A.*

Proof. Since ρ_x is a right congruence on S, it suffices to show that ρ_x is left compatible. Let $c \in S$ and suppose $a\rho_x b$, i.e., $xa = xb$. There exists y in A such that $yx = xc$ and therefore $xca = yxa = yxb = xcb$, that is, $(ca)\rho_x(cb)$.

Next, suppose that A is a subsemigroup of S and define the mapping θ from A to S/ρ_x by

$$\theta : a \rightarrow c\rho_x, \text{ for all } a \text{ in } A,$$

where c is an element of S such that $ax = xc$. If $a \in A$ and $ax = xc$, $ax = xd$, where $c, d \in S$, then $xc = xd$, i.e., $c\rho_x = d\rho_x$ and θ is well defined.

If $a, b \in A$ then $ab \in A$, and hence there exists an r in S such that $(ab)x = xr$. Corresponding to a and b there exist s, t in S such that $ax = xs$ and $bx = xt$. But then:

$$xr = (ab)x = a(xt) = x(st),$$

and therefore $r\rho_x = (st)\rho_x$. Hence:

$$(ab)\theta = r\rho_x = (st)\rho_x = (s\rho_x)(t\rho_x) = (a\theta)(b\theta),$$

i.e., θ is a homomorphism of A into S/ρ_x.

Lastly, for any s in S, there exists a in A such that $ax = xs$ and therefore θ is a homomorphism of A onto S/ρ_x.

THEOREM 3.20. *Let S be a semigroup and let I be a right ideal of S which contains a right identity x. Then the relation ρ_x defined by equation* (3.9) *is a congruence on S. Moreover the following are equivalent:*

(1) *I is a minimal right ideal of S.*

(2) *I is a group.*

If this is the case then, in addition,

(3) *ρ_x is the minimum right group congruence on S.*

(4) *ρ_x is the minimum group congruence on S.*

(5) *$\rho_x = \rho_y$ for all y in R, where R is the union of all the minimal right ideals of S.*

(6) *S/ρ_x is isomorphic to I.*

Proof. (1) *implies* (2). If I is a MRI of S, then I is a right group. If $e \in$

E_I then, by [**1**, Lemma 1.26, p. 37], e is a left identity for I. In particular, $ex = x$. On the other hand, $ex = e$ since x is a right identity for I. Hence $|E_x| = 1$ and therefore, by [**1**, Theorem 1.27, p. 38], I is a group.

(2) *implies* (1). For all i in I, $iI = I$. Thus I is a MRI of S.

Now assume that I is a MRI of S.

Proof of (3). By (2), I is a group and hence also a right group. Therefore, by Theorem 2.7, S/ρ_x is a homomorph of I, i.e., ρ_x is a RGC on S. If $(a, b) \in \rho_x$, then $xa = xb$, so $(xa)\sigma = (xb)\sigma$ for any right congruence σ on S. But $x\sigma$ is then a left identity for S/σ and therefore $a\sigma = b\sigma$, i.e., $(a, b) \in \sigma$. Thus $\rho_x \subseteq \sigma$.

(4). Combining the proof of (3) with condition (2), we conclude that ρ_x is the MGC on S.

(5). Since ρ_x is a congruence on S, it follows from Theorem 3.17 that $\rho_x = \rho_y$ for all y in R.

(6). By (1), $xS = I = Ix$ and hence by Theorem 3.19, $S/\rho_x^\natural$ is a homomorph of I. If $(i, j) \in \rho_x$ and $i, j \in I$, then $i = ix = jx = j$, i.e., ρ_x is one-to-one on I.

If a is an element of a regular semigroup S, then the set of all inverses of a is denoted by $V(a)$.

COROLLARY 3.21. *If S is a left group, say $S = G \times L$, where G is a group and L is a left zero semigroup, then for any $x \in S$, ρ_x is the minimum group congruence on S and S/ρ_x is isomorphic to G.*

Proof. For any a in S, aS is a right ideal of S containing a. If $b \in aS$, then $b = at$ for some t in S. By the dual of Lemma 2.2, S is regular and thus $bt' = att'$ for every t' in $V(t)$. Since tt' is idempotent, by the dual of Lemma 2.1, tt' is a right identity for S, so $bt' = a$. Thus $bS = aS$, so aS is a MRI of S. Therefore each element of S is contained in some MRI of S, i.e., $S = R$, where R denotes the union of all MRI's of S.

If e is an idempotent of S then by the above paragraph, eS is a MRI of S, and e is a right identity since $(eS)e = eS$. By Theorem 3.20, eS is a group, $\rho_x = \rho_R(=\rho_S)$ is the MRGC on S, and $S/\rho_x \cong eS$. By Lemma 2.2 (4), $eS \cong G$.

A *homogroup* is a semigroup S containing an ideal A which is a subgroup of S; A is sometimes called a *group ideal* of S.

COROLLARY 3.22. *Let S be a homogroup, with group ideal I. For every x in I, ρ_x is the minimum group congruence on S and S/ρ_x is isomorphic to I.*

THEOREM 3.23. *Let S be a semigroup which contains a right zero. Denote the set of all right zeros of S by R, and define ρ_R as in equation* (3.10). *Then:*

(1) *R is the* (unique) *minimal right ideal of S.*

(2) *Every right simple homomorph of S is a right zero semigroup.*

(3) *ρ_R is a congruence on S.*

(4) *If ρ_R is right simple then it is the minimum right group congruence on S, and S/ρ_R is isomorphic to R.*

Proof. The proof of (1) is evident. For (2), let σ be any right simple congruence on S. By Theorem 2.7, the induced homomorphism $\sigma^\natural$ maps R onto S/σ and therefore σ is a RZC.

(3) By Remark 3.11, it suffices to show that ρ_R is left compatible. For any c in S and r in R, $rc \in R$. Thus if $a \rho_R b$, then $(rc)a = (rc)b$ for all c in S, r in R. Hence $(ca, cb) \in \rho_r$ for all r in R so $(ca, cb) \in \rho_R$.

(4) If ρ_R is right simple, then since $r^2 \rho_R r$ for all r in R, ρ_r is a RGC. If σ is any RGC on S and $(a, b) \in \rho_R$, then $ra = rb$ for r in R implies that $(ra, rb) \in \sigma$. But $r^2 = r$ implies that $r\sigma \in E_{S/\sigma}$ and therefore $r\sigma$ is a left identity for S/σ. Thus $a\sigma b$ and $\rho_R \subseteq \sigma$.

Lastly, if $(s, t) \in \rho_R$ and $s, t \in R$ then $s = rs = rt = t$ for all r in R. Thus $\rho_R{}^\natural$ is one-to-one on R and $\rho_R{}^\natural$ maps R isomorphically onto S/ρ_R.

For comparison, recall that in Corollary 2.8, S having a *left* zero implied that the MRGC on S was ω.

The next theorem is closely related to Theorem 3.17 in that ρ_R is assumed to be the MRGC on S and the structure of S is then determined.

THEOREM 3.24. *Let S be a semigroup containing minimal right ideals and denote the union of the minimal right ideals of S by R. Suppose the relation ρ_R on S defined by*

$$\rho_R = \{(a, b) \in S \times S : ra = rb \text{ for all } r \in R\}$$

is the minimum right group congruence on S. Then:

(1) $E_S \neq \Box$.

(2) *Every minimal right ideal of S is isomorphic to S/ρ_R.*

(3) *Each equivalence class A of ρ_R contains a unique minimal right neat subset of S, that is, A contains exactly one element of each minimal right ideal of S.*

(4) *Let B be an index set for the collection of minimal right neat subsets of S specified in* (3). *Then ρ_R can be characterized as follows:*

$$\rho_R = \{(a, b) \in S \times S : K_i a = K_i b \text{ for each } i \text{ in } B\}.$$

Proof. For convenience, ρ_R will be denoted by ρ in this proof.

(1) Let $e\rho \in E_{S/\rho}$. Since S/ρ is right simple, $e\rho$ is a right neat subset of S and hence contains a MRN subset of S meeting each MRI of S. Let $x \in I \cap e\rho$ for some MRI, I, of S. Since $x \in e\rho$, then $xx = xe$ and therefore $xex = x^3$. But $e\rho$ is a left identity for S/ρ and hence $(ex)\rho x$, which implies $xex = x^2$. Combining this with $xex = x^3$ yields $x^2 = x^3$. Since $x^2 I = I$, then to each x in $I \cap e\rho$ there corresponds u in I such that $x^2 u = x$. Thus

$$x = x^2 u = x^3 u = x(x^2 u) = x^2.$$

So $x \in E_S$ and in particular, x is a left identity for I.

(2) Let I be a MRI of S and let x be a left identity element of I. For any i in I, $iI = I$ and hence, since $E_I \neq \square$, I is a right group. By Theorem 2.7, I is mapped onto S/ρ by the natural homomorphism $\rho^\natural$. If $a, b \in I$ and $a\rho b$, then $xa = xb$. But x is a left identity for I and therefore ρ is the identity relation on I, i.e., $I \cong S/\rho$.

(3) Let $a\rho$ be any ρ-class of S and suppose that $i, j \in I \cap a\rho$, where I and x are as defined in (2). Since $\rho = \rho_x$, then $xi = xj = xa$. Since x is a left identity for I, $i = j = xa$, that is, $|I \cap a\rho| = 1$. Hence $a\rho$ contains exactly one cross-section of the MRI's of S.

(4) If $(a, b) \in \rho$ then $xa = xb$ for all x in R. Hence for each i in B and for each x in K_i, $xa = xb$, i.e., $K_i a = K_i b$.

Conversely, if $a, b \in S$ are such that $K_i a = K_i b$ for each i in B, then there exist h, k in K_i such that $ha = kb$ and therefore, by Theorem 3.2 (iii), $h = k$. Thus $ha = hb$ for each i in B and for each h in K_i. By (3), $R = \cup \{K_i : i \text{ in } B\}$. Thus $ha = hb$ for each h in R, i.e., $(a, b) \in \rho$.

The following result is a partial converse to Theorem 3.24.

THEOREM 3.25. *Let S be a semigroup containing minimal right ideals. Assume that for some minimal right neat subset K of S, and for some x in K, ρ_x satisfies the following two conditions.*

(1) *Each ρ_x-class of S contains a unique minimal right neat subset of S.*

(2) *If $(a, b) \in \rho_x$, then $Ha = Hb$, where H is any of the unique minimal right neat subsets from condition* (1).

Then ρ_x is the minimum right group congruence on S.

Proof. Let C be the set of MRN subsets of S provided by hypothesis (1). Index C by a set B, say $C = \{K_i : i \in B\}$. Thus B is also an index set for the ρ_x-classes of S. We first show that:

$$R = \cup \{K_i : i \in B\}, \tag{3.26}$$

where R denotes the union of all MRI's of S. Clearly the right side of (3.26) is contained in the left. Conversely, if $r \in R$, then r lies in some MRN subset of S.

Since ρ_x is a right congruence on S, it remains to show that ρ_x is left compatible. By (3.26), x in K implies that $x \in K_i$ for some i in B. Let $(a, b) \in \rho_x$ and $c \in S$. The set $K_i c$ is a MRN subset by Theorem 3.2 (iv), and hence, by hypothesis (2), $K_i ca = K_i cb$. But then by Theorem 3.2 (iii), $xca = xcb$ and therefore ρ_x is a congruence. Moreover, by Theorem 3.17 (6), ρ_x is the MRGC on S.

At this point it would seem natural to ask if the existence of MRN subsets is a necessary and sufficient condition for a minimum right simple congruence to exist. The necessity of this condition is an open question, whereas this condition is not sufficient. Consider, for example, the full transformation semigroup T_X on the set $X = \{1, 2\}$. The constant transformations of X form a MRI of T_X, say R, but the only right simple congruences on T_X are ι and ω. In fact $\rho_R = \iota$.

References

1. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, Mathematical Surveys (No. 7), Amer. Math. Soc., Providence, Rhode Island, 1961.

2. ———, *The Algebraic Theory of Semigroups*, Vol. II, Mathematical Surveys (No. 7), Amer. Math. Soc., Providence, Rhode Island, 1967.

3. P. Dubreil, *Quelques Problemes d'Algebra lies a la theorie des demi-groupes*, Colloque d'Algebre superieure, Bruxelles (1956), 29–44.

4. Jean-Claude Fontaine, *Sur certaines equivalences definies dans un demigroupe et liees a l'equivalence reversible generalisee*, C. R. Acad. Sci. Paris, (Ser. A-B) **V264** (1967), 781–784.

5. P. Lefebvre, *Demi-groupes admettant des complexes nets a droite minimax*, Comptes Rendus Acad. Sci. Paris, **V247** (1958), 393–396.

6. ———, *Sur les groupes homomorphes a un demi-groupe; demi-groupes admettant un groupe homomorphe maximum*, C. R. Acad. Sci. Paris, **V248** (1959), 2277–2279.

Received October 13, 1972 and in revised form June 6, 1974. The results of this paper were included in the author's doctoral dissertation, directed by Professor Donald W. Miller of the University of Nebraska.

Glassboro State College

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the five editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California 90708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 54, No. 2 June, 1974