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An entire function $f(z)$ is said to have bounded value distribution (b.v.d.) if there exist constants p, R such that the equation $f(z) = w$ never has more than p roots in any disk of radius R . It was shown by W. K. Hayman that this is the case for a particular p and some $R > 0$ if and only if there is a constant $C > 0$ such that for all z

$$|f^{(p+1)}(z)| \leq C \max_{1 \leq \nu \leq p} |f^{(\nu)}(z)|,$$

so that $f'(z)$ has bounded index in the sense of Lepson.

The fact that $f'(z)$ has bounded index if $f(z)$ has b.v.d. follows readily from a classical result on p -valent functions. In the other direction Hayman proved that if

$$|f^{(n)}(z)| \leq \max_{0 \leq \nu \leq n-1} |f^{(\nu)}(z)|,$$

then $f(z)$ cannot have more than $n - 1$ zeros in $|z| \leq \sqrt{n}/e\sqrt{20}$. Here the order of magnitude is correct in the sense that $\sqrt{n}/e\sqrt{20}$ cannot be replaced by $\sqrt{3}\sqrt{n}$. The result when applied to $f(z) - w$ does show that $f'(z)$ has bounded index only if $f(z)$ has b.v.d. but it is clearly of interest to determine the largest disk containing at most $n - 1$ zeros of $f(z)$. We are able to replace $\sqrt{n}/e\sqrt{20}$ by $\sqrt{n}/e\sqrt{10}$.

The above mentioned result of Hayman appeared in [2]. He did not assume $f(z)$ to be entire but simply regular in $|z| < 2n$. To be precise he proved [2, Theorem 3] the following:

THEOREM A. *If $f(z)$ is regular in $|z| < 2n$, where it satisfies*

$$(1.1) \quad |f^{(n)}(z)| \leq \max_{0 \leq \nu \leq n-1} |f^{(\nu)}(z)|,$$

then $f(z)$ possesses at most $n - 1$ zeros in

$$(1.2) \quad |z| \leq \frac{\sqrt{n}}{e\sqrt{20}}.$$

In his proof of Theorem A Hayman made use of the following lemma.

LEMMA A. *Let $z_\nu, \nu = 1, 2, \dots, n$ be complex numbers such that $\max_{1 \leq \nu \leq n} |z_\nu| = \rho_0$. If*

$$(1.3) \quad \varphi(z) = \left\{ \prod_{v=1}^n (1 - z_v z) \right\}^\varepsilon = \sum_0^\infty b_k z^k$$

and $b_1 = \sum_{v=1}^n z_v = 0$, $\varepsilon = 1$ or -1 , then

$$(1.4) \quad |b_k| < (\sqrt{n})^k \rho_0^k, \quad k > 1.$$

The bound in (1.4) is not the best possible and this is one of the reasons why the conclusion of Theorem A is not precise. We observe that (1.4) can be considerably improved, viz. we have

LEMMA A'. *Under the hypotheses of Lemma A*

$$(1.5) \quad |b_k| \leq \left\{ \sqrt{\binom{n}{2}} \right\}^k \rho_0^k, \quad k > 1$$

Now Hayman's reasoning itself gives us the following improvement of Theorem A.

THEOREM A'. *Under the hypothesis of Theorem A $f(z)$ possesses at most $n - 1$ zeros in*

$$(1.6) \quad |z| \leq \frac{\sqrt{n}}{e\sqrt{10}}.$$

This refined version of Theorem A gives corresponding refinements in several of the other theorems proved by Hayman in [2]. For example, Theorems 4 and 6 of his paper may respectively be replaced by

THEOREM 4'. *Suppose that $f(z)$ is regular in $|z - z_0| < R$ and satisfies there*

$$(CR)^{p+1} \left| \frac{f^{(p+1)}(z)}{(p+1)!} \right| \leq \max_{1 \leq v \leq p} (CR)^v \left| \frac{f^{(v)}(z)}{v!} \right|$$

with $C \leq 1/2$. Then $f(z)$ is p -valent in $|z - z_0| \leq CR / \{e\sqrt{10} (p+1)^{1/2}\}$.

THEOREM 6'. *Consider the differential equation*

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0,$$

in the disk $D_0 = \{z \mid |z - z_0| < R\}$, where $0 < R \leq \infty$ and the functions a_1 to a_n are supposed to be regular and bounded in D_0 . Let t_0 be the positive root of the equation

$$\sum_{v=1}^n \alpha_v t^v = 1$$

where

$$\alpha_v = \sup_{z \in D_0} |a_v(z)|.$$

If $y(z)$ is a solution of the differential equation then $y(z)$ has at most $n - 1$ zeros in

$$|z - z_0| \leq R'_1 = \min \left\{ t_0 \frac{\sqrt{n}}{e\sqrt{10}}, \frac{R}{2e(10n)^{1/2}} \right\},$$

i.e. the differential equation is disconjugate in $|z - z_0| < R'_1$.

DEFINITION. Let \mathcal{P}_n denote the class of polynomials

$$p_n(z) = \prod_{v=1}^n (1 - z_v z)$$

which do not vanish in $|z| < 1$ and for which $p'_n(0) = \sum_{v=1}^n -z_v \equiv 0$.

Lemma A' may now be stated in the following equivalent form.

THEOREM 1. If

$$(1.7) \quad \varphi(z) = \{p_n(z)\}^\varepsilon = \sum_0^\infty b_{k,\varepsilon} z^k$$

where $p_n(z) \in \mathcal{P}_n$ and $\varepsilon = 1$ or -1 , then

$$(1.8) \quad |b_{k,\varepsilon}| \leq \{\sqrt{(n/2)}\}^k.$$

If n is even and $p_n(z) = (1 - e^{i\gamma} z^2)^{n/2}$ where γ is real then $|b_{2,1}| = |b_{2,-1}| = n/2$ which shows that (1.8) is the best possible result of its kind.

The bound in (1.8) is not sharp for $k \geq 3$ and it is clearly of interest to get precise estimates for $|b_{k,\varepsilon}|$ for each k . We are able to do it for $k \leq 4$.

THEOREM 2. Under the hypothesis of Theorem 1 we have

$$(1.9) \quad |b_{2,\varepsilon}| \leq n/2,$$

$$(1.10) \quad |b_{3,\varepsilon}| \leq n/3,$$

$$(1.11) \quad |b_{4,1}| \leq (n^2 - 2n)/8,$$

$$(1.12) \quad |b_{4,-1}| \leq (n^2 + 2n)/8.$$

The example $p_n(z) = (1 - z^2)^{n/2}$ where n is even shows that (1.9), (1.11) and (1.12) are sharp. To see that (1.10) is sharp we may consider $p_n(z) = (1 - z^3)^{n/3}$ where n is divisible by 3.

The following theorem shows that $|b_{2,\varepsilon}|$ and $|b_{3,\varepsilon}|$ cannot both be large at the same time.

THEOREM 3. *Under the hypothesis of Theorem 1 we have*

$$(1.13) \quad |b_{2,\varepsilon}| + |b_{3,\varepsilon}| \leq \frac{25}{48}n$$

$$(1.14) \quad |b_{2,\varepsilon}| + \frac{3}{4}|b_{3,\varepsilon}| \leq n/2$$

$$(1.15) \quad \frac{2\sqrt{2}}{3}|b_{2,\varepsilon}| + |b_{3,\varepsilon}| \leq n/2.$$

If k is fixed, $k > 4$ and n is large, the bound in (1.8) can also be sharpened.

THEOREM 4. *Let $p_n(z) \in \mathcal{P}_n$ and λ a real number $\neq 0$. If*

$$(1.16) \quad \varphi_\lambda(z) = \{p_n(z)\}^\lambda = \sum_0^\infty b_{k,\lambda} z^k$$

then for every given $0 < \delta < \pi$ there exists an integer n_0 depending on λ and δ such that

$$(1.17) \quad |b_{k,\lambda}| \leq 2 \frac{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1} \left\{ \sqrt{\left(\frac{|\lambda|n}{\delta}\right)} \right\}^k$$

provided $n > n_0$.

The proof of Theorem 4 depends on the fact that if $p_n(z) \in \mathcal{P}_n$ then

$$(1.18) \quad \omega(z) = \frac{1 - \{p_n(z)\}^{1/n}}{z^2}$$

is analytic in $|z| < 1$ and there exists a positive number ρ_0 independent of n such that

$$(1.19) \quad |\omega(z)| \leq \frac{1}{2} + \frac{1}{8}|z|^2 + |z|^4$$

for $|z| < \rho_0$. For the study of polynomials $p_n(z) \in \mathcal{P}_n$ it will be very helpful to get precise estimates for $|\omega(z)|$. The example

$$p(z) = (1 - z^2)^{n/2}, n \text{ even}$$

shows that

$$\max_{p(z) \in \mathcal{P}_n} |\omega(z)| \geq \frac{1}{2} + \frac{1}{8}|z|^2 + \frac{1}{16}|z|^4 + \frac{5}{128}|z|^6.$$

We prove

THEOREM 5. *If $p_n(z) \in \mathcal{P}_n$ then*

$$(1.20) \quad |\omega(z)| = \left| \frac{1 - \{p_n(z)\}^{1/n}}{z^2} \right| \leq \frac{1}{2} + \frac{1}{8}|z|^2 + \frac{1}{16}|z|^4 + \frac{3\sqrt{3}}{4}|z|^6$$

at least for $|z| \leq 1/2$.

The following corollary is obtained by applying Theorem 5 to the reciprocal polynomial $z^n p_n(1/z)$, and setting $\alpha = z^{-1}\omega(z^{-1})$.

COROLLARY 1. *Let*

$$p_n(z) = \prod_{k=1}^n (z - \alpha_k)$$

be a polynomial of degree n having all its zeros in $|z| \leq 1$. If the centre of gravity of the zeros lies at the origin then for $|z| > 2$ the equation

$$(1.21) \quad \sum_{k=1}^n \frac{1}{n} \log \left(1 - \frac{\alpha_k}{z} \right) = \log \left(1 - \frac{\alpha}{z} \right)$$

has a solution which satisfies

$$(1.22) \quad |\alpha| \leq \frac{1}{2|z|} + \frac{1}{8|z|^3} + \frac{1}{16|z|^5} + \frac{3\sqrt{3}}{4|z|^7}.$$

If $\alpha_k, k = 1, 2, \dots, n$ are complex numbers of absolute value ≤ 1 and $m_k = p_k/q_k, k = 1, 2, \dots, n$ are positive rational numbers such that $\sum_{k=1}^n m_k = 1, \sum_{k=1}^n m_k \alpha_k = 0$ then

$$\left\{ \prod_{k=1}^n (z - \alpha_k)^{p_k/q_k} \right\}^{q_1 q_2 \dots q_n}$$

is a polynomial of degree $q_1 q_2 \dots q_n$ having all its zeros in $|z| \leq 1$. Besides, the centre of gravity of the zeros (taking into account their multiplicity) lies at the origin. Hence by the above corollary the equation in α

$$\sum_{k=1}^n \frac{p_k q_1 q_2 \dots q_{k-1} q_{k+1} \dots q_n}{q_1 q_2 \dots q_n} \log \left(1 - \frac{\alpha_k}{z} \right) \equiv \sum_{k=1}^n m_k \log \left(1 - \frac{\alpha_k}{z} \right) = \log \left(1 - \frac{\alpha}{z} \right)$$

has a solution α which satisfies (1.22) at least for $|z| \geq 2$. It is clear that if some or all the numbers m_k are irrational then we get the same conclusion by a limiting process. Thus we have

COROLLARY 1'. *If we have $m_k > 0, \sum m_k = 1, |\alpha_k| \leq 1, \sum m_k \alpha_k = 0, |z| \geq 2$ (where $k = 1, 2, 3, \dots, n$) then there exists an α such that*

$$(1.22) \quad |\alpha| \leq \frac{1}{2|z|} + \frac{1}{8|z|^3} + \frac{1}{16|z|^5} + \frac{3\sqrt{3}}{4|z|^7}$$

with

$$(1.23) \quad \sum m_k \log \left(1 - \frac{\alpha_k}{z} \right) = \log \left(1 - \frac{\alpha}{z} \right).$$

This result was proved by Walsh (see [4], Lemma 2 and (1.10) on p. 358) except that he had

$$|\alpha| \leq \frac{1}{2|z|} + \frac{3}{2|z|^2}$$

for $|z| > 3$ instead of (1.22) which we prove to be valid for $|z| \geq 2$. As illustrated by Walsh (see [4], pp. 358–360) such a result is very useful for applications.

2. LEMMAS. We shall need the following subsidiary results.

LEMMA 1. *If*

$$f(z) = \sum_0^{\infty} a_k z^k$$

is analytic in $|z| < 1$, where $|f(z)| \leq 1$ then

$$(2.1) \quad |a_0|^2 + |a_k| \leq 1, \quad k \geq 1$$

and

$$(2.2) \quad \sum_0^{\infty} |a_k|^2 \leq 1.$$

For (2.1) we refer to [3, p. 172, exer. #9]. Inequality (2.2) follows from the fact that for $0 < r < 1$

$$\sum_0^{\infty} |a_k|^2 r^{2k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta \leq 1.$$

LEMMA 2. *Under the hypothesis of Lemma 1 we have*

$$(2.3) \quad \left| \sum_0^{\infty} \frac{a_k}{k+2} z^k \right| \leq \frac{1}{2} |a_0| + \frac{1}{3} (1 - |a_0|^2) |a_0| |z| \quad \text{for } |z| < 1.$$

Proof of Lemma 2. By Schwarz's lemma

$$|f(\zeta)| \leq \frac{|a_0| + |\zeta|}{1 + |a_0||\zeta|}$$

for $|\zeta| < 1$. Hence

$$\begin{aligned} \left| \sum_0^{\infty} \frac{a_k}{k+2} z^k \right| &= \left| \frac{1}{z^2} \int_0^z \zeta f(\zeta) d\zeta \right| \leq \frac{1}{|z|^2} \int_0^{|z|} |\zeta| \frac{|a_0| + |\zeta|}{1 + |a_0||\zeta|} d|\zeta| \\ &= \frac{1}{2} |a_0| + (1 - |a_0|^2) \sum_{k=1}^{\infty} (-1)^{k+1} \frac{|a_0|^{k-1}}{k+2} |z|^k \\ &\leq \frac{1}{2} |a_0| + \frac{1}{3} (1 - |a_0|^2) |a_0| |z|. \end{aligned}$$

LEMMA 3. *If*

$$g(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

is analytic in $|z| < 1$, *where*

$$(2.4) \quad \operatorname{Re} g(z) > 0$$

and

$$(2.5) \quad |g(z)| < M$$

then

$$(2.6) \quad |\alpha_k| \leq 2 \frac{M^2 - 1}{M^2 + 1}.$$

Proof of Lemma 3. The function $G(z) = F^{-1}(w)$ where

$$\begin{aligned} F(w) &= \left\{ \left(\frac{iM - w}{iM + w} \right)^2 - \left(\frac{iM - 1}{iM + 1} \right)^2 \right\} / \left\{ \left(\frac{iM - w}{iM + w} \right)^2 - \left(\frac{iM + 1}{iM - 1} \right)^2 \right\} \\ &= \frac{1}{2} \frac{M^2 + 1}{M^2 - 1} (w - 1) + \dots \end{aligned}$$

maps the unit disk $|z| < 1$ onto the semicircular disk

$$D^+ = \{w : \operatorname{Re} w > 0, |w| < M\}$$

such that $G(0) = 1, G'(0) = 2(M^2 - 1)/(M^2 + 1)$. Since the function $g(z)$ maps the unit disk into D^+ and the function $G(z)$ is convex univalent it follows from a well-known result (see e.g. [3], p. 238, exer. #6) that

$$|\alpha_k| \leq |G'(0)| = 2 \frac{M^2 - 1}{M^2 + 1}, \quad k \geq 1.$$

LEMMA 4. *If*

$$p_3(z) = \prod_{\nu=1}^3 (1 - z_\nu z) = \sum_0^3 b_{k,1} z^k \in \mathcal{P}_3$$

then

$$(2.7) \quad |b_{2,1}|^2 + |b_{3,1}|^2 \leq 1.$$

Proof of Lemma 4. Let $|z_1| = \max_{1 \leq \nu \leq 3} |z_\nu|$. The polynomial

$$\hat{p}_3(z) = p_3\left(\frac{z}{z_1}\right) = 1 + \hat{b}_{2,1} z^2 + \hat{b}_{3,1} z^3$$

also belongs to \mathcal{P}_3 and $|b_{2,1}| \leq |\hat{b}_{2,1}|, |b_{3,1}| \leq |\hat{b}_{3,1}|$. Hence it is enough to prove (2.7) for $\hat{p}_3(z)$. We have

$$\hat{p}_3(z) = 1 - (1 - \hat{z}_2 \hat{z}_3) z^2 - \hat{z}_2 \hat{z}_3 z^3$$

where $|\hat{z}_2| \leq 1$, $|\hat{z}_3| \leq 1$ and $1 + \hat{z}_2 + \hat{z}_3 = 0$. Since $\hat{z}_2 + \hat{z}_3 = -1$ we may suppose

$$\hat{z}_2 = -a + ib, \hat{z}_3 = -1 + a - ib, \quad 0 \leq a \leq 1/2.$$

Since $|\hat{z}_3| \leq 1$ we have $(1 - a)^2 + b^2 \leq 1$, i.e.

$$(2.8) \quad b^2 \leq 2a - a^2.$$

We write $\hat{z}_2\hat{z}_3 = (-1 + a - ib)(-a + ib) = x + iy$, where

$$x = a(1 - a) + b^2, y = b(2a - 1).$$

Then

$$\begin{aligned} |\hat{b}_{2,1}|^2 + |\hat{b}_{3,1}|^2 &= |1 - x - iy|^2 + |x + iy|^2 \\ &= 2(x^2 + y^2 - x) \\ &= 2\{(b^2 + a(1 - a))^2 + b^2(2a - 1)^2 - b^2 - a(1 - a)\} \\ &= 2\{(b^2 - a(1 - a))^2 - a(1 - a)\}. \end{aligned}$$

In view of (2.8) and since $0 \leq a \leq 1/2$, we have

$$(b^2 - a(1 - a))^2 \leq a^2 \leq a(1 - a),$$

and now Lemma 4 follows.

3. Proofs of theorems.

Proof of Theorems 1, 2, 3. It has been proved by Dieudonné [1, p. 7] that if

$$p_n(z) = \prod_{v=1}^n (1 - z_v z)$$

is a polynomial of degree n with all its zeros in $|z| \geq 1$ then in $|z| < 1$

$$(3.1) \quad \frac{p'_n(z)}{p_n(z)} = \frac{n}{z - \frac{1}{\Psi(z)}}$$

where $\Psi(z)$ is analytic and $|\Psi(z)| \leq 1$. We observe that if $p_n(z) \in \mathcal{P}_n$, i.e. $\sum_{v=1}^n z_v = 0$ then $\Psi(0) = 0$ and hence by Schwarz's lemma $\Psi(z) = z\psi(z)$ where $\psi(z)$ is analytic and $|\psi(z)| \leq 1$ in $|z| < 1$. Thus for polynomials $p_n(z) \in \mathcal{P}_n$ the representation (3.1) takes the form

$$(3.2) \quad \frac{p'_n(z)}{p_n(z)} = \frac{-nz\psi(z)}{1 - z^2\psi(z)}.$$

If $\varphi(z) = \{p_n(z)\}^\varepsilon = \sum_0^\infty b_{k,\varepsilon} z^k$ then

$$(3.3) \quad \frac{\varphi'(z)}{\varphi(z)} = \frac{-\varepsilon n z \psi(z)}{1 - z^2 \psi(z)}$$

$$(3.4) \quad z\phi'(z) = \{z^3 \phi'(z) - n\varepsilon z^2 \phi(z)\} \psi(z).$$

Setting $\psi(z) = \sum_{\nu=0}^{\infty} c_{\nu} z^{\nu}$ and comparing coefficients on the two sides of (3.4) we get

$$(3.5) \quad kb_{k,\varepsilon} = \sum_{\substack{\nu=0 \\ \nu \neq 1}}^{k-2} (-n\varepsilon + \nu) b_{\nu,\varepsilon} c_{k-2-\nu}, \quad k \geq 2.$$

In particular

$$(3.6) \quad 2b_{2,\varepsilon} = -n\varepsilon c_0, \quad 3b_{3,\varepsilon} = -n\varepsilon c_1$$

which give (1.9) and (1.10) immediately since the coefficients of a function $\psi(z)$ analytic and bounded by 1 in $|z| < 1$ are themselves bounded by 1.

Again from (3.5) we have

$$\begin{aligned} 4b_{4,\varepsilon} &= -n\varepsilon c_2 + (-n\varepsilon + 2)b_{2,\varepsilon} c_0 \\ &= -n\varepsilon c_2 - \frac{1}{2} n\varepsilon (-n\varepsilon + 2) c_0^2 && \text{using (3.6)} \\ &= -n\varepsilon \left\{ c_2 - \frac{1}{2} (n\varepsilon - 2) c_0^2 \right\}. \end{aligned}$$

By (2.1)

$$|b_{4,\varepsilon}| \leq \frac{n}{8} \{ (|n\varepsilon - 2| - 2) |c_0|^2 + 2 \}$$

which readily gives (1.11), (1.12) and completes the proof of Theorem 2.

Theorem 3 is an immediate consequence of (3.6) and (2.1).

Now we come to the proof of Theorem 1. From inequalities (1.9)–(1.12) it follows that Theorem 1 holds for $k \leq 4$. For a given $n \geq 4$ let (1.8) hold for $k \leq j - 1$. We shall show that it then holds for $k = j$ and (for $n \geq 4$) the theorem will follow by the principle of mathematical induction. By formula (3.5) we have

$$\begin{aligned} j|b_{j,\varepsilon}| &\leq \sum_{\substack{\nu=0 \\ \nu \neq 1}}^{j-2} |-n\varepsilon + \nu| |b_{\nu,\varepsilon}| |c_{j-2-\nu}| \\ &\leq (n + j - 2) \sum_{\nu=0}^{j-2} |b_{\nu,\varepsilon}| |c_{j-2-\nu}| \end{aligned}$$

$$\leq (n + j - 2) \left(\sum_{v=0}^{j-2} |b_{v,e}|^2 \right)^{1/2} \left(\sum_{v=0}^{j-2} |c_{j-2-v}|^2 \right)^{1/2}.$$

Using (2.2) and the induction hypothesis we deduce

$$\begin{aligned} j|b_{j,e}| &\leq (n + j - 2) \left\{ \sum_{v=0}^{j-2} (n/2)^v \right\}^{1/2} \\ &= (n + j - 2) (n/2)^{j/2} \left\{ \sum_{v=0}^{j-2} (2/n)^{j-v} \right\}^{1/2} \\ &< (n + j - 2) (n/2)^{j/2} \frac{2/n}{\sqrt{1 - (2/n)}} \\ &< j \left(\sqrt{\frac{n}{2}} \right)^j \quad \text{if } j \geq 5. \end{aligned}$$

This completes the proof of (1.8) for $n \geq 4$. If $n = 2$ or 3 we argue as follows.

It follows from (2.7) that if

$$p_3(z) = \sum_0^\infty b_{k,1} z^k \in \mathcal{P}_3$$

then

$$|b_{2,1}| \leq 1, |b_{3,1}| \leq 1.$$

Since $|b_{k,1}| = 0$ for $k \geq 4$ we trivially have

$$|b_{k,1}| < \left(\sqrt{\frac{3}{2}} \right)^k, \quad k \geq 2.$$

From (1.9), (1.10) and (1.12) we have

$$(3.7) \quad |b_{k,-1}| \leq \left(\sqrt{\frac{3}{2}} \right)^k \quad \text{for } k \leq 4.$$

Hence (3.7) will be proved for all k if we show that it holds for $k = j$ provided it holds for $k \leq j - 2$. So let (3.7) be true for $k \leq j - 2$. From the identity

$$\frac{1}{1 + b_{2,1}z^2 + b_{3,1}z^3} \equiv \sum_0^\infty b_{k,-1}z^k$$

we have

$$b_{j,-1} + b_{j-2,-1}b_{2,1} + b_{j+3,-1}b_{3,1} = 0.$$

Using this, Lemma 4, and the induction hypothesis, we deduce

$$\begin{aligned}
 |b_{j,-1}| &\leq (|b_{j-2,-1}|^2 + |b_{j-3,-1}|^2)^{1/2} (|b_{2,1}|^2 + |b_{3,1}|^2)^{1/2} \\
 &\leq (|b_{j-2,-1}|^2 + |b_{j-3,-1}|^2)^{1/2} \\
 &\leq \left(\left(\frac{3}{2}\right)^{j-2} + \left(\frac{3}{2}\right)^{j-3} \right)^{1/2} \\
 &< \left(\sqrt{\frac{3}{2}}\right)^j.
 \end{aligned}$$

This completes the proof of (1.8) for $n = 3$.

If

$$p_2(z) = \prod_{v=1}^2 (1 - z_v z) \in \mathcal{P}_2$$

then $z_2 = -z_1$. Hence $p_2(z) = 1 - (z_1 z)^2$ and

$$|b_{k,\varepsilon}| \leq |z_1|^k \leq 1 = \left(\sqrt{\frac{2}{2}}\right)^k, \quad k \geq 2.$$

Next we prove Theorem 5 since we shall need it (in a weaker form) for the proof of Theorem 4.

Proof of Theorem 5. It was shown by Dieudonné (see [1], p. 7) that if

$$p_n(z) = \prod_{v=1}^n (1 - z_v z)$$

is a polynomial of degree n having all its zeros in $|z| \geq 1$ then

$$\Omega(z) = \frac{1 - \{p_n(z)\}^{1/n}}{z}$$

is analytic in $|z| < 1$ and $|\Omega(z)| \leq 1$. If $p_n(z) \in \mathcal{P}_n$ then $\Omega(0) = 0$ and hence by Schwarz's lemma

$$(3.8) \quad \omega(z) = \frac{\Omega(z)}{z} = \frac{1 - \{p_n(z)\}^{1/n}}{z^2}$$

is analytic in $|z| < 1$ and $|\omega(z)| \leq 1$. From (3.8) we get

$$(3.9) \quad \frac{p'_n(z)}{p_n(z)} = \frac{-n\{2z\omega(z) + z^2\omega'(z)\}}{1 - z^2\omega(z)}.$$

The two representations (3.2) and (3.9) for $p'_n(z)/p_n(z)$ give us the identity

$$(3.10) \quad \{2\omega(z) + z\omega'(z)\} \equiv \{1 + z^2\omega(z) + z^3\omega'(z)\}\psi(z).$$

Setting

$$\omega(z) = \sum_{v=0}^{\infty} \alpha_v z^v, \quad \psi(z) = \sum_{v=0}^{\infty} c_v z^v,$$

and comparing coefficients on the two sides of (3.10) we get

$$\alpha_0 = \frac{1}{2} c_0, \quad \alpha_1 = \frac{1}{3} c_1,$$

$$(3.11) \quad \alpha_v = \frac{1}{v+2} c_v + \frac{1}{v+2} \sum_{\mu=0}^{v-2} (\mu+1) \alpha_\mu c_{v-2-\mu}, \quad v \geq 2.$$

In particular

$$\alpha_2 = \frac{1}{4} c_2 + \frac{1}{4} \alpha_0 c_0 = \frac{1}{4} c_2 + \frac{1}{8} c_0^2,$$

$$\alpha_3 = \frac{1}{5} c_3 + \frac{7}{30} c_0 c_1,$$

$$\alpha_4 = \frac{1}{6} c_4 + \left(\frac{5}{24} c_0 c_2 + \frac{1}{9} c_1^2 + \frac{1}{16} c_0^3 \right),$$

$$\alpha_5 = \frac{1}{7} c_5 + \left(\frac{13}{70} c_0 c_3 + \frac{17}{84} c_1 c_2 + \frac{157}{840} c_0^2 c_1 \right).$$

Thus

$$(3.12) \quad \begin{aligned} \omega(z) &= \sum_{v=0}^{\infty} \frac{1}{v+2} c_v z^v + \frac{1}{8} c_0^2 z^2 + \frac{7}{30} c_0 c_1 z^3 \\ &+ \left(\frac{5}{24} c_0 c_2 + \frac{1}{9} c_1^2 + \frac{1}{16} c_0^3 \right) z^4 + \left(\frac{13}{70} c_0 c_3 + \frac{17}{84} c_1 c_2 + \frac{157}{840} c_0^2 c_1 \right) z^5 \\ &+ \sum_{v=6}^{\infty} \left(\alpha_v - \frac{c_v}{v+2} \right) z^v. \end{aligned}$$

Now let $|z| \leq 1/2$. By (2.1) we have

$$(3.13) \quad \left| \frac{1}{8} c_0^2 z^2 + \frac{7}{30} c_0 c_1 z^3 + \frac{1}{30} c_0 c_2 z^4 \right|$$

$$\leq \left\{ \left(\frac{7}{60} + \frac{1}{120} \right) |c_0|^2 + \frac{7}{60} |c_1| + \frac{1}{120} |c_2| \right\} |z|^2 \leq \frac{1}{8} |z|^2,$$

$$(3.14) \quad \left| \left(\frac{1}{16} c_0 c_2 + \frac{1}{16} c_0^3 \right) z^4 \right| \leq \frac{1}{16} |z|^4,$$

$$\begin{aligned} & \left| \left(\frac{9}{80} c_0 c_2 + \frac{1}{9} c_1^2 \right) z^4 + \left(\frac{13}{70} c_0 c_3 + \frac{17}{84} c_1 c_2 + \frac{157}{840} c_0^2 c_1 \right) z^5 \right| \\ & \leq \left(\frac{9}{640} |c_2| + \frac{1}{72} |c_1| + \frac{13}{1120} |c_3| + \frac{17}{1344} |c_1| + \frac{157}{13440} |c_1| \right) |z| \\ & \leq \left(\frac{9}{640} + \frac{1}{72} + \frac{13}{1120} + \frac{17}{1344} + \frac{157}{13440} \right) (1 - |c_0|^2) |z| \end{aligned}$$

$$(3.15) \quad \leq \frac{1}{6} (1 - |c_0|^2) |z|.$$

Using (3.12)–(3.14) and Lemma 2 in (3.12) we get

$$\begin{aligned} |\omega(z)| & \leq \frac{1}{8} |z|^2 + \frac{1}{16} |z|^4 + \frac{1}{2} |c_0| + \frac{1}{2} (1 - |c_0|^2) |z| + \sum_{v=6}^{\infty} \left(|\alpha_v| + \frac{|c_v|}{v+2} \right) |z|^v \\ & \leq \frac{1}{2} + \frac{1}{8} |z|^2 + \frac{1}{16} |z|^4 + \sum_{v=6}^{\infty} |\alpha_v| |z|^v + \frac{1}{8} \sum_{v=6}^{\infty} |c_v| |z|^v. \end{aligned}$$

But by (2.2)

$$\sum_{v=6}^{\infty} |\alpha_v| |z|^v \leq \left(\sum_{v=0}^{\infty} |\alpha_v|^2 \right)^{1/2} \left(\sum_{v=6}^{\infty} |z|^{2v} \right)^{1/2} \leq \frac{|z|^6}{(1 - |z|^2)^{1/2}} \leq \frac{2}{\sqrt{3}} |z|^6,$$

$$|\omega(z)| \leq \frac{1}{2} + \frac{1}{8} |z|^2 + \frac{1}{16} |z|^4 + \frac{3\sqrt{3}}{4} |z|^6.$$

Hence

$$\sum_{v=6}^{\infty} |c_v| |z|^v \leq \left(\sum_{v=0}^{\infty} |c_v|^2 \right)^{1/2} \left(\sum_{v=6}^{\infty} |z|^{2v} \right)^{1/2} \leq \frac{|z|^6}{(1 - |z|^2)^{1/2}} \leq \frac{2}{\sqrt{3}} |z|^6.$$

This completes the proof of Theorem 5.

Proof of Theorem 4. By Theorem 5

$$p_n(z) = \{1 - z^2\omega(z)\}^n$$

where $\omega(z)$ is analytic in $|z| < 1$ and

$$|\omega(z)| \leq \frac{1}{2} + \frac{1}{4}|z|^2 \quad \text{for } |z| \leq \frac{1}{2}.$$

If λ is a real number $\neq 0$ and $n > 6/|\lambda|$ then by simple geometrical considerations $\text{Re}\phi_\lambda(z) > 0$ if $|z| < \rho_0$ where ρ_0 is the only positive root of the equation

$$(3.16) \quad \rho^4 + 2\rho^2 = 4\sin \frac{\pi}{2|\lambda|n}.$$

In other words, $\text{Re} \phi_\lambda(\rho_0 z) > 0$ for $|z| < 1$. Besides, in $|z| < 1$

$$|\varphi_\lambda(\rho_0 z)| \leq \left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{|\lambda|n}.$$

Hence by Lemma 3

$$|b_{k,\lambda}| \rho_0^k \leq 2 \frac{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1}.$$

This gives

$$|b_{k,\lambda}| \leq 2 \frac{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1} \left(\frac{1}{\sqrt{1 + 4\sin \frac{\pi}{2|\lambda|n} - 1}}\right)^k$$

from which the desired result follows at once.

It may be noted that for fixed λ

$$\frac{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} - 1}{\left(1 + \sin \frac{\pi}{2|\lambda|n}\right)^{2|\lambda|n} + 1} \rightarrow \frac{e^\pi - 1}{e^\pi + 1} \quad \text{as } n \rightarrow \infty.$$

4. Some remarks.

REMARK 1. Theorem 2 can be easily extended to read as follows.

THEOREM 2'. *Let*

$$p_n(z) = \prod_{\nu=1}^n (1 - z_\nu z)$$

be a polynomial of degree n not vanishing in $|z| < 1$ and let

$$p'_n(0) = p''_n(0) = \dots = p_n^{(l)}(0) = 0.$$

If

$$\varphi(z) = \{p_n(z)\}^\varepsilon = \sum_0^\infty b_{k,\varepsilon} z^k$$

where $\varepsilon = 1$ or -1 then

$$|b_{k,\varepsilon}| \leq n/k \quad (l + 1 \leq k \leq 2l + 1),$$

and

$$|b_{2l+2,1}| \leq \frac{n}{2(l+1)^2} (n - l - 1), \quad |b_{2l+2,-1}| \leq \frac{n}{2(l+1)^2} (n + l + 1).$$

For the proof we simply need to observe that in $|z| < 1$

$$(4.1) \quad \frac{p'_n(z)}{p_n(z)} = \frac{-nz'\psi(z)}{1 - z^{l+1}\psi(z)}$$

where $\psi(z)$ is analytic and $|\psi(z)| \leq 1$ for $|z| < 1$.

REMARK 2. The radii of starlikeness and of convexity of the family

$$\{z[p_n(z)]^\alpha : p_n(z) \neq 0 \text{ in } |z| < 1, p_n(0) = 1\}$$

were determined by Dieudonné [1] with the help of the representation (3.1) for $p'_n(z)/p_n(z)$. In precisely the same way we may use (4.1) to determine the radii of starlikeness and of convexity of the family

$$\{z[p_n(z)]^\alpha : p_n(z) \neq 0 \text{ in } |z| < 1, p_n(0) = 1, p'_n(0) = p'_n(0) = \dots = p_n^{(l)}(0) = 0\}.$$

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