RANDOM POINTS IN A SIMPLEX

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The expected values of certain functions of N points chosen at random in an n dimensional simplex or parallelotope are considered, and a decomposition of such integrals is obtained by use of a generalised form of Crofton's Theorem.

Explicit expressions are found for the moments of the area of the triangle formed by three points chosen at random in a triangle or parallelogram.

I. Introduction. In Euclidean n-space a convex body K_n of volume V is given, and $n + 1$ points

$$\mathbf{x}_1, \mathbf{x}_2, \ldots\ldots\ldots, \mathbf{x}_{n+1}$$

are chosen independently, at random in K_n.

Let $C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1})$ denote the volume of the convex hull of the $n + 1$ points, which with probability one is an n-simplex, and let

$$D(K_n) = C(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1})/V.$$

Let $V^h_{K_n}$ be the hth moment of $D(K_n)$

$$V^h_{K_n} = E([D(K_n)]^h).$$

Since a ratio of volumes, and thus a uniform distribution over a body, is preserved under affine transformation, it follows that $V^h_{K_n}$ is an affine invariant of Kn.

The problem of finding $V^h_{K_n}$ is almost trivial for the case $n = 1$, for K_1 is a line segment.

$$V^h_{K_1} = \frac{2}{(h+1)(h+2)}.$$

For the case of $n = 2$, the problem of finding the first moment $V^1_{K_2}$, for various plane convex figures K_2, was investigated by Sylvester and others during the 1860's and has come to be known as Sylvester's Problem (see[3] for references).

For the equivalence class of triangles Δ_2 it is known that

$$V_{\Delta_2}^1 = \frac{1}{12}$$

and for the ellipse B_2

$$V_{B_2}^1 = \frac{35}{48}\pi^2.$$

In higher dimensions the problem is more difficult. Klee [4] considers the problem, and in particular asks for $V_{\Delta_n}^1$ where Δ_n is the n-simplex. Kingman [5] shows that for the n-dimensional ellipsoid B_n

$$V_{B_n} = \left(\frac{n+1}{\frac{1}{2}(n+1)}\right)^{n+1} \left/ \left(\frac{(n+1)^2}{\frac{1}{2}(n+1)^2}\right) 2^n.\right.$$

He also shows that the second moment $V_{K_n}^2 = (n+1)I/n! \ V^{n+2}$, where I is the determinant of the inertia matrix of K_n regarded as a body of uniform density.

One way of tackling Sylvester's problem in two dimensions has been with the use of Crofton's Theorem (see [1] and [3]). This sets up a differential equation involving the expected value of $D(K_2)$ when one of the points, x_1 say, is chosen according to some probability law on the boundary of K_2, and x_2 and x_3 are chosen independently, at random from the interior of K_2. This reduces the total number of degrees of freedom of the random points by one, i.e. it reduces the number of integrations involved by one.

In this paper it is shown how a generalized form of Crofton's Theorem can be repeatedly applied, to simplify the problem of finding $V_{K_n}^h$ when K_n is the n-simplex Δ_n or the n dimensional parallelotope P_n.

In particular, in two dimensions it is shown for the triangle Δ_2, how $V_{\Delta_2}^h$ can be expressed as a linear combination of two basic expectations, in each of which the number of degrees of freedom of the random points is reduced by three. These basic expectations are found, and it is shown that

$$V_{\Delta_2}^h = \frac{12}{(h+1)^3(h+2)^3(h+3)(2h+5)}\left\{6(h+1)^2 + (h+2)^2 \sum_{r=0}^{h} \frac{1}{\binom{h}{r}^2}\right\}$$

A similar decomposition into two basic expectations holds for the parallelogram P_2, and it is shown that

$$V_{P_2}^h = \frac{3}{(h+1)(h+2)^3(h+3)^2 2^{h-2}}\left\{(h+1) + 2(h+2)\sum_{r=1}^{h+1}\frac{1}{r}\right\}$$

It is shown how in principle, a decomposition of $V^h_{K_n}$ into a linear combination of basic expectations in each of which the number of degrees of freedom is reduced by $n + 1$, could be obtained when K_n is either the n-simplex Δ_n or the n-dimensional parallelotope P_n.

However for $n > 2$ the evaluation of these basic expectations seems difficult.

In II an analytic formulation of the problem for the n-simplex Δ_n is developed along with a suitable notation.

In III the generalized Crofton theorem is used to obtain expressions for $V^h_{K_2}$ for the triangle and the parallelogram.

In IV the problem in higher dimensions is considered, and some numerical results given.

II. Random points in the n-simplex Δ_n.   The ratio of the volumes of two bodies in n-space is invariant under a nonsingular affine transformation. It follows that a uniform distribution over a body will be preserved under such a transformation.

Since any nondegenerate n-simplex Δ_n is affinely equivalent to a regular n-simplex S_n it follows that

$$D(\Delta_n) = C(\mathbf{y}_1, \dots, \mathbf{y}_{n+1})/(\text{volume of } S_n)$$

where $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n+1}$ are $n + 1$ points chosen independently at random in S_n.

Let $X_1, X_2, \dots, X_k$ be k independent random variables with distribution function

$$F(x) = \begin{cases} 0 & , \ x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}$$

and let

$$Y_j = \frac{X_j}{\sum\limits_{i=1}^{k} X_i}, \qquad\qquad (j = 1, \dots, k).$$

The k-dimensional vector random variable

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix}$$

has a uniform distribution over the regular $(k - 1)$-simplex

$$S_{k-1} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : x_j \geq 0 \ (j = 1, \ldots, k), \ \sum_{j=1}^{k} x_j = 1 \right\}$$

and furthermore $\mathbf{Y}$ is independent of the denominator

$$\sum_{i=1}^{k} X_i \, .$$

(This is a particular case of a k-variate Dirichlet distribution (see [10]).)

Suppose $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n+1}$ are the Cartesian column vectors of $n+1$ points in S_n. It can be shown that the ratio of the volume of the convex hull of these $n+1$ points to the volume of S_n is

$$|\det (\mathbf{y}_1, \mathbf{y}_2, \ldots \mathbf{y}_{n+1})|$$

and hence $D(\Delta_n)$ can be expressed

$$D(\Delta_n) = \frac{1}{\prod_{j=1}^{n+1} \left(\sum_{i=1}^{n+1} X_{ij} \right)} \left| \det((X_{ij}))_{i,j=1,\ldots,n+1} \right|$$

where X_{ij} $(i, j = 1, \ldots, n+1)$ are $(n+1)^2$ independent random variables with distribution function $F(x)$, and furthermore it can be shown that $D(\Delta_n)$ is independent of its denominator

$$\prod_{j=1}^{n+1} \left(\sum_{i=1}^{n+1} X_{ij} \right) .$$

Since the sum of independent exponential random variables has a gamma distribution it follows that the hth moment of $D(\Delta_n)$, will be

$$V_{\Delta_n}^h = \frac{E \left\{ \left| \det((X_{ij}))_{i,j=1,\ldots,n+1} \right|^h \right\}}{[(n+h)!/n!]^{n+1}} \, .$$

The second moment of the determinant of a matrix of independent, identically distributed random variables is known and thus

$$V_{\Delta_n}^2 = \frac{n!}{(n+1)^n (n+2)^n} \, .$$

Higher even order moments $V_{\Delta_n}^{2h}$ may admit a solution, by similar methods, but odd order moments look very difficult due to the presence of the absolute value sign.

However, as pointed out by Kingman [5] the even order moments determine the distribution of $D(\Delta_n)$ and thus determine the odd order moments in principle, although this hardly seems to offer a practical method of solving Klee's problem of finding $V_{\Delta_n}^1$.

It is obvious that if

$$X_1 = 0$$

and $X_j (j = 2, \ldots, n + 1)$ are independent r.vs with df. $F(x)$, then the $(n + 1)$-dimensional vector r.v.

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{n+1} \end{pmatrix}$$

where

$$Y_j = \frac{X_j}{\sum\limits_{i=1}^{n+1} X_i} \,, \qquad (j = 1, \ldots, n)$$

represents a point at random in the $(n - 1)$ simplex S_{n-1} which forms the "face" of S_n opposite to the vertex

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Similarly if $X_1 = X_2 = 0$, all other X_j independent r.vs. with df. $F(x)$, then $\mathbf{Y}$ represents a point at random in the $(n - 2)$-simplex S_{n-2} which forms that part of the boundary of S_n opposite to the "edge" joining vertices

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $X_{1, i_1} = X_{2, i_2} = \ldots = X_{k, i_k} = 0$, $k \leq n + 1$ for some sequence $(i_1, i_2, \ldots, i_k)$ and suppose all other X_{ij} are r.vs with d.f. $F(x)$.

Define the vectors

$$Y_j = \begin{pmatrix} Y_{1,j} \\ \vdots \\ Y_{n+1,j} \end{pmatrix} \qquad \text{for } j = 1, \dots, n+1$$

where

$$Y_{ij} = \frac{X_{ij}}{\Sigma_{i=1}^{n+1} X_{ij}}$$

and define the matrix $M_n(i_1, \dots, i_k) = (Y_1, \dots, Y_{n+1})$.

This matrix represents $n + 1$ random points each in some simplex of the boundary S_n or in S_n itself. Note that no two points can have a common zero barycentric coordinate, since M_n has at most one zero in each row.

We shall be concerned only with functions of $n + 1$ points in S_n which:

(a) are symmetric in the sense of being invariant under permutations of the points.

(b) depend only on intrinsic properties of the points and their relative positions and not on their positions relative to S_n.

Such a function will be invariant under permutations of rows and columns of the configuration matrix $M_n(i_1, \dots, i_k)$.

It should be fairly clear that by permuting rows and columns of the matrix M_n it can be arranged that the sequence $(i_1, \dots, i_k)$ satisfies the following.

(1)   it is nondecreasing.

(2)   for each r, $1 \le r \le i_k$, there exists j, $1 \le j \le k$, such that $i_j = r$.

(3)   for each r, $1 \le r \le i_k$ define $m(r)$ as the number of i_j's which equal r. Then for each r, s such that $1 \le r < s \le i_k$ it follows that $m(r) \ge m(s)$.

Under the operations of permuting points or relabelling vertices, any sequence of k positive integers, $k \le n + 1$, satisfying (1), (2) and (3) above will determine an equivalence class in the set of configurations of $n + 1$ random points chosen either in S_n or in a simplex of the boundary of S_n in such a way that no two points have a common zero barycentric coordinate.

For a function ϕ satisfying (a) and (b) and any sequence $(i_1, \dots, i_k)$ satisfying (1), (2) and (3) we define

$$\mu_n(i_1, \dots, i_k) = E\{\phi(M_n^*(i_1, \dots, i_k)\}$$

where $M_n^*(i_1, \dots, i_k)$ is any matrix obtained by permuting rows and columns of $M_n(i_1, \dots, i_k)$.

We also define $\mu_n = E\{\phi(Y_1, \dots, Y_{n+1})\}$ if all X_{ij} have d.f. $F(x)$, $i, j = 1, \dots, n + 1$.

To illustrate the notation we show the case of $n = 2$. The dots in the matrix represent the nonzero $Y_{i,j}$.

Expected Value of ϕ	Configuration Matrix	S_2	Situation
μ_2	$\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$		3 points at random in S_2.
$\mu_2(1)$	$M_2(1) = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$		2 points at random in S_2, one at random on an edge.
$\mu_2(1,1)$	$M_2(1,1) = \begin{bmatrix} 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$		2 points at random in S_2, one fixed at a vertex.
$\mu_2(1,2)$	$M_2(1,2) = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}$		one point at random in S_2, one at random on each of two edges.
$\mu_2(1,1,2)$	$M_2(1,1,2) = \begin{bmatrix} 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \end{bmatrix}$		one point at random in S_2, one fixed at a vertex, and one at random on opposite edge.
$\mu_2(1,2,3)$	$M_2(1,2,3) = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & 0 \end{bmatrix}$		one point at random on each of 3 edges.

It will be shown how μ_n can be decomposed into a linear combination of basic expectations

$$\mu_n(i_1, \dots, i_{n+1}).$$

III. **Application of a generalized form of Crofton's theorem.** Crofton's Theorem (see [1], [3]) gives a differential relationship between the expected value of a function of N points chosen independently at random in a domain D, and the expected value of that function when one of the points is constrained to the boundary of D and the other $N-1$ are selected at random in D.

In [6] the following more general theorem is proved.

THEOREM. *Suppose that for $j = 1, \ldots, k$, D_j is a domain in n_j-dimensional Euclidean space, with volume V_j, and that N_j points are chosen independently at random in D_j.*

Let μ be the expected value of some numerical function of the $\Sigma_{j=1}^{k} N_j$ points, which depends only on intrinsic properties of the points and their relative positions and not on the domains D_j or the positions of the points relative to them, and suppose that under an affine transformation μ changes only through $V_1, \ldots, V_k$.

If each domain D_j is incremented by an infinitesimal around ΔV_j, $j = 1, \ldots, k$, in such a way as to preserve affine equivalence with the original situation, then

$$\Delta \mu = \sum_{j=1}^{k} N_j (\mu_j^* - \mu) \frac{\Delta V_j}{V_j}$$

where Δ_μ is the increment in μ, and for $j = 1, \ldots, k$, μ_j^* is the expected value of the function when one point is chosen at random in the increment of D_j, $N_j - 1$ points are chosen at random in D_j, and N_l points are chosen at random in D_l, for $l \neq j$, $l = 1, \ldots, k$.

The theorem is now applied in finding the expected value of functions of three random points in the triangle Δ_2.

Let V be the area of a regular (equilateral) triangle S_2, and let s be the length of the sides. Let ϕ be a symmetric function of three points in S_2 depending only on intrinsic properties of the points, and let μ_2 be the expected value of ϕ, which we shall assume changes under an affine transformation only through V.

We apply the theorem with three points in one domain (S_2), incrementing S_2 by a narrow band of area ΔV parallel to one of the edges. (This is simply Crofton's original theorem.)

 ΔV

Using the notation developed in II, we have

$$\Delta \mu_2 = 3(\mu_2(1) - \mu_2) \frac{\Delta V}{V}$$

from which we obtain

(1) $\frac{d}{dV} \{ V^3 \mu_2 \} = 3 \mu_2(1) V^2.$

We now apply the theorem again for $\mu_2(1)$. The situation is now that there are two points at random in S_2, and one point at random in the side of S_2. Incrementing S_2 along an edge other than the one containing a point we obtain

$$\Delta\mu_2(1) = 2[\mu_2(1,2) - \mu_2(1)]\frac{\Delta V}{V} + 1\cdot[\mu_2(1,1) - \mu_2(1)]\frac{\Delta s}{s}$$

and since $V = s^2\sqrt{3}/4$ it follows that $\Delta V/V = 2\Delta s/s$ and hence

(2) $$\frac{d}{dV}\{V^{5/2}\mu_2(1)\} = [2\mu_2(1,2) + \frac{1}{2}\mu_2(1,1)]V^{3/2}.$$

Applying the theorem again for $\mu_2(1, 2)$ incrementing S_2 along the edge containing no point gives

$$\Delta\mu_2(1,2) = [\mu_2(1,2,3) - \mu_2(1,2)]\frac{\Delta V}{V} + [\mu_2(1,1,2) - \mu_2(1,2)]\frac{\Delta s}{s}$$

$$+ [\mu_2(1,1,2) - \mu_2(1,2)]\frac{\Delta s}{s}$$

which gives

(3) $$\frac{d}{dV}\{V^2\mu_2(1,2)\} = [\mu_2(1,2,3) - \mu_2(1,1,2)]V.$$

We apply the theorem again for $\mu_2(1, 1)$ incrementing S_2 along the edge opposite to the vertex at which the point is fixed.
This gives

$$\Delta\mu_2(1,1) = 2[\mu_2(1,1,2) - \mu_2(1,1)]\frac{\Delta V}{V}$$

which gives

(4) $$\frac{d}{dV}\{V^2\mu_2(1,1)\} = 2\mu_2(1,1,2)V.$$

If the expectations $\mu_2(1, 2, 3)$ and $\mu_2(1, 1, 2)$ can be determined for a given function of the random points, then in principle it should be possible to determine μ_2 by integrating equations (1) through (4).

Since a distribution function can be expressed as the expected value of an indicator random variable, in principle we can determine the distribution function of a given function of the random points if we can determine it in the two constrained cases.

Before considering these constrained expectations we make some re-
marks concerning the way in which the theorem has been applied.

Firstly we have always incremented S_2 along an edge to which no
points are constrained. A close examination of the proof of the theorem [6]
will reveal that it would not be valid to apply the theorem if this were not
the case. This is because the expectation in the case with the enlarged
domain must decompose, to first order, into two parts—one is μ^* the
constrained expectation and the other reduces to μ, the expectation with the
original domains. If we were to increment S_2 along an edge containing a
point the latter part would be fundamentally different from μ.

Secondly we have always incremented S_2 parallel to an edge. This
ensures that the enlarged domain is still a regular triangle and that in the
resulting constrained case the point will be chosen at random from the
edge. However the regularity of the enlarged domain is not essential, since
we assume that within an affine equivalence class μ depends only on V.
Thus it would be sufficient that the enlarged domain be any triangle. This
would be the case if S_2 were enlarged by a wedge-shaped increment.

In this case the resulting constrained point will have a distribution with
a linear density function along the edge, and also $\Delta V/V = \Delta s/s$. In this way
it is possible to decompose $\mu_2(1, 1, 2)$ further, but not $\mu_2(1, 2, 3)$ because of
our earlier remark about the validity of the theorem's application.

We now solve (1) to (4) for the case when ϕ is a function which varies
with V proportionally to V^h.

In this case $\mu_2 = \hat{\mu}_2 V^h$

$$\mu_2(1) = \hat{\mu}_2(1)V^h$$

etc., where the $\hat{\mu}_2$'s are independent of V. We can rewrite (1) to (4) as

$$\frac{d}{dV}\{\hat{\mu}_2 V^{h+3}\} = 3\hat{\mu}_2 V^{h+2}$$

$$\frac{d}{dV}\{\hat{\mu}_2(1) V^{5/2+h}\} = [2\hat{\mu}_2(1,2) + \frac{1}{2}\hat{\mu}_2(1,1)]V^{h+3/2}$$

$$\frac{d}{dV}\{\hat{\mu}_2(1,2) V^{h+2}\} = [2\hat{\mu}_2(1,2,3) + \hat{\mu}_2(1,1,2)]V^{h+1}$$

$$\frac{d}{dV}\{\hat{\mu}_2(1,1)\,V^{h+2}\} = 2\hat{\mu}_2(1,1,2)\,V^{h+1}.$$

Solving these and eliminating $\hat{\mu}_2(1)$, $\hat{\mu}_2(1,2)$ and $\hat{\mu}_2(1,1)$ gives

$$\hat{\mu}_2 = \frac{6}{(h+2)(h+3)(2h+5)}[3\hat{\mu}_2(1,1,2) + 2\hat{\mu}_2(1,2,3)]$$

and a similar equation holds for the μ_2, $\mu_2(1,1,2)$ and $\mu_2(1,2,3)$.

It seems that this is not altogether a new result. In a footnote [2] Crofton mentions that J. J. Sylvester has discovered "remarkable results" which he calls a "decomposition of probabilities." This is essentially (5) with $h = 0$, the μ_2's being in this case probabilities, which can be thought of as expectations of indicator functions, independent of V.

There is a vague general account of a paper Sylvester read to the British Association [7] concerning the result. He used the result in a solution to a problem in the *Educational Times*, to find $V^1_{\Delta_2}$ [8] and proposes a question in a later issue of that journal [9] which asks for a proof. No proof was forthcoming and as far as can be discovered Sylvester never gave any further details.

When $\phi = $ (area of the triangle formed by three random points)^h

$$V^h_{\Delta_2} = E\left(\frac{\varphi}{V^h}\right) = \hat{\mu}_2.$$

The basic expectations $\hat{\mu}_2(1,1,2)$ and $\hat{\mu}_2(1,2,3)$ are easily found in this case. By the methods of II

$$\hat{\mu}_2(1,1,2) \approx \frac{E\left\{\left|\det\begin{bmatrix} 0 & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ X_{31} & 0 & X_{33} \end{bmatrix}\right|^h\right\}}{E\{X_{31}^h\}\,E\{(X_{12}+X_{22})^h\}\,E\{(X_{13}+X_{23}+X_{33})^h\}}$$

$$= \frac{V^h_{K_1}\,E\{(X_{13}+X_{23})^h\}}{E\{(X_{13}+X_{23}+X_{33})^h\}}$$

since

$$V^h_{K_1} = E\left\{\left|\det\begin{bmatrix} X_{12} & X_{13} \\ X_{22} & X_{23} \end{bmatrix}\right|^h\right\}\Big/ E\{(X_{12}+X_{22})^h\}\,E\{(X_{13}+X_{23})^h\}.$$

Thus

(7) $\quad \hat{\mu}(1, 1, 2) = \dfrac{2}{(h+1)(h+2)} \dfrac{\Gamma(h+2)}{\Gamma(2)} \dfrac{\Gamma(3)}{\Gamma(h+3)} = \dfrac{4}{(h+1)(h+2)^2}$

Also

$$\hat{\mu}_2(1,2,3) = \frac{E\left\{\left|\det \begin{bmatrix} 0 & X_{12} & X_{13} \\ X_{21} & 0 & X_{23} \\ X_{31} & X_{32} & 0 \end{bmatrix}\right|^h\right\}}{E\{(X_{21} + X_{31})^h\}\, E\{(X_{12} + X_{32})^h\}\, E\{(X_{13} + X_{23})^h\}}$$

$$= E\{(X_{12}X_{23}X_{31} + X_{13}X_{21}X_{32})^h\}/[(h+1)!]^3$$

$$= \frac{1}{[(h+1)!]^3} \sum_{r=0}^{h} \binom{h}{r} E\{X_{12}^r X_{23}^r X_{31}^r X_{13}^{h-r} X_{21}^{h-r} X_{32}^{h-r}\}$$

(8) $\qquad = \dfrac{1}{(h+1)^3} \displaystyle\sum_{r=0}^{h} \dfrac{1}{\binom{h}{r}^2}.$

From (6), (7), and (8),

$$V_{\Delta_2}^h = \frac{12}{(h+1)^3(h+2)^3(h+3)(2h+5)} \left\{ 6(h+1)^2 + (h+2)^2 \sum_{r=0}^{h} \frac{1}{\binom{h}{r}^2} \right\}.$$

A similar decomposition can be obtained for the affine equivalence class of parallelograms, P_2. Repeated application of the theorem for functions proportional to V^h gives

$$\mu = \frac{12}{(h+2)^2(h+3)^2}(\mu^* + 2\mu^{**})$$

where μ^* is the expectation when a point is fixed at each of two opposite vertices, and a third point is chosen at random from the interior, and μ^{**} is the expectation when one point is fixed at a vertex and a point is chosen at random from each of the two sides non-incident with that vertex.

The hth moment of the area of the triangle formed by the three points in each of the above situations is easily found by considering the configurations in the square. From this we obtain

$$V_{P_2}^h = \frac{3}{(h+1)(h+2)^3(h+3)^2 2^{h-2}} \left\{ (h+1) + 2(h+2) \sum_{r=1}^{h+1} \frac{1}{r} \right\}.$$

IV. Application of the theorem in higher dimensions. In higher dimensions the problem does not admit such a simple solution.

Consider the general case of the simplex S_n in n dimensions. The generalized Crofton theorem can be repeatedly applied. Bearing in mind the remarks in III about the validity of the application of the theorem we see that S_n can be incremented parallel to a "face" named by the vertex opposite to it only if none of the $n + 1$ points has zero barycentric coordinate for that vertex. Thus at each iteration of the theorem a new expectation arises for a situation with a configuration matrix (as described in II) with one extra zero in a row which did not previously contain one.

When each row of the configuration matrix contains exactly one zero we will have exhausted the possibilities obtainable by incrementing S_n parallel to one of its faces. We will call these the basic expectations. Of course it will be possible to decompose some of these basic expectations further by incrementing S_n in other ways, but these cases seem rather difficult to deal with.

If we are considering only intrinsic, symmetric functions of the $n + 1$ points it will be seen that each basic expectation can be represented

$$\mu_n(i_1, i_2, \ldots, i_{n+1})$$

where $i_j, j = 1, \ldots, n + 1$ satisfy the properties (1), (2), (3) of II, with the further restriction that they do not all equal one. And furthermore it will be seen that each such sequence represents a basic expectation which can be arrived at by successive decomposition, using the theorem.

Thus the number of basic expectations will be the number of such sequences

viz $$p(n + 1) - 1$$

where $p(m)$ is the number of ways the integer m can be partitioned into integral parts, the order of the parts being immaterial.

For example, 3 is capable of partition as $(1, 1, 1)$, $(1, 2)$, (3) leading to the basic expectations $\mu_2(1, 2, 3)$ and $\mu_2(1, 1, 2)$ for the case $n = 2$. (The last partition (3) is discarded since it yields 3 zeros in one column.)

Again 4 is capable of partition as $(1, 1, 1, 1)$, $(1, 1, 2)$, $(1, 3)$, $(2, 2)$, (4), and in the case $n = 3$ we have the following corresponding basic expectations

$\mu_3(1, 2, 3, 4)$, $\mu_3(1, 1, 2, 3)$, $\mu_3(1, 1, 1, 2)$ and $\mu_3(1, 1, 2, 2)$.

A closed form expression for $p(m)$ is not known in general, and it has not been possible to obtain a general decomposition formula for the n-dimensional case.

For any n it is, in principle, possible to apply the theorem repeatedly and write down the solutions to the differential equations leading to a decomposition.

For example for $n = 3$, if s is the length of an edge, A the area of a face and V the volume of S_3, then

$$3\frac{ds}{s} = \frac{3}{2}\frac{dA}{A} = \frac{dV}{V}$$

and by applying the theorem repeatedly as in II we obtain seven differential equations in the μ_3's which on solution for functions varying as V^h give:

$$\mu_3 = \frac{4}{3(h+3)(h+4)(3h+10)(3h+11)}\left\{\begin{array}{l}170\mu_3(1,1,1,2) + 178\mu_3(1,1,2,2) \\ + 606\mu_3(1,1,2,3) + 36\mu_3(1,2,3,4)\end{array}\right\}.$$

When we consider the volume of the convex hull of the four points we can show

$$\mu_3(1, 1, 1, 2) = \frac{3}{64} \quad \text{and} \quad \mu_3(1, 2, 3, 4) = \frac{1}{27}$$

but the other two basic expectations have not been evaluated. They are the expected value of the volume of the tetrahedron formed by four points when there is:

(a)   one point at random on each of two opposite edges of S_3, and two points at random in the interior of S_3

(b)   one point at random in an edge of S_3, and one point at random in each of the two faces not incident with that edge, and the fourth point at random in the interior of S_3.

For higher values of n an algorithm could easily be developed to obtain the decomposition into the linear combination of the $p(n + 1) - 1$ basic expectations. However in dimensions higher than two it seems that the basic expectations are difficult to evaluate.

Similarly for the n-dimensional parallelotope P_n a decomposition could be achieved by repeated iteration of the theorem. We give some numerical values for $V_{\Delta_n}^h$. Firstly we give exact values of $V_{\Delta_n}^2$ calculated from the formula in II.

n	1	2	3	4	5
$V^2_{\Delta_n}$	1/6	1/72	3/4000	8/270,000	9.189×10^{-7}

n	6	7	8	9	10
$V^2_{\Delta_n}$	2.334×10^{-8}	5.027×10^{-10}	9.374×10^{-12}	1.539×10^{-13}	2.260×10^{-15}

For $n = 2$ the first four moments $V^h_{\Delta_2}$ of the distribution of $D(\Delta_2)$ are calculated from the formula in III.

h	1	2	3	4
$V^h_{\Delta_2}$	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{3}{1375}$	$\frac{1}{9000}$

For $n = 3, \ldots, 10$ we give estimates of the first four moments $V^h_{\Delta_n}$ of the distribution of $D(\Delta_n)$ based on Monte-Carlo studies with samples of size 10,000.

n	$V^1_{\Delta_n}$	$V^2_{\Delta_n}$	$V^3_{\Delta_n}$	$V^4_{\Delta_n}$
3	1.763×10^{-2}	7.747×10^{-4}	5.971×10^{-5}	6.559×10^{-6}
4	3.124×10^{-3}	2.712×10^{-5}	4.345×10^{-7}	1.023×10^{-8}
5	5.185×10^{-4}	9.029×10^{-7}	3.960×10^{-9}	3.364×10^{-11}
6	7.991×10^{-5}	2.440×10^{-8}	2.085×10^{-11}	3.699×10^{-14}
7	1.109×10^{-5}	5.125×10^{-10}	5.993×10^{-14}	1.185×10^{-17}
8	1.437×10^{-6}	1.093×10^{-11}	3.927×10^{-16}	3.324×10^{-20}
9	1.780×10^{-7}	1.398×10^{-13}	2.631×10^{-19}	7.923×10^{-25}
10	2.035×10^{-8}	1.954×10^{-15}	4.731×10^{-22}	1.863×10^{-28}

The first four moments for the parallelogram P_2 are calculated from the formula in II

h	1	2	3	4
$V^h_{P_2}$	$\frac{11}{144}$	$\frac{53}{4800}$	$\frac{149}{72,000}$	$\frac{9}{19,600}$

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