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**APPROXIMATION AND INTERPOLATION FOR SOME SPACES
OF ANALYTIC FUNCTIONS IN THE UNIT DISC**

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APPROXIMATION AND INTERPOLATION FOR SOME SPACES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Let U be a bounded open subset of the complex plane \mathbb{C} such that U and $\mathbb{C} \setminus \bar{U}$ are connected. (If $B \subset \mathbb{C}$, \bar{B} denotes its closure in \mathbb{C} .) $H^\infty(U)$ is the space of all bounded analytic functions defined on U . Let $S \subset U$ be the zero set of a nonzero function in $H^\infty(U)$.

Necessary and sufficient conditions on S are given for the existence of an open set $0 \supset \bar{U} \setminus (\bar{S} \setminus S)$ such that $H^\infty(0)$ and $H^\infty(U)$ have the same restrictions to S . If U is the unit disc $D = \{z : |z| < 1\}$ and S is as above, the following result holds for all the Hardy spaces $H^p(D)$, $0 < p \leq \infty$: Given $g \in H^p(D)$, there is a function f analytic in $\mathbb{C} \setminus (\bar{S} \setminus S)$ such that $f|_D \in H^p(D)$ and $f = g$ on S .

If S and U are as above, $H^\infty(U)|_S$ denotes the set of restrictions $f|_S$ of all $f \in H^\infty(U)$. If $S = \{z_n\} \subset D$ satisfies $\sum_n (1 - |z_n|) < \infty$, Detraz [3] proved the following result

(*): *There exists an open set $0 \supset \bar{D} \setminus (\bar{S} \setminus S)$ such that*
$$H^\infty(0)|_S = H^\infty(D)|_S.$$

In this paper we give two extensions of this result. First we show that (*) holds for domains of a somewhat more general type than the unit disc D . Consider the following statement which is very similar to (*):

(**) *There exists an open set V such that $\bar{V} \setminus (\bar{S} \setminus S) \subset D$ and*
$$H^\infty(V)|_S = H^\infty(D)|_S.$$

It turns out that conditions (*) and (**) are equivalent, even with D replaced by a somewhat more general set.

We shall make some use of the theory of the classical H^p spaces. We refer to [4] or [9] in this connection. Before stating our first result, we mention some more notation. If f is a complex valued function defined for each $z \in B$ we put $\|f\|_B = \sup \{|f(z)|, z \in B\}$. If $U \subset \mathbb{C}$ is open, $H^\infty(U)$ is a Banach algebra with sup norm on U and we denote by M the maximal ideal space of $H^\infty(U)$. The maximal ideals $m \in M$ are identified with the multiplicative functionals on $H^\infty(U)$ they correspond to. If $S \subset U$ is relatively closed and I denotes the set of all $f \in H^\infty(U)$ which are zero on

S , we define $\tilde{S} = \{m \in M: m(f) = 0 \quad f \in I\}$. (Cf. p. 345 in [3]). We have a projection $\Pi: M \rightarrow \bar{U}$ given by $m \rightarrow m(e)$ where $e \in H^\infty(U)$ is the function $z \rightarrow z$. For a detailed study of M we refer to [7] and Ch. 10 in [9]. Other results like (*) can be found in [1], [5], [8], [11] and [14].

With the notation as above we now state:

THEOREM 1. *Let U be the interior of a compact set X and assume both U and $\mathbb{C} \setminus X$ are connected. If $S \subset U$ is the zero set of a nonzero function in $H^\infty(U)$, the following statements are equivalent:*

- (i) *There exists an open set $0 \supset \bar{U} \setminus (\bar{S} \setminus S)$ such that $H^\infty(0)|_S = H^\infty(U)|_S$*
- (ii) *There exists an open set V such that $S \subset V \subset U$, $\bar{V} \setminus (\bar{S} \setminus S) \subset U$ and $H^\infty(V)|_S = H^\infty(U)|_S$*
- (iii) $\Pi(\tilde{S}) \subset \tilde{S}$.

REMARK. The author is indebted to the referee for an example where (iii) fails. For details of this example see the final remarks. If the boundary ∂U of U is a Jordan arc, it is easy to verify that (iii) holds, but considerably weaker conditions on ∂U also imply (iii).

Proof: If $\tilde{S} \supset \partial U$, the theorem trivially holds with $0 = V = U$. Assume now $(\partial U) \setminus \tilde{S} \neq \emptyset$. We prove the implications (ii) \Rightarrow (i), (i) \Rightarrow (iii) and (iii) \Rightarrow (ii). We assume first that (ii) is true and consider the restriction map $R: H^\infty(0) \rightarrow H^\infty(V)|_S$ where $0 \supset \bar{U} \setminus (\bar{S} \setminus S)$ is some open set and where $H^\infty(V)|_S$ has the quotient norm induced from $H^\infty(V)$. We need to prove that R maps $H^\infty(0)$ onto $H^\infty(V)|_S$. It is sufficient to find constants $L > 0$ and $\epsilon \in (0, 1)$ such that the image by R of the L -ball in $H^\infty(0)$ is ϵ -dense in the unit ball in $H^\infty(V)|_S$. (See for example Lemma 1.4. in [11].) Choose f in the unit ball of $H^\infty(V)|_S$. By (ii) and the open mapping theorem there is a constant c_1 independent of f , and $f_1 \in H^\infty(U)$ such that $f_1|_S = f$ and $\|f_1\|_U \leq c_1$. By Lemma 3.2 in [11] we can choose 0 such that for each $g \in H^\infty(U)$ there exists $g_1 \in H^\infty(0)$ such that

- (1) $\|g_1\|_0 \leq c_2 \|g\|_U$
- (2) $\|g - g_1\|_V \leq (2c_1)^{-1} \|g\|_U$

where c_2 is independent of g . (That we actually can apply Lemma 3.2 in [11] in this situation follows from well known estimates of analytic capacity. See for example the proof of Theorem 7.4 on page 213 in [6]). If we replace g by (f_1) in (1) and (2), we see that with $\epsilon = 1/2$ and $L = c_1 c_2$, Lemma 1.4 in [11] can be applied.

To see that (i) \Rightarrow (iii) we first observe that the restriction map R

defined above is not one-to-one. If it was, $\|f\|_0$ and $\|f\|_U$ would be equivalent norms on $H^\infty(0)$ by (i) and the open mapping theorem, and that is absurd. Hence there is some function $h \in H^\infty(0)$ which is zero on S but not identically zero in U . Choose $m \in M$ such that $\Pi(m) = z_0 \in \bar{U} \setminus \bar{S}$. Since h is analytic near z_0 we can write $h - h(z_0) = (z - z_0)h_1$ where $h_1 \in H^\infty(0)$. If we apply m on the right side we get zero and therefore $m(h) = h(z_0)$. Since we clearly can assume $h(z_0) \neq 0$ we have proved that $m \notin \bar{S}$ and (iii) follows.

It remains to prove that (iii) \Rightarrow (ii) and here we apply Carleson's lemma. (See [2] or on page 203 in [4].) Let $\varphi: U \rightarrow D$ be a conformal map and put $S_1 = \varphi(S)$. By (iii) S_1 must be countable and we let B denote the Blaschke product corresponding to S_1 . For definition and basic properties of Blaschke products we refer to [4] page 20 or [9] page 66. From these properties it is easy to see that $V_1 = \{z : |B(z)| < 2^{-1}\}$ satisfies $\bar{V}_1 \setminus (\bar{S}_1 \setminus S_1) \subset D$ and Carleson's lemma ([4] page 203) combined with a simple normal family argument, gives that $H^\infty(D)|_{S_1} = H^\infty(V_1)|_{S_1}$. If we define $V = \varphi^{-1}(V_1)$, it only remains to prove that $\bar{V} \setminus (\bar{S} \setminus S) \subset U$. Put $g = B \circ \varphi$. Choose an arbitrary point $z_0 \in (\partial U) \setminus \bar{S}$. If we can show that $|g(w_n)| \rightarrow 1$ whenever $\{w_n\}_{n=i}^\infty \subset U$ converges to z_0 , the proof will be complete.

Let $\{z_n\}$ be an arbitrary sequence in U converging to z_0 . We denote by J the ideal of all $h \in H^\infty(U)$ satisfying $\lim h(z_n) = 0$. We want to show that $g \notin J$. Let m denote some maximal ideal containing J . Since J contains the translation $z \rightarrow z - z_0$ we get that $\Pi(m) = z_0$. If $g \in J$ and $f \in H^\infty(U)$ vanishes on S , we can write $f = gf_1$, with $f_1 \in H^\infty(U)$. (see Thm. 2.8 on page 24 in [4]) and hence we get $m(f) = m(g)m(f_1) = 0$. This implies $m \in \bar{S}$ which is impossible by (iii) and since $\Pi(m) = z_0$. We can therefore assume that $|g| > t$ on $U_t = \bar{U} \cap \{z : |z - z_0| < t\}$ for some $t > 0$.

The proof is completed using some well known facts about $H^\infty(U)$ which we shall not prove. But references will be given below. We fix a point $w \in U$ and let λ denote the harmonic measure on ∂U which represents w . There is a (unique) function $g^* \in L^\infty(\lambda)$ whose harmonic extension to U equals g . (See for example [15] page 26.) We now claim:

(a) Since $|B| = 1$ a.e. on ∂D with respect to linear measure, $|g^*| = 1$ a.e. with respect to λ .

(b) Define g_1 on ∂U_t by $g_1 = g$ on $(\partial U_t) \cap U$ and $g_1 = g^*$ on $(\partial U_t) \setminus U$. Then the harmonic extension of g_1 to U_t equals the restriction g_2 of g to U_t . We can also assume that $|g_1| = 1$ on $(\partial U_t) \setminus U$.

Since $|g| > t$ on U_t we have from Jensen's inequality ([6] page 33–34) and (b) that the harmonic extension of $\log|g_1|$ to U_t equals $\log|g_2| = \log|g|$. But if $\{w_n\} \subset U$ converges to z_0 , we get that $\log|g_2(w_n)| \rightarrow 0$ since z_0 is regular for the Dirichlet problem for U_t . Since $g_2 = g$ in U_t this completes

the proof that (iii) \Rightarrow (ii). The claims (a) and (b) above are easy to justify using well known theory about harmonic measure and algebras of analytic functions. A convenient reference is the introductory part of [7]. (See in particular Lemma 2.2 and Lemma 4.4 in [7].)

We shall now prove that (*) holds for all the Hardy spaces $H^p(D)$, $0 < p \leq \infty$ and with $0 = C \setminus (\bar{S} \setminus S)$. We first prove a general result which may be of independent interest.

THEOREM 2. *Let A be a Banach space of functions on D with norm $N(\cdot)$. Assume A contains the polynomials in z and there exists constants M_n , $n = 1, 2, \dots$ such that:*

$$(1) \quad N(p|_D) \leq M_n \sup \{|p(z)| : |z| \leq 1 + n^{-1}\} \text{ for } n = 1, 2, \dots$$

if p is a polynomial. For each $z \in D$ assume the map $f \rightarrow f(z)$ is continuous on A .

Let $S \subset D$ and assume there exists an open set $0 \supset \bar{D} \setminus (\bar{S} \setminus D)$ such that each $g \in A|_S$ extends to a function f analytic in 0 such that $f|_D \in A$. Then such a function exists which even extends to be analytic in $C \setminus (\bar{S} \setminus D)$.

REMARKS. Note that (1) implies $f|_D \in A$ whenever f is analytic in a neighbourhood of \bar{D} and that we have estimates like (1) also for such functions.

Proof of Theorem 2. Denote by A_1 all analytic functions in 0 whose restriction to D are in A . We topologize A_1 by saying that a sequence $\{f_n\} \in A_1$ converges to $f \in A_1$ if and only if $N((f_n - f)|_D) \rightarrow 0$ and $\|f - f_n\|_K \rightarrow 0$ if K is a compact subset of 0 .

With this topology A_1 is a Frechet space and we can apply the open mapping theorem to the restriction map $A_1 \rightarrow A|_S$ where $A|_S$ has the quotient norm induced from A . $A|_S$ is then a Banach space since the set of functions in A vanishing on S must be closed by hypothesis. Choose an open set $0_1 \supset \bar{D} \setminus (\bar{S} \setminus D)$ such that $\bar{0}_1 \setminus \bar{D} \subset 0$. By the open mapping theorem there exists a constant M and constants M_K for each compact subset K of $\bar{0}_1 \setminus (\bar{S} \setminus D)$ such that each g in the unit ball of $A|_S$ extends to $h \in A_1$ such that

$$(i) \quad N(h|_D) \leq M$$

$$(ii) \quad |h| \leq M_K \text{ on } K \text{ if } K \subset \bar{0}_1 \setminus (\bar{S} \setminus D) \text{ is compact}$$

Now redefine h by setting $h \equiv 0$ in $C \setminus \bar{0}_1$. When we in the rest of the proof of Theorem 2 claim that a property holds independent of h , we shall mean

that this property holds for all $h \in A_1$ satisfying (i) and (ii) as above and extended to \mathbb{C} as above.

We can and shall assume 0_1 has the following property:

(2) $C \setminus \bar{0}_1$ is connected and there exists a constant L such that each $z \in C \setminus \bar{0}_1$ can be connected to a point in \bar{S} by an arc $\gamma_z \subset C \setminus \bar{0}_1$ such that (length of γ_z) $\leq L \operatorname{dist}(z, \bar{S} \setminus D)$.

With the notation as above the following lemma completes the proof of Theorem 2:

LEMMA 1. *Given $t > 0$ there exists constants C_K for each compact subset K of $C \setminus (\bar{S} \setminus S)$ such that for each function h as above we can find h_1 analytic in $C \setminus (\bar{S} \setminus D)$ such that $h_1|_D \in A$ with the following properties:*

- (a) $N((h - h_1)|_D) \leq t$
- (b) $|h_1| \leq C_K$ on each compact subset K of $C \setminus (\bar{S} \setminus D)$.

Indeed if Lemma 1 is proved, Theorem 2 follows by the same iteration argument as in the proof of Lemma 1.4 in [11].

The first part of the proof of Lemma 1 is very similar to the proof of Lemma 3.2 in [11], but for completeness we give most of the details.

Let $\{K_n\}_{n=1}^\infty$ be compact sets, $\{V_n\}_{n=1}^\infty$ open sets with the following properties:

- (i) $K_n \subset V_n, n = 1, 2, \dots$
- (ii) $\bar{V}_n \cap \bar{D} = \emptyset, n = 1, 2, \dots$
- (iii) $\bar{V}_n \cap \bar{V}_m = \emptyset$ if $|n - m| > 1$
- (iv) $(\partial 0_1) \setminus \bar{D} = \bigcup_n K_n$

(v) For each compact set $F \subset C \setminus (\bar{S} \setminus D)$, $F \cap \bar{V}_n = \emptyset$ if n is sufficiently large.

Fix n : Put $K = K_n, V = V_n$ and let $\varepsilon = \varepsilon_n$ be a positive number. Let $\delta > 0$ be given. Then cover C by open discs $\Delta_k = \Delta(z_k, \delta)$ (of radius δ and centered at z_k) and choose continuously differentiable functions ϕ_k (supported at Δ_k) as in the scheme for approximation described on page 210 in [6].

Let T_{ϕ_k} be the integral operator on $L^\infty(dx dy)$ defined by

$$\begin{aligned} T_{\phi_k}(f)(w) &= \frac{1}{\pi} \iint \frac{f(w) - f(z)}{w - z} \frac{\partial \phi}{\partial z} dx dy \\ &= f(w) \phi_k(w) + \frac{1}{\pi} \iint \frac{f(z)}{z - w} \frac{\partial \phi_k}{\partial z} dx dy \end{aligned}$$

We mention that $T_{\phi_k}(f)$ is analytic outside the support of ϕ_k and wherever f is and that $T_{\phi_k}(f)$ is continuous wherever f is. Also $f - T_{\phi_k}(f)$ is analytic in the interior of the set where ϕ_k attains the value 1. (See on page 28–29 in [6] for more details.)

Put $G_k = T_{\phi_k}(h)$ where h is as above. We are only interested in those k for which $\bar{\Delta}_k \cap K \neq \emptyset$. Assume this happens if and only if $1 \leq k \leq N$.

Then $h - \sum_1^N G_k$ is analytic near K since $\sum_1^N G_k = T_{(\sum_1^N \phi_k)}(h)$ and $\sum_1^N \phi_k \equiv 1$ in a neighborhood of K . We can assume $\delta > 0$ is so small that $\{z : |z - z_k| \leq 2\delta\} \subset V$ for $1 \leq k \leq N$.

Now there exist functions H_k , $k = 1, \dots, N$ analytic outside a compact subset of $D_k = \{w : |w - z_k| < 2\delta\} \setminus \bar{\Delta}_1$ such that $G_k - H_k$ has a triple zero in the Taylor expansion at infinity, and in our situation (since $C \setminus \bar{\Delta}_1$ is connected) we obtain $\|H_k\| < C_1 \|h\|_V$ where C_1 is an absolute constant. (See [6], Theorem 7.4 on page 213 and the proof of it). The important fact is that C_1 is independent of h .

We now list the facts which will be needed to prove Lemma 1.

(a) One can choose δ depending only on ε and $\text{dist}(K, C \setminus V)$ so small that the function $f = \sum_1^N (G_k - H_k)$ satisfies

$$\|f\|_{C \setminus V} < \varepsilon \|h\|_V$$

and we also have $\|f\|_\infty \leq C_2 \|h\|_V$ where C_2 is independent of h . ($\|f\|_\infty$ denotes ess.sup. of $|f|$ with respect to plane measure.)

(b) The functions H_k can be written as

$$H_k = \alpha_k(h)F_{k,1} + \beta_k(h)F_{k,2}$$

where $F_{k,1}$ and $F_{k,2}$ both are analytic outside a compact subset of D_k , they are independent of h and $\|F_{k,1}\|_\infty + \|F_{k,2}\|_\infty \leq 20$.

Here $\alpha_k(h)$ and $\beta_k(h)$ are complex numbers depending linearly on h and we have

$$(3) \quad |\alpha_k(h)| + |\beta_k(h)| \leq C_3 \|h\|_V$$

where C_3 is independent of h . (See the proof of Theorem 3.1 in [11] for more details about this.)

The functions $F_{k,1}$ and $F_{k,2}$ can now be approximated as well as we please in $C \setminus D_k$ by rational functions $R_{k,1}$ and $R_{k,2}$ with their poles in D_k $k = 1, 2, \dots, N$ so that if we define

$$(4) \quad f^* = \sum_{k=1}^N G_k - \alpha_k(h)R_{k,1} + \beta_k(h)R_{k,2}$$

then we have

$$(5) \quad \|f^*\|_{C \setminus V} < \varepsilon \|h\|_V < \varepsilon C_V$$

where C_V is a constant depending only on V . The existence of C_V comes from property (ii) of h listed above, and since $\bar{V} \cap \bar{D} = \emptyset$.

Note that from the remark following Theorem 2 there exists a constant C'_V also depending only on V such that from (5) we have

$$(6) \quad N(f^*|_D) \leq \varepsilon C'_V \|h\|_V < \varepsilon C_V C'_V.$$

Let now n vary and carry out this construction with $V = V_{2n-1}$ and $\varepsilon = \varepsilon_n$, $n = 1, 2, \dots$. In this way we obtain functions f_n^* , $n = 1, 2, \dots$ with the same properties as f^* has above. We can choose ε_n independent of h such that

$$(6') \quad \|f_n\|_{C \setminus V_{2n-1}} + N(f_n^*|_D) < t(2^{-2})2^{-n}$$

where t is the number in Lemma 1.

Now define $h' = h - \sum_n f_n^*$. By (6) and property (iii) of $\{V_n\}$, h' has the following property

$$(7) \quad h'|_D \in A \text{ and } N((h' - h)|_D) < t \cdot 2^{-2}$$

We now wish to repeat this construction with h replaced by h' and V_{2n-1} by V_{2n} , $n = 1, 2, \dots$. We have to be a bit careful because h' can be unbounded in V_{2n} for some n . But for $n = 1, 2, \dots$ it is easy to see that we can find open sets $W_n \subset V_{2n}$ such that $K_{2n} \subset W_n$ and such that none of the rational functions $R_{k,1}$ or $R_{k,2}$ used in the definition of f_n^* , $n = 1, 2, \dots$ has poles in W_n . But then it follows that there exists constants E_n , $n = 1, 2, \dots$ independent of h and h' such that

$$(8) \quad \|h'\|_{W_n} \leq E_n \quad \text{for } n = 1, 2, \dots$$

We can now repeat the above construction with h replaced by h' and V_{2n-1} replaced by W_n for $n = 1, 2, \dots$. We obtain functions g_n^* analytic in $C \setminus W_n$ in the same way as we obtained f_n^* .

Define $h^* = h - \sum_n f_n^* - \sum_n g_n^*$. In the same way as we obtained (7) we get

$$(9) \quad h^*|_D \in A \quad \text{and} \quad N((h^* - h)|_D) < t \cdot 2^{-1}.$$

From the properties of the T_φ -operator mentioned above one can also deduce that h^* is analytic in $\mathbf{C} \setminus (\bar{S} \setminus D)$ except for the poles of the rational functions $R_{k,1}$ and $R_{k,2}$ corresponding to each f_n^* and each g_n^* .

Let now K be a compact subset of $\mathbf{C} \setminus (\bar{S} \setminus D)$ and let Σ'_n denote summation over those n for which $\bar{W}_n \cap K = \emptyset$ and $V_{2n-1} \cap K = \emptyset$. It is easy to see that there exists a constant E_K depending on K but not on h such that

$$(10) \quad \|h - \sum'_n (f_n^* + g_n^*)\|_K \leq E_K.$$

We conclude that our function h^* satisfies almost Lemma 1. We get rid of the rational functions $R_{k,1}$ and $R_{k,2}$ by the following lemma

LEMMA 2. *Suppose $\eta > 0$ is given. Let p be a rational function with poles only at the points z_1, \dots, z_m in $\mathbf{C} \setminus \bar{D}_1$. Then there exists a function s analytic in $\mathbf{C} \setminus (\bar{S} \setminus D)$ and an open set $W \subset \mathbf{C} \setminus (\bar{S} \setminus D)$ such that*

- (i) $s|_D \in A$ and $\|s - p\|_{\mathbf{C} \setminus W} + N((s - p)|_D) < \eta$
- (ii) $\text{dist}(z, \bar{S} \setminus D) < 2L \max_{1 \leq k \leq m} \text{dist}(z_k, \bar{S} \setminus D)$ for

each $z \in W$, where L is as in condition (II) mentioned above.

Proof. It is clearly sufficient to prove this lemma when $m = 1$. We choose a polygonal arc $\gamma = \gamma_{z_1}$ as in condition (2).

Divide γ into subarcs γ_k with endpoints z_k and z_{k+1} , $k = 1, 2, \dots$ such that z_{k+1} is the only common point of γ_k and γ_{k+1} for each k .

Choose connected open sets $U_k \supset \bar{\gamma}_n$ for $k = 1, 2, \dots$ and rational functions p_k , $k = 1, 2, \dots$ (with $p = p_1$) with poles only at z_k such that

$$\|p_{k+1} - p_k\|_{\mathbf{C} \setminus U_k} + N((p_{k+1} - p_k)|_D) < \eta 2^{-k}$$

$k = 1, 2, \dots$. Since each U_k is connected and since we can assume $\bar{U}_k \cap \bar{D} = \emptyset$ this is easy to obtain. We can also assume $\bar{U}_k \cap K = \emptyset$ if k is sufficiently large and K is a given compact subset of $\mathbf{C} \setminus (\bar{S} \setminus D)$. Since the length of γ is less than $L \text{dist}(z_1, \bar{S} \setminus D)$ it is easy to see that we can choose U_k $k = 1, 2, \dots$ such that $W = \cup_k U$ satisfies (ii). But then p_k converges to a function s which satisfies our requirements.

It is now relatively easy to complete the proof of Lemma 1. Each of the functions f_n^* and g_n^* can be written as finite sums of the form (4). (For g_n^* one must replace h by h' in (4).) The rational functions $R_{k,1}$ and $R_{k,2}$ are independent of h and we have also bounds for the constants $\alpha_k(h)$ and

$\beta_k(h)$ which are independent of h . (See [3] and the remark following (5).) If one applies Lemma 2 with care and approximate the functions $R_{k,1}$ and $R_{k,2}$ by functions $S_{k,1}$ and $S_{k,2}$ analytic in $\mathbb{C} \setminus (\bar{S} \setminus D)$ using that lemma, we get “new” functions f_n^{**} and g_n^{**} by replacing $R_{k,1}$ and $R_{k,2}$ by $S_{k,1}$ and $S_{k,2}$ in the expressions of the form (4) for f_n^* and g_n^* . Define then

$$(11) \quad h_1 = h - \sum_n (f_n^{**} + g_n^{**}).$$

Note from property (v) of $\{V_n\}$ that if $U \supset (\bar{S} \setminus D)$ is open then there exists a number N such that the poles of the rational functions $R_{k,1}$ and $R_{k,2}$ corresponding to f_n^* and g_n^* , must be contained in U if $n \geq N$. From this fact and (ii) in Lemma 2 it is easy to see that the series (11) will converge uniformly on compact subsets of $\mathbb{C} \setminus (\bar{S} \setminus D)$. From (9) and (10) it follows that h_1 will satisfy Lemma 1 if Lemma 2 is applied carefully. We don't want to go into further details about this.

Using Theorem 2 we shall now prove:

THEOREM 3. *Assume $S = \{z_n\} \subset D$ satisfies $\sum_n (1 - |z_n|) < \infty$. If $0 < p \leq \infty$ and $f \in H^p(D)|_S$, there exists g analytic in $\mathbb{C} \setminus (\bar{S} \setminus D)$ such that $g|_D \in H^p(D)$ and $g = f$ on S .*

Proof. Assume first Theorem 3 is proved for $1 < p \leq \infty$. If $g \in H^p(D)$ has no zeros in D , g has a k 'th root for some integer k such that $kp > 1$. By assumption we can find f in $H^{kp}(D)$ which interpolates this k 'th root on S and extends to be analytic in $\mathbb{C} \setminus (\bar{S} \setminus D)$. But $f^k|_D \in H^p(D)$ and interpolates g on S . Since an arbitrary function in $H^p(D)$ can be written as the sum of two functions in $H^p(D)$ with no zeros in D , ([4], page 79) Theorem 3 will be true for all $p > 0$ if it holds for $1 < p \leq \infty$. By Theorem 2 we need only prove the following for $1 < p \leq \infty$:

(***) *There exists an open set $0 \supset \bar{D} \setminus (\bar{S} \setminus D)$ such that each f in $H^p(D)|_S$ extends to a function h analytic in 0 such that $h|_D \in H^p(D)$.*

If $p = \infty$ this is just the result (*) proved by Detraz [3]. Her methods seem to work also if $1 < p < \infty$, but some additional results from the theory of H^p -spaces are needed. We give here a different proof for $1 < p < \infty$.

We first need an approximation result for $H^p(D)$ similar to Lemma 3.2 in [11]. If $f \in H^p(D)$, $1 < p < \infty$, $\|f\|_p$ denotes its norm in $H^p(D)$.

LEMMA 3. Assume $1 < p < \infty$. There exists a constant C_p depending only on p such that for each $\varepsilon > 0$ and each relatively closed set $F \subset D$ we can find an open set $0 \supset \bar{D} \setminus (\bar{F} \setminus D)$ with the following properties:

- Given $f \in H^p$ there exists g analytic in 0 such that $g|_D \in H^p$ and
- (a) $\sup\{|f(z) - g(z)|, z \in F\} < \varepsilon \|f\|_p$,
 - (b) $\|g|_D\|_p \leq C_p \|f\|_p$
 - (c) for each set $K \subset 0$ with $\text{dist}(K, \bar{F} \setminus D) > 0$ we have $\sup\{|g(z)|, z \in K\} < C_K \|f\|_p$ where C_K is independent of f .

To prove Lemma 3 it is convenient first to establish the following:

LEMMA 4. Assume $1 < p < \infty$ and $f \in H^p(D)$. If φ is a measurable function on the unit circle T we define

$$S\varphi f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \text{Re}f(e^{i\theta}) \varphi(e^{i\theta}) d\theta$$

if z is outside the closed support $\text{supp } \varphi$ of φ . Assume $0 \leq \varphi \leq 1$.

If $K \subset \mathbb{C}$ and $\text{dist}(K, \text{supp } \varphi) > 0$ we have $\sup\{|S\varphi f(z)|, z \in K\} \leq M_p \text{dist}(K, \text{supp } \varphi)^{-1} \|f\|_p$ where M_p is a constant depending only on p .

Proof of Lemma 4. Since we on T have $\text{Re } S\varphi f = \varphi \text{Re } f$, Lemma 4 is an immediate consequence of a well known theorem on M. Riesz ([4], Thm. 4.1, page 54) and Hölder's inequality.

Proof of Lemma 3. We choose open plane sets V_j , $j = 1, 2, \dots$ satisfying:

- (i) $T \setminus \bar{F} \subset \bigcup_1^\infty V_j$,
- (ii) $\bar{V}_j \cap \bar{V}_i = \emptyset$ if $|i - j| > 1$,
- (iii) $\bar{F} \cap \bar{V}_j = \emptyset$ for $j = 1, 2, \dots$,

and (iv) if $K \subset \mathbb{C} \setminus (\bar{F} \setminus D)$ is compact there are at most finitely many j such that $K \cap \bar{V}_j \neq \emptyset$.

We also choose functions $\varphi_j \in C^\infty(T)$ such that $0 \leq \varphi_j \leq 1$, $\text{supp } \varphi_j \subset V_j$ and $\sum_1^\infty \varphi_j = 1$ on $T \setminus \bar{F}$.

Given $f \in H^p$ we put $f_j = S\varphi_j(f)$, $j = 1, 2, \dots$ where $S\varphi_j(f)$ is defined as in Lemma 4. From the arguments used to prove Lemma 4 it is easy to see that we can choose numbers $r_j \in (0, 1)$, $j = 1, 2, \dots$ independent of f such that the functions $h_j : z \rightarrow f(r_j z)$ satisfies

$$(1): \sup\{|f_j(z) - h_j(z)| : z \in \mathbb{C} \setminus V_j\} < \varepsilon 2^{-j} \|f\|_p \text{ for } j = 1, 2, \dots$$

Define $g = f - \sum_{j=1}^{\infty} (f_j - h_j)$. By (1), (a) in Lemma 3 is valid. Consider a point $w \in T \setminus \bar{F}$. There exists by (i) and (ii) a number k and a disc $\Delta(w)$ centered at w such that $\overline{\Delta(w)} \cap \bar{V}_j = \emptyset$ if $j \notin \{k, k+1\}$.

Write

$$\begin{aligned} g &= (f - f_k - f_{k+1}) + (h_k + h_{k+1}) + \left(\sum_{j=k, k+1} (h_j - f_j) \right) \\ &= F_1 + F_2 + F_3 \end{aligned}$$

say. Here F_1 can be written as $S\varphi f$ where $\varphi = 1 - \varphi_k - \varphi_{k+1}$ must have compact support disjoint from $\overline{\Delta(w)}$. So F_1 is analytic in $\Delta(w)$ and by Lemma 4 $\sup\{|F_1(z)|, z \in \Delta(w)\} \leq C_w \|f\|_p$ where C_w depends only on $\text{dist}(\text{supp } \varphi, \Delta(w))$. Clearly also F_3 is analytic in $\Delta(w)$ and by (1) $\sup\{|F_3(z)|, z \in \Delta(w)\} \leq \varepsilon \|f\|_p$. Put $t = \max\{r_k, r_{k+1}\}$. Then F_2 is analytic in $\{z : |z| < t^{-1}\}$.

Define $D(w) = \Delta(w) \cap \{z : |z| < (1 + t^{-1})2^{-1}\}$. Again by Lemma 4 we obtain $\sup\{|F_2(z)|, z \in D(w)\} \leq C_w^1 \|f\|_p$ where C_w^1 depends only on t .

Let $D_j = D(w_j)$, $j = 1, 2, \dots$ denote a locally finite covering of $T \setminus \bar{F}$ by such sets. We define $0 = D \cup (\cup_j D_j)$.

To verify (c) in Lemma 3 let $K \subset 0$ have positive distance from $\bar{F} \setminus D$. Then we can write $K = K_1 \cup K_2$ where \bar{K}_1 is a compact subset of D and $K_2 \subset \cup_1^N D_j$ for some number N . It is easy to verify (c) on K_1 and K_2 separately.

It remains to verify (b). Consider the point $w \in T \setminus \bar{F}$ again. We have $|\text{Re } g(w)| \leq \varepsilon \|f\|_p + |h_k(w)| + |h_{k+1}(w)|$

$$\begin{aligned} &\leq \varepsilon \|f\|_p + \sup_{0 < r < 1} |f_k(rw)| + \sup_{0 < r < 1} |f_{k+1}(rw)| \\ &\leq \varepsilon \|f\|_p + 2 \sup_{0 < r < 1} u(rw) = \varepsilon \|f\|_p + \eta(w) \end{aligned}$$

where u is the harmonic extension to D of $|f|$.

Finally let $w \in \bar{F} \setminus D$. We can clearly assume $\bar{V}_j \cap rz = \emptyset$ for all j , all $z \in \bar{F} \setminus D$ and all $r \in (0, 1)$. But this implies

$$|\text{Re } g(w)| \leq \varepsilon \|f\|_p + |\text{Re } f(w)|.$$

By a theorem of Hardy and Littlewood $\|\eta\|_p \leq A_p \|f\|_p$ where A_p depends only on p . But then $\|\text{Re } g\|_p \leq K_p \|f\|_p$ where K_p depends only p and by the

theorem of M. Riesz used in the proof of Lemma 4, (b) follows. The Hardy-Littlewood result is in [4, Thm. 1.9, p. 12].

To complete the proof of the above claim about $H^p(D)$ we need a result similar to (***) for $H^p(D)$ when $1 < p < \infty$.

We need some notation. Let Γ be a simple closed rectifiable curve and denote by 0_Γ the bounded component of $C \setminus \Gamma$. Let μ denote the arc length measure associated with Γ . So $\mu(E)$ is the length of $E \cap \Gamma$ for each Borel set E . If $1 < p < \infty$, $H^p(\Gamma)$ denotes the closure in $L^p(\mu)$ of the polynomials in z . The functions in $H^p(\Gamma)$ can be extended to analytic functions in 0_Γ by Cauchy's integral formula and we shall assume them extended in this way.

LEMMA 5. *Let $S = \{z_n\} \subset D$ satisfy $\sum_n (1 - |z_n|) < \infty$. Then there exists a contour Γ such that $\bar{0}_\Gamma \setminus (\bar{S} \setminus S) \subset D$, $0_\Gamma \supset S$ and $H^p(\Gamma)|_S = H^p(D)|_S$ for $1 \leq p \leq \infty$.*

Proof. This result is essentially contained in Carleson's lemma ([4], page 203) and the proof we give has all its basic ideas contained in the proof of Carleson's lemma. Let $B(z)$ be the Blaschke product corresponding to S and let B_N consist of the first N factors in the product defining B . Let

$$S_1 = \{z \in D : |B(z)| \leq 2^{-1}\}. \text{ Then } \bar{S}_1 \setminus S_1 = \bar{S} \setminus S.$$

Let now $T \setminus \bar{S}$ consist of the disjoint arcs J_n , $n = 1, 2, \dots$. For each n we choose a simple arc $I_n \subset D \setminus \{0\}$ with endpoints equal to the endpoints of J_n and with the radial projection onto T equal to J_n . We wish to do this in such a way that the arclength measure associated with $\cup_n I_n$ is a Carleson measure. (See [4] page 157 for definition.) We indicate one way of doing this. Assume for simplicity that $J_n = \{e^{i\theta} : -a < \theta < a\}$ for some $a \in (0, \pi)$. Let $\{a_k\} \subset (0, a)$ and $\{r_k\} \subset (1 - a/\pi, 1)$ be monotonic sequences converging to a and 1 respectively. Assume that $R_k = \{re^{i\theta} : |\theta| < a_k, r_k < r < 1\}$ is disjoint from S_1 and $1 - r_k < a - a_k$ for all k . Define $I_n = D \cap \partial(\cup_k R_k)$. It is easy to verify that $\{I_n\}$ has all the required properties.

Define $\Gamma = (\bar{S} \setminus S) \cup (\cup_n I_n)$. Fix an integer N and choose $f \in H^p(\Gamma)$. As in [4] page 204 and 139–140, we get that the function g_N in $H^p(D)$ of minimal norm which interpolates f on $\{z_1, \dots, z_N\}$ must satisfy

$$(11) \quad \|g_N\|_p \leq |(2\pi i)^{-1} \int_\Gamma h(z) f(z) (B_N(z))^{-1} dz|$$

for some $h \in H^q(D)$ of norm one and where $p^{-1} + q^{-1} = 1$. Since $|B_N| \geq |B| \geq 2^{-1}$ on Γ and the arc length measure associated with $\Gamma \cap D$ is a Carleson measure we get by using Hölder's inequality that

$$(12) \quad \|g_N\|_P \leq C_1 \|f\|_{L^p(\mu)} \text{ where } C_1$$

depends only on Γ . (See Theorem 9.3 on page 157 in [4].) A subsequence of $\{g_N\}$ converges uniformly on compact subsets of D to a function g which satisfies Lemma 5.

The result (***) for $1 < p < \infty$ is now easy to prove. It follows from Lemma 3 and Lemma 5 in the same way as we proved (ii) \Rightarrow (i) in Theorem 1.

We finally apply Theorem 2 to a result of Vinogradov [12]. Again let $S = \{z_n\} \subset D$. We shall need the following condition on S :

$$(C) \quad \inf_k \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| \frac{z_n - z_k}{1 - \bar{z}_k z_n} \right| > 0.$$

This is a condition which is necessary for solving many interpolation problems. See [2], [13] and [14] for example.

Denote by BV_1 all sequences $\{a_n\}_{n=1}^{\infty}$ such that $\sum_1^{\infty} |a_{n+1} - a_n| < \infty$. BV_1 is a Banach space with norm

$$\|\{a_n\}_{n=1}^{\infty}\| = |a_1| + \sum_1^{\infty} |a_{n+1} - a_n|.$$

We also let B_1 denote the Banach algebra of all analytic functions in D whose derivative belongs to $H^1(D)$. The norm on B_1 is given by $N(f) = \|f\|_D + \|f'\|_1$.

If $S = \{z_n\}_{n=1}^{\infty} \subset D$ satisfies (C) and converges to 1 non-tangentially, (which means that $|1 - z_n| \leq \lambda(1 - |z_n|)$, $n = 1, 2, \dots$ for some $\lambda > 0$) Vinogradov proved that $B_1|_S = BV_1$.

Our result is:

THEOREM 4. *Assume $S = \{z_n\}$ satisfies (C) and converges to 1 non-tangentially. For each $\{a_n\} \in BV_1$ there exists f analytic in $\mathbb{C} \setminus \{1\}$ interpolating $\{a_n\}$ at $\{z_n\}$ such that f is bounded in $\{w : |1 + w| \leq 2\}$ and $f'|_D \in H^1$.*

Proof. We first prove that each $g \in B_1|_S$ extends to a bounded analytic function h in $\{w : |1 + w| < 2\}$ with $h'|_D \in H^1$.

Define $\phi(z) = (1 + z)/2$, $z \in \mathbb{C}$. By the theorem of Vinogradov it is sufficient to show that $\{\phi(z_n)\}_{n=1}^{\infty}$ satisfies (C). (Observe that $f \in B_1 \Rightarrow h = f_0 \phi|_{B_1}$). Clearly $w_n = \phi(z_n) \rightarrow 1$ non-tangentially.

By a recent result of Kam-Fook Tse [12], Theorem 1, page 352, it is sufficient to find $t > 0$ such that

$$\inf_{i,j} \left| \frac{w_i - w_j}{1 - \bar{w}_j w_i} \right| \geq t.$$

Since $\{z_n\}$ satisfies (C) this is easy and we omit it. But then we can deduce Theorem 4 from Theorem 2.

Final remarks. We now give the example showing that (iii) in Theorem 1 may fail. Let $R = \{z = x + iy : 0 < x < 1, -1 < y < 1\}$ and define $R_n = \{z = x + iy : 2^{-3n-2} \leq x \leq 2^{-3n-1}, |y| > \varepsilon_n\}$ for $n = 1, 2, \dots$ where $\{\varepsilon_n\}$ is a sequence to be specified. Let $I_n = (2^{-3n-4}, 2^{-3n-2})$ and choose a finite set of points $S_n \subset I_n$ with the following property: If f is an analytic function vanishing on S_n and bounded by one on the rectangle $D_n = \{z = x + iy : x \in I_n, |y| < 1\}$ then $|f(2^{-3n-3} + iy)| < n^{-1}$ if $|y| < 1 - n^{-1}$. Let now $U = R \setminus \bigcup_n R_n$ and $S = \bigcup_n S_n$. Clearly $\bar{S} \setminus S = \{0\}$ and if $f \in H^\infty(U)$ then $f(2^{-3n} + iy) \rightarrow 0$ as $n \rightarrow \infty$ if $|y| < 1$. It follows that $\Pi(\bar{S})$ includes the segment $\{x = 0, -1 < y < 1\}$. It only remains to show that $\{\varepsilon_n\}$ can be chosen such that S is the zero set of a nonzero function h in $H^\infty(U)$. Let g_n correspond to S_n and D_n in the same way as g corresponded to S and V in the proof of Theorem 1. Define $g_n \equiv 1$ outside D_n . Using Vitushkin's scheme for approximation ([6], page 210) it is easy to find functions h_n such that $h_n g_n$ is analytic near the endpoints of I_n , h_n is analytic where g_n is and $|1 - h_n(z)| < 2^{-n}$ if $\text{dist}(z, I_n)$ is less than $n^{-1} 2^{-3n}$. (Approximate $\log(g_n)$ near the endpoints of I_n and take exponentials and call this function h_n .) Moreover $\sup\{|h_n(z)|, z \in C\} \leq A$ where A is an absolute constant. It follows that the infinite product consisting of all the factors $h_n g_n$, $n = 1, 2, \dots$ is analytic in $\bigcup_n D_n$ and in a neighbourhood of the closure of I_n for $n = 1, 2, \dots$. So if the ε_n tend sufficiently rapidly to zero, h will be in $H^\infty(U)$ and S will be zero set of h .

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REFERENCES

1. E. J. Akutowicz and L. Carleson, *The analytic continuation of interpolatory functions*, J. Analyse Math., **7** (1959-60).
2. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921-930.
3. J. Detraz, *Algebres de Fonctions analytique dans le disque*, Ann. Sci. Ecole Norm. Sup., **3** (1970), 313-352.
4. P. L. Duren, *Theory of H^p -Spaces*, Academic Press, 1970.
5. J. P. Earle, *On the interpolation of bounded sequences by bounded functions*, J. London

Math. Soc., (2), 2 (1970), 544–548.

6. T. W. Gamelin, *Uniform Algebras*, Prentice Hall, Englewood Cliffs, N. J. (1967).

7. ———, *The Shilov boundary of $H^\infty(U)$* , To appear.

8. E. A. Heard and J. H. Wells, *An interpolation problem for subalgebras of H^∞* , Pacific J. Math., **28** (1969), 543–553.

9. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N. J. 1962.

10. D. Sarason, *Weak-star density of polynomials*, J. für die reine und angewandte mathematik, (1972).

11. A. Stray, *Approximation and interpolation*, Pacific J. Math., **40** (1972).

12. Kam-Fook Tse, *Non-tangential interpolating sequences and interpolation by normal functions*, Proc. Amer. Math. Soc., **29** (1971).

13. S. A. Vinogradov, *Interpolation and zeros of power series with a sequence of coefficients from L^p* , Soviet Math. Dokl., **6** (1965), 57.

14. ———, *Paley features and Rudin Carleson interpolation theorems for some classes of analytic functions*, Soviet Math. Dokl., **9** (1968), 111–114.

15. J. Wermer, *Seminar über Funktionen Algebren*, Springer Verlag, Berlin (1964).

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