

Pacific Journal of Mathematics

**CONTINUOUS SPECTRA OF A SINGULAR SYMMETRIC
DIFFERENTIAL OPERATOR ON A HILBERT SPACE OF
VECTOR-VALUED FUNCTIONS**

ROBERT LEE ANDERSON

CONTINUOUS SPECTRA OF A SINGULAR SYMMETRIC DIFFERENTIAL OPERATOR ON A HILBERT SPACE OF VECTOR-VALUED FUNCTIONS

ROBERT ANDERSON

Let H be the Hilbert space of complex vector-valued functions $f: [a, \infty) \rightarrow C^2$ such that f is Lebesgue measurable on $[a, \infty)$ and $\int_a^\infty f^*(s)f(s)ds < \infty$. Consider the formally self adjoint expression $\iota(y) = -y'' + Py$ on $[a, \infty)$, where y is a 2-vector and P is a 2×2 symmetric matrix of continuous real valued functions on $[a, \infty)$. Let D be the linear manifold in H defined by

$$D = \{f \in H: f, f' \text{ are absolutely continuous on compact subintervals of } [a, \infty), f \text{ has compact support interior to } [a, \infty) \text{ and } \iota(f) \in H\}.$$

Then the operator L defined by $L(y) = \iota(y)$, $y \in D$, is a real symmetric operator on D . Let L_0 be the minimal closed extension of L . For this class of minimal closed symmetric operators this paper determines sufficient conditions for the continuous spectrum of self adjoint extensions to be the entire real axis. Since the domain, D_0 , of L_0 is dense in H , self adjoint extensions of L_0 do exist.

A general background for the theory of the operators discussed here is found in [1], [3], and [5]. The theorems in this paper are motivated by the theorems of Hinton [4] and Eastham and El-Deberky [2]. In [4], Hinton gives conditions on the coefficients in the scalar case to guarantee that the continuous spectrum of self adjoint extensions covers the entire real axis. Eastham and El-Deberky [2] study the general even order scalar operator.

DEFINITION 1. Let \tilde{L} denote a self adjoint extension of L_0 . Then we define the *continuous spectrum*, $C(\tilde{L})$, of \tilde{L} to be the set of all λ for which there exists a sequence $\langle f_n \rangle$ in $D_{\tilde{L}}$, the domain of \tilde{L} , with the properties:

- (i) $\|f_n\| = 1$ for all n ,
- (ii) $\langle f_n \rangle$ contains no convergent subsequence (i.e., is not compact), and
- (iii) $\|(\tilde{L} - \lambda)f_n\| \rightarrow 0$ as $n \rightarrow \infty$.

For the self adjoint operator \tilde{L} we have the following well-known lemma.

LEMMA 1. *The continuous spectrum of \tilde{L} is a subset of the real numbers.*

Proof. Let $\lambda = \alpha + i\beta$ where $\beta \neq 0$. Then for all $f \in D_{\tilde{L}}$ we can see by expanding $\|(\tilde{L} - \lambda)f\|^2$ that

$$\|(\tilde{L} - \lambda)f\|^2 \geq |\beta|^2 \|f\|^2,$$

which implies $\lambda \notin C(\tilde{L})$.

THEOREM 2. Let $L(y) = y'' + P(t)y$ for $a \leq t < \infty$, where $P(t) = \begin{bmatrix} \alpha(t) & \gamma(t) \\ \gamma(t) & \beta(t) \end{bmatrix}$ where $\gamma(t)$ is positive and has two continuous derivatives. Let $g(t) > 0$ be one of $\alpha(t)$ or $\beta(t)$, where both α and β are continuous on $[a, \infty)$ and $g(t)$ has a continuous derivative. Then if for some sequence of intervals $\{A_m\}$ where $A_m \subseteq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ and $a_m \rightarrow \infty$, the following are satisfied:

- (i) $\min_{x \in A_m} \{g(x)\} \rightarrow \infty$,
- (ii) $\int_{A_m} ((g'(x))^2)/(g(x)) dx = o(a_m)$,
- (iii) $\int_{A_m} g(x) dx = o(a_m^3)$,
- (iv) $\int_{A_m} [\gamma(x)]^2 dx = o(a_m)$,

we can conclude that $C(\tilde{L})$ is $(-\infty, \infty)$.

Proof. We will establish the theorem for $g(t) = \alpha(t)$ since the other case follows in exactly the same way.

Note that to prove the theorem then we need only show that for any real number μ there is a sequence $\langle f_m \rangle$ in $D(\tilde{L})$ such that $\|f_m\| = 1$, $f_m \rightarrow 0$ a.e., f_m vanishes outside A_m and $\|(\tilde{L} - \mu)f_m\| \rightarrow 0$ as $m \rightarrow \infty$.

Let $\langle h_m \rangle$ be defined by

$$(1) \quad h_m(t) = \begin{cases} [1 - \{(t - c_m)/a_m\}^2]^3 & \text{for } |t - c_m| \leq a_m \\ 0 & \text{for } |t - c_m| > a_m \end{cases}.$$

Then define $\langle f_m(t) \rangle$ by

$$(2) \quad f_m(t) = h_m(t) \begin{bmatrix} b_{m1} e^{iQ_1(t)} \\ b_{m2} e^{iQ_2(t)} \end{bmatrix},$$

where Q_1, Q_2 are real functions with two continuous derivatives and b_{m1}, b_{m2} are normalization constants.

To find $|b_m| = \sqrt{b_{m1}^2 + b_{m2}^2}$ we have

$$\begin{aligned} 1 = \|f_m\|^2 &= \int_{c_m - a_m}^{c_m + a_m} |b_m|^2 h_m^2(t) dt = |b_m|^2 \int_{-a_m}^{a_m} \left[1 - \left(\frac{x}{a_m}\right)^2\right]^6 dx \\ &= |b_m|^2 \int_{-1}^1 a_m [1 - y^2]^6 dy = |b_m|^2 (2a_m) \left[1 + \sum_{r=1}^6 \binom{6}{r} (2r + 1)^{-1}\right]. \end{aligned}$$

Hence for some positive constant K

$$(3) \quad |b_m|^2 = K(2a_m)^{-1},$$

and

$$(4) \quad |f_m(t)| \leq |b_m| = \sqrt{K}/\sqrt{2}a_m.$$

Hence

$$(5) \quad f_m \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

$$(6) \quad |h_m^{(r)}(t)| \leq K_r(a_m)^{-r},$$

where K_r does not depend on t or m .

Since $f_m \in D(\tilde{L})$, we have

$$\begin{aligned} (\tilde{L} - \mu I)f_m &= f_m'' + (P - \mu I)f_m \\ &= \begin{bmatrix} f_{m1}'' + (\alpha - \mu)f_{m1} + \gamma f_{m2} \\ f_{m2}'' + (\beta - \mu)f_{m2} + \gamma f_{m1} \end{bmatrix} \\ (\tilde{L} - \mu I)f_m &= \begin{bmatrix} \{-Q_1'^2 + (\alpha - \mu)\}f_{m1} + \gamma f_{m2} + iQ_1''f_{m1} \\ \{-Q_2'^2 + (\beta - \mu)\}f_{m2} + \gamma f_{m1} + iQ_2''f_{m2} \end{bmatrix} \\ &\quad + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ b_{m2}e^{iQ_2}h_m'' + 2iQ_2'b_{m2}e^{iQ_2}h_m' \end{bmatrix}. \end{aligned}$$

Now if Q_1 is chosen so that

$$Q_1'^2 = \alpha - \mu, \quad Q_1'' = \frac{\alpha'}{2\sqrt{\alpha - \mu}},$$

and b_{m2} is chosen to be identically zero we have that

$$(\tilde{L} - \mu I)f_m = \begin{bmatrix} iQ_1''f_{m1} \\ \gamma f_{m1} \end{bmatrix} + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ 0 \end{bmatrix}.$$

By the way Q_1 is chosen,

$$\|(\tilde{L} - \mu I)f_m\| \leq \left\| \left(\frac{\alpha'}{2\sqrt{\alpha - \mu}} \right) f_m \right\| + \|\gamma f_m\| + \|b_m h_m''\| + \|2Q_1' b_m h_m'\|.$$

Now, by (ii)

$$\left\| \frac{\alpha'}{2\sqrt{\alpha - \mu}} f_m \right\| \leq \left[\frac{K}{a_m} \int_{A_m} \left(\frac{\alpha'}{2\sqrt{\alpha - \mu}} \right)^2 \right]^{1/2} = o(1) \quad \text{as } m \rightarrow \infty.$$

By condition (iv),

$$\|\gamma f_m\| \leq \left(\frac{K}{a_m} \int_{A_m} |\gamma|^2 \right)^{1/2} = o(1) \quad \text{as } m \rightarrow \infty.$$

Next, by (iii), (3) and (6)

$$\begin{aligned} \|Q'_m h'_m\| &= \left(\int_{A_m} (\alpha - \mu) \frac{K}{2a_m} \cdot \frac{K_1^2}{a_m^2} \right)^{1/2} \\ &= K_1 K^{1/2} \left(\frac{1}{2a_m^3} \int_{A_m} (\alpha - \mu) \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty. \end{aligned}$$

Then, by (3), (6), and the Cauchy-Schwartz Inequality

$$\begin{aligned} \|b_m h''_m\| &\leq \left(\int_{A_m} |b_m|^2 \right)^{1/2} \left(\int_{A_m} |h''_m|^2 \right)^{1/2} \\ &\leq \sqrt{K/2} \left(\int_{A_m} (K_r^2/a_m^2) \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty. \end{aligned}$$

Hence it follows that

$$\|(\tilde{L} - \mu I)f_m\| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty,$$

which is what we were to show.

COROLLARY 3. *If $P(t) = \begin{bmatrix} at^\sigma & ct^\gamma \\ ct^\gamma & bt^\beta \end{bmatrix}$ on some half-line $d \leq t < \infty$ in Theorem 2 and*

(i) $a, c > 0$ with $\delta < 0$, $0 < \sigma < 2$, or

(ii) $b, c > 0$ with $\delta < 0$, $0 < \eta < 2$

then $C(\tilde{L}) = (-\infty, \infty)$.

THEOREM 4. *Suppose $L(y)$ is as in Theorem 2, where $\gamma(t)$ is positive and has two continuous derivatives. If for some sequence of intervals $\{A_m\}$, where $A_m = [c_m - a_m, c_m + a_m]$, $A_m \subseteq [a, \infty)$ and $a_m \rightarrow \infty$, the following are satisfied:*

(i) $\min_{t \in A_m} \{\gamma(t)\} \rightarrow \infty$,

(ii) $\int_{A_m} ((\gamma'(t))^2)/(\gamma(t)) dt = o(a_m)$,

(iii) $\int_{A_m} \gamma(t) dt = o(a_m^3)$,

(iv) $\int_{A_m} \alpha^2(t) dt$ and $\int_{A_m} \beta^2(t) dt$ are $o(a_m)$,

then $C(\tilde{L}) = (-\infty, \infty)$.

Proof. In the proof of Theorem 2 choose $Q_1^2 = Q_2^2 = \gamma(t) - \mu$, so that $f_{m1} = f_{m2}$. Then $Q_1' = Q_2' = (\gamma'(t))/(2\sqrt{\gamma(t) - \mu})$ and applying conditions (i) – (iv) as before where $g(t)$ is replaced by $\gamma(t)$ we get that $\|(\tilde{L} - \mu I)f_m\| \rightarrow 0$ as $m \rightarrow \infty$.

COROLLARY 5. *Let $P(t) = \begin{bmatrix} at^\sigma & ct^\delta \\ ct^\delta & bt^\eta \end{bmatrix}$ in Theorem 4. If $c > 0$, $0 < \delta < 2$ and $\sigma, \eta < 0$ then $C(\tilde{L}) = (-\infty, \infty)$.*

Let H be the Hilbert space $\tilde{L}_2([a, \infty), w)$ of complex vector-valued functions $f: [a, \infty) \rightarrow \mathbf{C}^2$ such that $\|f\|^2 = \int_a^\infty w(f^*f) < \infty$, where w is positive and $w \in C^{(2)}[a, \infty)$. Let $l(y) \equiv (1/w)y'' + Py$. Then define L_0 as before and let \tilde{L} be a self adjoint extension of L_0 .

THEOREM 6. *Suppose there is a sequence of intervals, $A_m \subseteq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ where $a_m \rightarrow \infty$ as $m \rightarrow \infty$, such that*

$$(i) \quad \int_{A_m} (\alpha(w')^2)/w^3 = o(|A_m|), \quad \int_{A_m} \alpha/w = o(|A_m|^3), \quad \min_{t \in A_m} \alpha(t) \rightarrow \infty,$$

$$(ii) \quad \int_{A_m} (w')^4/w^5 = o(|A_m|), \quad \int_{A_m} (w''/w^2)^2 = o(|A_m|),$$

$$(iii) \quad \int_{A_m} 1/w^2 = o(|A_m|^5),$$

$$\int_{A_m} ((wa')^3)/(\alpha w^3) = o(|A_m|), \text{ and}$$

$$(iv) \quad \int_{A_m} \gamma^2 = o(|A_m|)$$

as $m \rightarrow \infty$. Then $C(\tilde{L}) = (-\infty, \infty)$.

Note that (ii) implies that $\int_{A_m} (w'/w^2)^2 = o(|A_m|^3)$ by $(w'/w^2)^2 = (w')^2/w^3 \cdot 1/w$ and Cauchy-Schwartz Inequality.

Proof. As is the previous theorem define

$$f_m = \begin{bmatrix} f_{m1} \\ f_{m2} \end{bmatrix} \quad \text{where} \quad f_{m2} = 0 \quad \text{and} \quad f_{m1} = (b_m e^{iQ} h_m) w^{-1/2}.$$

Then again $b_m^2 = K/a_m$ and $|f_{m1}| \leq b_m w^{-1/2} = (K/(wa_m))^{1/2}$. Calculating

$$\begin{aligned} f'_{m1} &= w^{-1/2} b_m e^{iQ} h'_m + f_{m1} [iQ' - 1/2w^{-1}w'] \\ f''_{m1} &= f_{m1} [-(Q')^2 - iQ'w^{-1}w' + 3/4w^{-2}(w')^2 - 1/2w^{-1}w'' + iQ''] \\ &\quad + b_m e^{iQ} [2w^{-1/2}iQ'h'_m - w^{-3/2}w'h'_m + w^{-1/2}h''_m]. \end{aligned}$$

Then $(\tilde{L} - \mu I)f_m = (1/w)f''_m + Pf_m$, where the top element is

$$\begin{aligned} \frac{1}{w}f''_{m1} + (\alpha - \mu)f_{m1} &= \frac{f_{m1}}{w} [-(Q')^2 + (\alpha - \mu)w] \\ &\quad + \frac{f_{m1}}{w} \left[-iQ'w^{-1}w' + \frac{3}{4}w^{-2}(w')^2 - \frac{1}{2}w^{-1}w'' + iQ'' \right] \\ &\quad + b_m e^{iQ} [w^{-3/2}] [2iQ'h'_m - w^{-1}w'h'_m + h''_m] \\ &= \frac{f_{m1}}{w} [-(Q')^2 + (\alpha - \mu)w] + \frac{f_{m1}}{w^3} \left[-iQ'ww' + \frac{3}{4}(w')^2 - \frac{1}{2}ww'' + w^2iQ'' \right] \\ &\quad + b_m e^{iQ} w^{-3/2} [2iQ'h'_m - w^{-1}w'h'_m + h''_m]. \end{aligned}$$

Of course, the second element of $(L - \mu I)f_m$ is γf_{m1} . By choosing $(Q')^2 = (\alpha - \mu)w$ we have that by (i)

$$Q' = [(\alpha - \mu)w]^{1/2} = O((\alpha w)^{1/2}) \quad \text{as } t \longrightarrow \infty .$$

$$Q'' = O\left(\frac{[\alpha w']}{\sqrt{\alpha w}}\right) \quad \text{as } t \longrightarrow \infty .$$

Then by the calculations above

$$(7) \quad \begin{aligned} \|\tilde{L} - \mu I\| f_m &\leq \left\| \frac{f_{m1}}{w^2} Q' w' \right\| + \frac{3}{4} \left\| \frac{f_{m1}}{w^3} (w')^2 \right\| + \frac{1}{2} \left\| f_{m1} \frac{w''}{w^2} \right\| \\ &+ \left\| \frac{f_{m1} Q''}{w} \right\| + 2 \|b_m w^{-3/2} Q' h'_m\| \\ &+ \|b_m w^{-5/2} w' h'_m\| \\ &+ \|b_m w^{-3/2} h''_m\| + \|\gamma f_{m1}\| . \end{aligned}$$

Since $|f_{m1}|^2 \leq K/(w a_m)$ and $(Q')^2 = (\alpha - \mu)w$,

$$\begin{aligned} &\|f_{m1} w^{-2} Q' w'\| \\ &\leq \left(\frac{K}{a_m} \int_{A_m} (\alpha - \mu) w^{-3} (w')^2 \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty \quad \text{by (i)} . \end{aligned}$$

Similarly,

$$\|f_{m1} w^{-3} (w')^2\| \leq \left(\frac{K}{a_m} \int_{A_m} [(w')^2 w^{-3}] \right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By the definition of Q and f_{m1} ,

$$\|f_{m1} w^{-1} Q''\| = O\left(\int_{A_m} \frac{K [(\alpha w)']^2}{a_m \alpha w^3} \right)^{1/2} = o(1) \quad \text{by (iii)} .$$

And by condition (ii),

$$\|f_{m1} w^{-2} w''\| \leq \left(\frac{K}{a_m} \int_{A_m} [(w'')^2 w^{-4}] \right)^{1/2} = o(1) .$$

Since $|b_m|^2 = K/a_m$ and $|h'_m| \leq K_1/a_m$,

$$\|b_m w^{-3/2} Q' h'_m\| \leq \left((K K_1^2 / a_m^3) \int_{A_m} \left(\frac{\alpha - \mu}{w} \right) \right)^{1/2} = o(1) \quad \text{by (i)} .$$

Similarly, by the remark at the end of the theorem,

$$\|b_m w^{-5/2} w' h'_m\| \leq \left((K K_1^2 / a_m^3) \int_{A_m} (w')^2 w^{-4} \right)^{1/2} = o(1) .$$

Since $|h''_m| \leq K_2/a_m^2$,

$$\|b_m w^{-3/2} h''_m\| \leq \left((K K_2^2 / a_m^5) \int_{A_m} w^{-2} \right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By (iv),

$$\|\gamma f_{m1}\| \leq \left((K/a_m) \int_{A_m} \gamma^2 \right)^{1/2} = o(1) \text{ as } m \longrightarrow \infty .$$

Hence, by the above calculations and (7),

$$\|(\tilde{L} - \mu I)f_m\| \longrightarrow 0 \text{ as } m \longrightarrow \infty .$$

Since this is what we were to show, this concludes the proof.

REFERENCES

1. N. Dunford and J. T. Schwartz, *Linear Operators, Part II: Spectral Theory*, Interscience-Wiley, New York, 1963.
2. M. S. P. Eastham and A. A. El-Deberky, *The Spectrum of differential operators with large coefficients*, J. London Math. Soc., **2** (1970), 257-266.
3. I. M. Glazman, *Direct Methods of Qualitative spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, 1965.
4. D. Hinton, *Continuous spectra of second-order differential operators*, Pacific J. Math., **33**, No. 3 (1970), 641-643.
5. M. A. Naimark, *Linear Differential Operators, Part II*, Ungar, New York, 1968.

Received February 22, 1974, and in revised form October 8, 1974,

DEERING MILLIKEN, INC.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 55, No. 1

September, 1974

Robert Lee Anderson, <i>Continuous spectra of a singular symmetric differential operator on a Hilbert space of vector-valued functions</i>	1
Michael James Cambern, <i>The isometries of $L^p(X, K)$</i>	9
R. H. Cameron and David Arne Storvick, <i>Two related integrals over spaces of continuous functions</i>	19
Gary Theodore Chartrand and Albert David Polimeni, <i>Ramsey theory and chromatic numbers</i>	39
John Deryck De Pree and Harry Scott Klein, <i>Characterization of collectively compact sets of linear operators</i>	45
John Deryck De Pree and Harry Scott Klein, <i>Semi-groups and collectively compact sets of linear operators</i>	55
George Epstein and Alfred Horn, <i>Chain based lattices</i>	65
Paul Erdős and Ernst Gabor Straus, <i>On the irrationality of certain series</i> . . .	85
Zdeněk Frolík, <i>Measurable uniform spaces</i>	93
Stephen Michael Gagola, Jr., <i>Characters fully ramified over a normal subgroup</i>	107
Frank Larkin Gilfeather, <i>Operator valued roots of abelian analytic functions</i>	127
D. S. Goel, A. S. B. Holland, Cyril Nasim and B. N. Sahney, <i>Best approximation by a saturation class of polynomial operators</i>	149
James Secord Howland, <i>Puiseux series for resonances at an embedded eigenvalue</i>	157
David Jacobson, <i>Linear GCD equations</i>	177
P. H. Karvellas, <i>A note on compact semirings which are multiplicative semilattices</i>	195
Allan Morton Krall, <i>Stieltjes differential-boundary operators. II</i>	207
D. G. Larman, <i>On the inner aperture and intersections of convex sets</i>	219
S. N. Mukhopadhyay, <i>On the regularity of the P^n-integral and its application to summable trigonometric series</i>	233
Dwight Webster Read, <i>On (J, M, m)-extensions of Boolean algebras</i>	249
David Francis Rearick, <i>Multiplicativity-preserving arithmetic power series</i>	277
Indranand Sinha, <i>Characteristic ideals in group algebras</i>	285
Charles Thomas Tucker, II, <i>Homomorphisms of Riesz spaces</i>	289
Kunio Yamagata, <i>The exchange property and direct sums of indecomposable injective modules</i>	301