STRONGLY REGULAR GRAPHS AND GROUP DIVISIBLE DESIGNS

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The counting techniques of the author's earlier work on strongly regular graphs are used to prove the converse of a result of R. C. Bose and S. S. Shrikhande on geometric and pseudo-geometric graphs \((q^2 + 1, q + 1, 1)\).

0. Introduction. In the present paper, we use the counting techniques of the author's earlier work [5] to prove the converse of a result of R.C. Bose and S.S. Shrikhande [3] on geometric and pseudo-geometric graphs \((q^2 + 1, q + 1, 1)\).

Section 1 is devoted to preliminaries on strongly regular graphs and group divisible designs. We also give a brief description of the problem under consideration and a statement of our main result Theorem 1.1. Section 2 contains the proof of Theorem 1.1.

We refer to [3] for the necessary background. Throughout this paper \(I\) will denote an identity matrix and \(J\) a square matrix of all ones. Also \(j\) and \(O\) will denote row vectors of all ones and zeros respectively. Finally, \(|S|\) denotes the cardinality of the set \(S\).

1. Preliminary results and the statement of the main result Theorem 1.1.

A strongly regular graph [1] is a graph on \(v\) vertices, without loops or multiple edges and whose standard \((0, 1)\) adjacency matrix \(A\) satisfies

\[
AJ = JA = n_1 J
\]

and

\[
A^2 = n_1 I + \lambda A + \mu(J - I - A).
\]

The parameters of a strongly regular graph are then denoted by

\[
v, n_1, \lambda, \mu.
\]
Let $v = mn$ objects (= treatments) be partitioned into $m$ disjoint sets $S_i (i = 1, 2, \ldots, m)$, each containing $n$ objects. Let two objects be called adjacent if and only if they belong to the same set $S_i$. We then get a strongly regular graph, which is traditionally called a group divisible (G.D.) association scheme. The parameters of a G.D. scheme are given by

$$v = mn, n_1 = n - 1, \lambda = n - 2, \mu = 0 \quad (n \geq 2).$$

We observe that for a G.D. scheme, the $mn \times mn$ adjacency matrix (= association matrix) $C$ has the form

$$C = \text{diag}[J_n - I_n, J_n - I_n, \ldots, J_n - I_n].$$

Suppose now that we have a G.D. scheme on $v = mn$ treatments as above. A G.D. design $D(v, b, r, k, m, n, \lambda_1, \lambda_2)$ is an arrangement of these $v$ treatments $t_1, t_2, \ldots, t_v$ into $b$ distinct subsets $B_1, B_2, \ldots, B_b$ (called blocks) satisfying the following conditions:

1. $|B_i| = k \quad (i = 1, 2, \ldots, b)$
2. Each treatment occurs in exactly $r$ blocks.
3. Two treatments from the same set $S_i$ appear together in exactly $\lambda_1$ blocks and two treatments from distinct sets $S_i$ and $S_j$ occur together in exactly $\lambda_2$ blocks.

The parameters of a G.D. design are denoted by

$$v, b, r, k, m, n, \lambda_1, \lambda_2.$$

A G.D. design $D$ is called semi-regular group divisible (S.R.G.D.) if $r > \lambda_1$ and $rk = \lambda_2 v$. Bose and Connor [2] have shown that for a S.R.G.D. design, $m$ divides $k$ and each block contains $k/m$ treatments from each set $S_i (i = 1, 2, \ldots, m)$.

We now indicate the problem considered in the present paper. Let $D$ be a S.R.G.D. design with parameters (1.6). Let $t_1, t_2, \ldots, t_v$ and $B_1, B_2, \ldots, B_b$ denote the treatments and blocks of $D$ respectively. Suppose $D$ has the additional property that there exist distinct nonnegative integers $\mu_1$ and $\mu_2$ satisfying $|B_i \cap B_j| \in \{\mu_1, \mu_2\} (i \neq j)$. We construct the block graph $B$ of $D$ as follows. Take the vertices of $B$ to be the blocks of $D$. Define blocks $B_i, B_j (i \neq j)$ to be adjacent if and only if $|B_i \cap B_j| = \mu_1$.

Let $N$ denote the usual $v \times b$ ($0, 1$) incidence matrix of $D$. Let $C$ be given by (1.5). Define
We note that $A$ is a symmetric $(0, 1)$ matrix of size $b + v + 1$, and has zero trace. Therefore $A$ is the adjacency matrix of a graph. We wish to find necessary and sufficient conditions on the parameters of $D$, so that $A$ is strongly regular.

In [3], the converse situation was investigated. There, one starts with a very specific strongly regular graph, namely a pseudo-geometric graph $(q^2 + 1, q + 1, 1)$ $(q \geq 2)$. (See [1] for a general discussion of geometric and pseudo-geometric graphs $(r, k, \ell)$). The adjacency matrix $A$ of this graph can be brought to the form (1.7), where $C, N, B$ are now $(0, 1)$ matrices of the appropriate form. If further, $A$ has the properties $(P)$ and $(P^*)$ as in the notation of [3], then it was shown that $N$ is the incidence matrix of a S.R.G.D. design $D$ and $C$ is given by (1.5). Moreover the blocks of $D$ have two intersection cardinalities $\mu_1, \mu_2$ and $B$ is the block graph of $D$.

Specifically, the parameters of the S.R.G.D. design $D$ were shown to be

$$
(1.8) \begin{align*}
&v = q(q^2 + 1), \ b = q^4, \ r = q^3, \ k = q^2 + 1, \ m = q^2 + 1, \\
n = q, \ \lambda_1 = 0, \ \lambda_2 = q^2, \ \mu_1 = 1, \ \mu_2 = q + 1
\end{align*}
$$

In this paper, we shall show that there are only two parametrically possible strongly regular graphs $A$, of the form (1.7), which can be obtained from S.R.G.D. designs in the above manner. One of these graphs is pseudo-geometric $(q^2 + 1, q + 1, 1)$.

The full content of our main result is the following:

**Theorem 1.1** Let $N$ be the incidence matrix of a S.R.G.D. design $D$ with parameters $v = mn$, $b$, $r$, $k$, $\lambda_1$, $\lambda_2$ having $m$ sets of $n$ treatments each. Suppose any two distinct blocks of $D$ intersect in $\mu_1$ or $\mu_2$ $(\neq \mu_1)$ treatments. Let $C$ be the association matrix of $D$ and let $B$ be the adjacency matrix of the blocks of $D$.

Then,

$$A = \begin{bmatrix}
0 & j_v & O_b \\
j_v' & C & N \\
O_b' & N' & B
\end{bmatrix}$$
represents a strongly regular graph if and only if the parameters of $D$ are given by

\[ v = q(q^2 + 1), b = q^4, r = q^3, k = q^2 + 1, m = q^2 + 1, n = q, \lambda_1 = 0, \lambda_2 = q, \mu_1 = 1, \mu_2 = q + 1 \quad (q \geq 2) \]

or

\[ v = 2n, b = n^2, r = n, k = 2, m = 2, n, \lambda_1 = 0, \lambda_2 = 1, \mu_1 = 1, \mu_2 = 0 \quad (n \geq 2). \]

Moreover, the corresponding strongly regular graphs $A$ are respectively pseudo-geometric $(q^2 + 1, q + 1, 1)$ or pseudo-geometric $(2, n + 1, 1)$.

2. Proof of Theorem 1.1. Let $D(v, b, r, k, m, n, \lambda_1, \lambda_2)$ be a S.R.G.D. design based on $m$ sets of $n$ treatments each. Let $t_1, t_2, \ldots, t_v$ and $B_1, B_2, \ldots, B_b$ denote the treatments and blocks of $D$. We assume further that any two distinct blocks of $D$ intersect in $\mu_1$ or $\mu_2$ (≠ $\mu_1$) treatments. Then the parameters of $D$ can be taken to be

\[ v = mn, b, r, k, m, n, \lambda_1, \lambda_2, \mu_1, \mu_2. \]

Let $N, B, C$ and $A$ be as in the statement of Theorem 1.1. Let $m_i$ denote the number of blocks intersecting a given block in $\mu_i$ treatments ($i = 1, 2$). Then clearly

\[ m_1 + m_2 = b - 1 \]

and

\[ m_1 \mu_1 + m_2 \mu_2 = k(r - 1). \]

Therefore, $B$ has constant row sum $m_1$ given by

\[ m_1 = \frac{k(r - 1) + \mu_2(1 - b)}{\mu_1 - \mu_2}. \]

Now, since $D$ is a S.R.G.D. design with incidence matrix $N$, we know from [2], that $NN'$ has eigenvalues $rk, r - \lambda_1$ and $rk - \lambda_2, \nu = 0$, with multiplicities $1, m(n - 1)$ and $m - 1$ respectively. Hence $N'N$ has eigenvalues $rk, r - \lambda_1$ and 0 with multiplicities $1, m(n - 1)$ and $b - m(n - 1) - 1$ respectively. But, we have
(2.5) \[ N'N = kI + \mu_1 B + \mu_2 (J - I - B). \]

Hence, from Frobenius’ theorem on commuting matrices, \( B \) has eigenvalues \( \theta_0, \theta_1, \theta_2 \) given by

\[
\theta_0 = \frac{k(r - 1) + \mu_2(1 - b)}{\mu_1 - \mu_2} = m_1, \quad \text{with multiplicity 1}
\]

\[
\theta_1 = \frac{(r - \lambda_1) + (\mu_2 - k)}{\mu_1 - \mu_2}, \quad \text{with multiplicity } m(n-1)
\]

\[
\theta_2 = \frac{(\mu_2 - k)}{(\mu_1 - \mu_2)}, \quad \text{with multiplicity } b - m(n-1) - 1.
\]

Thus, from Lemma 5, [4], \( B \) is strongly regular \((b, m_1, \alpha, \beta)\), where

\[
\alpha = m_1 + \theta_1 + \theta_2 + \theta_1 \theta_2, \quad \beta = m_1 + \theta_1 \theta_2.
\]

Let

(2.10) \[
A = \begin{bmatrix}
0 & j_v & O_b \\
\hat{j}_v' & C & N \\
O_b' & N' & B
\end{bmatrix}.
\]

Suppose \( A \) is strongly regular \((b + v + 1, n_1, \lambda, \mu)\). Any row sum of \( A \) is either \( v, n + r \) or \( k + m_1 \). Hence, for regularity we must have

\[
n_1 = v = n + r = k + m_1.
\]

Next, by considering any two treatments or any two blocks which are adjacent or nonadjacent, easy counting arguments in (2.10) give

\[
\lambda = n - 1 = (n - 1) + \lambda_1 = \mu_1 + \alpha
\]

and

\[
\mu = k = 1 + \lambda_2 = \mu_2 + \beta.
\]
From (2.12), we see that $\lambda_1 = 0$. This together with the Bose-Connor property mentioned in §1, implies that every block contains exactly one treatment from each set. Hence the parameters (2.1) of $D$ can be taken as

\begin{align*}
(2.14) \quad v &= mn, \quad b = n^2 \lambda_2, \quad r = \lambda_2 n, \quad k = m, \quad n, \quad \lambda_1 = 0, \quad \lambda_2, \quad \mu_1, \quad \mu_2.
\end{align*}

Next, consider a treatment $t_i$ and a block $B_j$ such that $t_i \in B_j$. Denoting $N = (n_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, we have from (2.10),

$$
\lambda = |\{l : c_{il} = 1 = n_{ij}, \; 1 \leq l \leq v\}| + |\{l : n_{il} = 1 = b_{ij}, \; 1 \leq l \leq b\}|.
$$

Using the Bose-Connor property, we get

\begin{align*}
(2.16) \quad \lambda &= |\{B_i : l \neq j, \; t_i \in B_i \text{ and } |B_i \cap B_j| = \mu_1\}|.
\end{align*}

Let $B_j = \{t_i, y_1, y_2, ..., y_{k-1}\}$, $B_i = \{t_i, x_1, x_2, ..., x_{k-1}\}$ (say). Since $\lambda_1 = 0$, each pair $(t_i, y_p)$, $1 \leq p \leq k - 1$ occurs $\lambda_2$ times in the blocks of $D$. Counting the distribution of these pairs in two ways, we get

\begin{align*}
(2.17) \quad \lambda &= \frac{(k - 1) (\lambda_2 - 1) - (\mu_2 - 1) (r - 1)}{\mu_1 - \mu_2}.
\end{align*}

Next, consider a treatment $t_i$ and a block $B_j$ such that $t_i \notin B_j$. Then using the Bose-Connor property, a similar type of counting yields

\begin{align*}
(2.18) \quad \mu &= \frac{(k - 1) \lambda_2 + (\mu_1 - \mu_2) - r \mu_2}{\mu_1 - \mu_2}.
\end{align*}

Then, (2.12), (2.17) and (2.13), (2.18) imply that

\begin{align*}
(2.19) \quad (n - 1)(\mu_1 - \mu_2) &= (k - 1)(\lambda_2 - 1) - (\mu_2 - 1)(r - 1)
\end{align*}

and

\begin{align*}
(2.20) \quad (k - 1)(\mu_1 - \mu_2) &= (k - 1)\lambda_2 - r \mu_2.
\end{align*}

Then, (2.19) and (2.20) give

\begin{align*}
(2.21) \quad \mu_1 + r - k &= (n - k + 1)(\mu_1 - \mu_2)
\end{align*}
and

\[(2.22) \quad \mu_2 + r - k = (n - k)(\mu_1 - \mu_2). \]

Next, using (2.13), (2.12), (2.11) and (2.9), we obtain

\[(2.23) \quad (\mu_1 + r - k)((\mu_1 - \mu_2)^2 + \mu_1 - k) = 0 \]

and

\[(2.24) \quad (\mu_2 + r - k)((\mu_1 - \mu_2)^2 + \mu_2 - k) + (\mu_1 - \mu_2)^2(n - k) = 0. \]

Thus,

\[(2.25) \quad (\mu_1 - \mu_2)(n - k + 1)((\mu_1 - \mu_2)^2 + \mu_1 - k) = 0. \]

and

\[(2.26) \quad (\mu_1 - \mu_2)(n - k)((\mu_1 - \mu_2)^2 + \mu_1 - k) = 0. \]

Since \(\mu_1 \neq \mu_2\), this gives

\[(2.27) \quad (\mu_1 - \mu_2)^2 + \mu_1 - k = 0. \]

Putting \(\mu_1 - \mu_2 = g\) in (2.27) gives

\[(2.28) \quad \mu_1 = k - g^2 \]

\[(2.29) \quad \mu_2 = k - g^2 - g. \]

Substituting these values in (2.22) we get

\[(2.30) \quad (n + g)\lambda_2 = (n + g)g. \]

Hence, either

\[\lambda_2 = g \quad (> 0) \quad \text{case (a)} \]

or

\[n = -g \quad (\geq 2) \quad \text{case (b)} \]
If case (a) holds, then 
\[ k = m = 1 + \lambda_2 = 1 + g \quad \text{and} \quad \mu_1 = g + 1 - g^2, \]
\[ \mu_2 = 1 - g^2. \]
But \( \mu_1 \neq \mu_2 \) and \( \mu_1 \geq 0, \mu_2 \geq 0 \) then imply that \( D \) has parameters

\[
(2.31) \quad \begin{cases}
  v = mn = 2n, & b = n^2, \quad r = n, \quad k = 2, \quad m = 2, \quad n \\
  \lambda_1 = 0, & \lambda_2 = 1, \quad \mu_1 = 1, \quad \mu_2 = 0.
\end{cases}
\]

Also, the parameters of \( \Lambda \) are then

\[
(2.32) \quad b + v + 1 = (n + 1)^2, \quad n_1 = 2n, \quad \lambda = n - 1, \quad \mu = 2 \quad (n \geq 2).
\]

Thus \( \Lambda \) is pseudo-geometric \((2, n + 1, 1)\).

Finally if case (b) holds, put

\[
(2.33) \quad n = -g = q \quad \text{(say)}.
\]

Then \ref{eq2.20}, together with \( \lambda_2 \neq 0, n \geq 2 \) implies that \( D \) has parameters

\[
(2.34) \quad \begin{cases}
  v = q(q^2 + 1), & b = q^4, \quad r = q^3, \quad k = q^2 + 1, \quad m = q^2 + 1, \\
  n = q, & \lambda_1 = 0, \quad \lambda_2 = q^2, \quad \mu_1 = 1, \quad \mu_2 = q + 1, \quad (q \geq 2).
\end{cases}
\]

And in this case, it is easily seen that \( \Lambda \) has parameters

\[
(2.35) \quad \begin{cases}
  b + v + 1 = (q + 1)(q^3 + 1), & n_1 = q(q^2 + 1), \\
  \lambda = q - 1, & \mu = q^2 + 1.
\end{cases}
\]

Thus, in this case \( \Lambda \) is pseudo-geometric \((q^2 + 1, q + 1, 1)\).

We have therefore established that if \( \Lambda \) is strongly regular, then \( D \) has parameters given by \ref{eq2.21} or \ref{eq2.22}. Moreover \( \Lambda \) is then pseudo-geometric \((2, n + 1, 1)\) or \((q^2 + 1, q + 1, 1)\) respectively.

Conversely it can be easily shown that if \( D \) has parameters given by \ref{eq2.21} or \ref{eq2.22}, then \( \Lambda \) is strongly regular and is pseudo-geometric \((2, n + 1, 1)\) or \((q^2 + 1, q + 1, 1)\) respectively.

This completes the proof of Theorem 1.1.

**Remarks.**

(i) The existence of S.R.G.D. designs \( D \) with parameters of case (1) in Theorem 1.1 and partial geometries \((q^2 + 1, q + 1, 1)\) is known for \( q \) a prime or prime power (See [1] and [3]).

(ii) The design \( D \) with parameters of case (2) in Theorem 1.1 is
known for any integer $n$ and is constructed as follows: Arrange $n^2$ treatments in an $n \times n$ array as

$$L = \begin{bmatrix}
1 & 2 & \cdots & n \\
 n+1 & n+2 & \cdots & 2n \\
 \vdots & \vdots & \ddots & \vdots \\
 n^2-n+1 & n^2-n+2 & \cdots & n^2
\end{bmatrix}$$

Write down $2n$ blocks corresponding to the rows and columns of $L$. We get a design $E$ where the blocks are the columns in

$$S_1$$

\begin{align*}
1 & \quad n+1 \quad \cdots \quad n^2-n+1 \\
2 & \quad n+2 \quad \cdots \quad n^2-n+2 \\
\vdots & \quad \vdots \quad \cdots \quad \vdots \\
n & \quad 2n \quad \cdots \quad n^2
\end{align*}

$$S_2$$

\begin{align*}
1 & \quad 2 \quad \cdots \quad n \\
n+1 & \quad n+2 \quad \cdots \quad 2n \\
\vdots & \quad \vdots \quad \cdots \quad \vdots \\
n^2-n+1 & \quad n^2-n+2 \quad \cdots \quad n^2
\end{align*}

The required design $D$ is the dual of $E$. It is easily seen that in this case the line graph $L_2(n+1)$ of the complete bipartite graph $K(n+1, n+1)$ has the same parameters as the graph $A$.

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