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**CONTINUOUS SPECTRA OF A SINGULAR SYMMETRIC
DIFFERENTIAL OPERATOR ON A HILBERT SPACE OF
VECTOR-VALUED FUNCTIONS**

ROBERT LEE ANDERSON

CONTINUOUS SPECTRA OF A SINGULAR SYMMETRIC DIFFERENTIAL OPERATOR ON A HILBERT SPACE OF VECTOR-VALUED FUNCTIONS

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Let H be the Hilbert space of complex vector-valued functions $f: [a, \infty) \rightarrow C^2$ such that f is Lebesgue measurable on $[a, \infty)$ and $\int_a^\infty f^*(s)f(s)ds < \infty$. Consider the formally self adjoint expression $\iota(y) = -y'' + Py$ on $[a, \infty)$, where y is a 2-vector and P is a 2×2 symmetric matrix of continuous real valued functions on $[a, \infty)$. Let D be the linear manifold in H defined by

$$D = \{f \in H: f, f' \text{ are absolutely continuous on compact subintervals of } [a, \infty), f \text{ has compact support interior to } [a, \infty) \text{ and } \iota(f) \in H\}.$$

Then the operator L defined by $L(y) = \iota(y)$, $y \in D$, is a real symmetric operator on D . Let L_0 be the minimal closed extension of L . For this class of minimal closed symmetric operators this paper determines sufficient conditions for the continuous spectrum of self adjoint extensions to be the entire real axis. Since the domain, D_0 , of L_0 is dense in H , self adjoint extensions of L_0 do exist.

A general background for the theory of the operators discussed here is found in [1], [3], and [5]. The theorems in this paper are motivated by the theorems of Hinton [4] and Eastham and El-Deberky [2]. In [4], Hinton gives conditions on the coefficients in the scalar case to guarantee that the continuous spectrum of self adjoint extensions covers the entire real axis. Eastham and El-Deberky [2] study the general even order scalar operator.

DEFINITION 1. Let \tilde{L} denote a self adjoint extension of L_0 . Then we define the *continuous spectrum*, $C(\tilde{L})$, of \tilde{L} to be the set of all λ for which there exists a sequence $\langle f_n \rangle$ in $D_{\tilde{L}}$, the domain of \tilde{L} , with the properties:

- (i) $\|f_n\| = 1$ for all n ,
- (ii) $\langle f_n \rangle$ contains no convergent subsequence (i.e., is not compact), and
- (iii) $\|(\tilde{L} - \lambda)f_n\| \rightarrow 0$ as $n \rightarrow \infty$.

For the self adjoint operator \tilde{L} we have the following well-known lemma.

LEMMA 1. *The continuous spectrum of \tilde{L} is a subset of the real numbers.*

Proof. Let $\lambda = \alpha + i\beta$ where $\beta \neq 0$. Then for all $f \in D_{\tilde{L}}$ we can see by expanding $\|(\tilde{L} - \lambda)f\|^2$ that

$$\|(\tilde{L} - \lambda)f\|^2 \geq |\beta|^2 \|f\|^2,$$

which implies $\lambda \notin C(\tilde{L})$.

THEOREM 2. *Let $L(y) = y'' + P(t)y$ for $a \leq t < \infty$, where $P(t) = \begin{bmatrix} \alpha(t) & \gamma(t) \\ \gamma(t) & \beta(t) \end{bmatrix}$ where $\gamma(t)$ is positive and has two continuous derivatives. Let $g(t) > 0$ be one of $\alpha(t)$ or $\beta(t)$, where both α and β are continuous on $[a, \infty)$ and $g(t)$ has a continuous derivative. Then if for some sequence of intervals $\{A_m\}$ where $A_m \subseteq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ and $a_m \rightarrow \infty$, the following are satisfied:*

- (i) $\min_{x \in A_m} \{g(x)\} \rightarrow \infty$,
- (ii) $\int_{A_m} ((g'(x))^2)/(g(x)) dx = o(a_m)$,
- (iii) $\int_{A_m} g(x) dx = o(a_m^3)$,
- (iv) $\int_{A_m} [\gamma(x)]^2 dx = o(a_m)$,

we can conclude that $C(\tilde{L})$ is $(-\infty, \infty)$.

Proof. We will establish the theorem for $g(t) = \alpha(t)$ since the other case follows in exactly the same way.

Note that to prove the theorem then we need only show that for any real number μ there is a sequence $\langle f_m \rangle$ in $D(\tilde{L})$ such that $\|f_m\| = 1$, $f_m \rightarrow 0$ a.e., f_m vanishes outside A_m and $\|(\tilde{L} - \mu)f_m\| \rightarrow 0$ as $m \rightarrow \infty$.

Let $\langle h_m \rangle$ be defined by

$$(1) \quad h_m(t) = \begin{cases} [1 - \{(t - c_m)/a_m\}^2]^3 & \text{for } |t - c_m| \leq a_m \\ 0 & \text{for } |t - c_m| > a_m \end{cases}.$$

Then define $\langle f_m(t) \rangle$ by

$$(2) \quad f_m(t) = h_m(t) \begin{bmatrix} b_{m1} e^{iQ_1(t)} \\ b_{m2} e^{iQ_2(t)} \end{bmatrix},$$

where Q_1, Q_2 are real functions with two continuous derivatives and b_{m1}, b_{m2} are normalization constants.

To find $|b_m| = \sqrt{b_{m1}^2 + b_{m2}^2}$ we have

$$\begin{aligned} 1 = \|f_m\|^2 &= \int_{c_m - a_m}^{c_m + a_m} |b_m|^2 h_m^2(t) dt = |b_m|^2 \int_{-a_m}^{a_m} \left[1 - \left(\frac{x}{a_m}\right)^2\right]^6 dx \\ &= |b_m|^2 \int_{-1}^1 a_m [1 - y^2]^6 dy = |b_m|^2 (2a_m) \left[1 + \sum_{r=1}^6 \binom{6}{r} (2r + 1)^{-1}\right]. \end{aligned}$$

Hence for some positive constant K

$$(3) \quad |b_m|^2 = K(2\alpha_m)^{-1},$$

and

$$(4) \quad |f_m(t)| \leq |b_m| = \sqrt{K}/\sqrt{2\alpha_m}.$$

Hence

$$(5) \quad f_m \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty,$$

$$(6) \quad |h_m^{(r)}(t)| \leq K_r(\alpha_m)^{-r},$$

where K_r does not depend on t or m .

Since $f_m \in D(\tilde{L})$, we have

$$\begin{aligned} (\tilde{L} - \mu I)f_m &= f_m'' + (P - \mu I)f_m \\ &= \begin{bmatrix} f_{m1}'' + (\alpha - \mu)f_{m1} + \gamma f_{m2} \\ f_{m2}'' + (\beta - \mu)f_{m2} + \gamma f_{m1} \end{bmatrix} \\ (\tilde{L} - \mu I)f_m &= \begin{bmatrix} \{-Q_1'^2 + (\alpha - \mu)\}f_{m1} + \gamma f_{m2} + iQ_1''f_{m1} \\ \{-Q_2'^2 + (\beta - \mu)\}f_{m2} + \gamma f_{m1} + iQ_2''f_{m2} \end{bmatrix} \\ &\quad + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ b_{m2}e^{iQ_2}h_m'' + 2iQ_2'b_{m2}e^{iQ_2}h_m' \end{bmatrix}. \end{aligned}$$

Now if Q_1 is chosen so that

$$Q_1'^2 = \alpha - \mu, \quad Q_1'' = \frac{\alpha'}{2\sqrt{\alpha - \mu}},$$

and b_{m2} is chosen to be identically zero we have that

$$(\tilde{L} - \mu I)f_m = \begin{bmatrix} iQ_1''f_{m1} \\ \gamma f_{m1} \end{bmatrix} + \begin{bmatrix} b_{m1}e^{iQ_1}h_m'' + 2iQ_1'b_{m1}e^{iQ_1}h_m' \\ 0 \end{bmatrix}.$$

By the way Q_1 is chosen,

$$\|(\tilde{L} - \mu I)f_m\| \leq \left\| \left(\frac{\alpha'}{2\sqrt{\alpha - \mu}} \right) f_m \right\| + \|\gamma f_m\| + \|b_m h_m''\| + \|2Q_1' b_m h_m'\|.$$

Now, by (ii)

$$\left\| \frac{\alpha'}{2\sqrt{\alpha - \mu}} f_m \right\| \leq \left[\frac{K}{\alpha_m} \int_{A_m} \left(\frac{\alpha'}{2\sqrt{\alpha - \mu}} \right)^2 \right]^{1/2} = o(1) \quad \text{as} \quad m \rightarrow \infty.$$

By condition (iv),

$$\|\gamma f_m\| \leq \left(\frac{K}{\alpha_m} \int_{A_m} |\gamma|^2 \right)^{1/2} = o(1) \quad \text{as} \quad m \rightarrow \infty.$$

Next, by (iii), (3) and (6)

$$\begin{aligned} \| Q_1' b_m h_m' \| &= \left(\int_{A_m} (\alpha - \mu) \frac{K}{2a_m} \cdot \frac{K_1^2}{a_m^2} \right)^{1/2} \\ &= K_1 K^{1/2} \left(\frac{1}{2a_m^3} \int_{A_m} (\alpha - \mu) \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty . \end{aligned}$$

Then, by (3), (6), and the Cauchy-Schwartz Inequality

$$\begin{aligned} \| b_m h_m'' \| &\leq \left(\int_{A_m} |b_m|^2 \right)^{1/2} \left(\int_{A_m} |h_m''|^2 \right)^{1/2} \\ &\leq \sqrt{K/2} \left(\int_{A_m} (K_r^2/a_m^2) \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty . \end{aligned}$$

Hence it follows that

$$\| (\tilde{L} - \mu I) f_m \| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty ,$$

which is what we were to show.

COROLLARY 3. *If $P(t) = \begin{bmatrix} at^\sigma & ct^\eta \\ ct^\eta & bt^\delta \end{bmatrix}$ on some half-line $d \leq t < \infty$ in Theorem 2 and*

(i) $a, c > 0$ with $\delta < 0$, $0 < \sigma < 2$, or

(ii) $b, c > 0$ with $\delta < 0$, $0 < \eta < 2$

then $C(\tilde{L}) = (-\infty, \infty)$.

THEOREM 4. *Suppose $L(y)$ is as in Theorem 2, where $\gamma(t)$ is positive and has two continuous derivatives. If for some sequence of intervals $\{A_m\}$, where $A_m = [c_m - a_m, c_m + a_m]$, $A_m \subseteq [a, \infty)$ and $a_m \rightarrow \infty$, the following are satisfied:*

(i) $\min_{t \in A_m} \{\gamma(t)\} \rightarrow \infty$,

(ii) $\int_{A_m} ((\gamma'(t))^2)/(\gamma(t)) dt = o(a_m)$,

(iii) $\int_{A_m} \gamma(t) dt = o(a_m^3)$,

(iv) $\int_{A_m} \alpha^2(t) dt$ and $\int_{A_m} \beta^2(t) dt$ are $o(a_m)$,

then $C(\tilde{L}) = (-\infty, \infty)$.

Proof. In the proof of Theorem 2 choose $Q_1'^2 = Q_2'^2 = \gamma(t) - \mu$, so that $f_{m1} = f_{m2}$. Then $Q_1'' = Q_2'' = (\gamma'(t))/(2\sqrt{\gamma(t) - \mu})$ and applying conditions (i) – (iv) as before where $g(t)$ is replaced by $\gamma(t)$ we get that $\| (\tilde{L} - \mu I) f_m \| \rightarrow 0$ as $m \rightarrow \infty$.

COROLLARY 5. *Let $P(t) = \begin{bmatrix} at^\sigma & ct^\delta \\ ct^\delta & bt^\eta \end{bmatrix}$ in Theorem 4. If $c > 0$, $0 < \delta < 2$ and $\sigma, \eta < 0$ then $C(\tilde{L}) = (-\infty, \infty)$.*

Let H be the Hilbert space $\tilde{L}_2([a, \infty), w)$ of complex vector-valued functions $f: [a, \infty) \rightarrow \mathcal{C}^2$ such that $\|f\|^2 = \int_a^\infty w(f^*f) < \infty$, where w is positive and $w \in C^{(2)}[a, \infty)$. Let $l(y) \equiv (1/w)y'' + Py$. Then define L_0 as before and let \tilde{L} be a self adjoint extension of L_0 .

THEOREM 6. *Suppose there is a sequence of intervals, $A_m \subseteq [a, \infty)$, $A_m = [c_m - a_m, c_m + a_m]$ where $a_m \rightarrow \infty$ as $m \rightarrow \infty$, such that*

$$(i) \quad \int_{A_m} (\alpha(w')^2)/w^3 = o(|a_m|), \quad \int_{A_m} \alpha/w = o(|A_m|^3), \quad \min_{t \in A_m} \alpha(t) \rightarrow \infty,$$

$$(ii) \quad \int_{A_m} (w')^4/w^6 = o(|A_m|), \quad \int_{A_m} (w''/w^2)^2 = o(|A_m|),$$

$$(iii) \quad \int_{A_m} 1/w^2 = o(|A_m|^5),$$

$$\int_{A_m} ((wa)')^2/(\alpha w^3) = o(|A_m|), \text{ and}$$

$$(iv) \quad \int_{A_m} \gamma^2 = o(|A_m|)$$

as $m \rightarrow \infty$. Then $C(\tilde{L}) = (-\infty, \infty)$.

Note that (ii) implies that $\int_{A_m} (w'/w^2)^2 = o(|A_m|^3)$ by $(w'/w^2)^2 = (w')^2/w^3 \cdot 1/w$ and Cauchy-Schwartz Inequality.

Proof. As is the previous theorem define

$$f_m = \begin{bmatrix} f_{m1} \\ f_{m2} \end{bmatrix} \quad \text{where} \quad f_{m2} = 0 \quad \text{and} \quad f_{m1} = (b_m e^{iQ} h_m) w^{-1/2}.$$

Then again $b_m^2 = K/a_m$ and $|f_{m1}| \leq b_m w^{-1/2} = (K/(wa_m))^{1/2}$. Calculating

$$f_{m1}' = w^{-1/2} b_m e^{iQ} h_m' + f_{m1} [iQ' - 1/2w^{-1}w']$$

$$f_{m1}'' = f_{m1} [-(Q')^2 - iQ'w^{-1}w' + 3/4w^{-2}(w')^2 - 1/2w^{-1}w'' + iQ'']$$

$$+ b_m e^{iQ} [2w^{-1/2} iQ' h_m' - w^{-3/2} w' h_m' + w^{-1/2} h_m''] .$$

Then $(\tilde{L} - \mu I)f_m = (1/w)f_m'' + Pf_m$, where the top element is

$$\frac{1}{w} f_{m1}'' + (\alpha - \mu) f_{m1} = \frac{f_{m1}}{w} [-(Q')^2 + (\alpha - \mu)w]$$

$$+ \frac{f_{m1}}{w} \left[-iQ'w^{-1}w' + \frac{3}{4}w^{-2}(w')^2 - \frac{1}{2}w^{-1}w'' + iQ'' \right]$$

$$+ b_m e^{iQ} [w^{-3/2} [2iQ' h_m' - w^{-1}w' h_m' + h_m'']]$$

$$= \frac{f_{m1}}{w} [-(Q')^2 + (\alpha - \mu)w] + \frac{f_{m1}}{w^3} \left[-iQ'ww' + \frac{3}{4}(w')^2 - \frac{1}{2}ww'' + w^2 iQ'' \right]$$

$$+ b_m e^{iQ} w^{-3/2} [2iQ' h_m' - w^{-1}w' h_m' + h_m''] .$$

Of course, the second element of $(L - \mu I)f_m$ is γf_{m1} . By choosing $(Q')^2 = (\alpha - \mu)w$ we have that by (i)

$$Q' = [(\alpha - \mu)w]^{1/2} = O((\alpha w)^{1/2}) \quad \text{as } t \longrightarrow \infty .$$

$$Q'' = O\left(\frac{[\alpha w]'}{\sqrt{\alpha w}}\right) \quad \text{as } t \longrightarrow \infty .$$

Then by the calculations above

$$(7) \quad \begin{aligned} \|\tilde{L} - \mu I\| &\leq \left\| \frac{f_{m1} Q' w'}{w^2} \right\| + \frac{3}{4} \left\| \frac{f_{m1} (w')^2}{w^3} \right\| + \frac{1}{2} \left\| f_{m1} \frac{w''}{w^2} \right\| \\ &+ \left\| \frac{f_{m1} Q''}{w} \right\| + 2 \|b_m w^{-3/2} Q' h'_m\| \\ &+ \|b_m w^{-5/2} w' h'_m\| \\ &+ \|b_m w^{-3/2} h''_m\| + \|\gamma f_{m1}\| . \end{aligned}$$

Since $|f_{m1}|^2 \leq K/(w\alpha_m)$ and $(Q')^2 = (\alpha - \mu)w$,

$$\begin{aligned} &\|f_{m1} w^{-2} Q' w'\| \\ &\leq \left(\frac{K}{\alpha_m} \int_{A_m} (\alpha - \mu) w^{-3} (w')^2 \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty \quad \text{by (i)} . \end{aligned}$$

Similarly,

$$\|f_{m1} w^{-3} (w')^2\| \leq \left(\frac{K}{\alpha_m} \int_{A_m} [(w')^2 w^{-3}]^2 \right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By the definition of Q and f_{m1} ,

$$\|f_{m1} w^{-1} Q''\| = O\left(\int_{A_m} \frac{K [(\alpha w)']^2}{\alpha_m \alpha w^3}\right)^{1/2} = o(1) \quad \text{by (iii)} .$$

And by condition (ii),

$$\|f_{m1} w^{-2} w''\| \leq \left(\frac{K}{\alpha_m} \int_{A_m} [(w'')^2 w^{-4}] \right)^{1/2} = o(1) .$$

Since $|b_m|^2 = K/\alpha_m$ and $|h'_m| \leq K_1/\alpha_m$,

$$\|b_m w^{-3/2} Q' h'_m\| \leq \left((KK_1^2/\alpha_m^3) \int_{A_m} \left(\frac{\alpha - \mu}{w}\right)^2 \right)^{1/2} = o(1) \quad \text{by (i)} .$$

Similarly, by the remark at the end of the theorem,

$$\|b_m w^{-5/2} w' h'_m\| \leq \left((KK_1^2/\alpha_m^3) \int_{A_m} (w')^2 w^{-4} \right)^{1/2} = o(1) .$$

Since $|h''_m| \leq K_2/\alpha_m^2$,

$$\|b_m w^{-3/2} h''_m\| \leq \left((KK_2^2/\alpha_m^5) \int_{A_m} w^{-2} \right)^{1/2} = o(1) \quad \text{by (ii)} .$$

By (iv),

$$\|\gamma f_{m_1}\| \leq \left((K/a_m) \int_{A_m} \gamma^2 \right)^{1/2} = o(1) \quad \text{as } m \longrightarrow \infty .$$

Hence, by the above calculations and (7),

$$\|(\tilde{L} - \mu I)f_m\| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty .$$

Since this is what we were to show, this concludes the proof.

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