Pacific Journal of Mathematics

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Vol. 55, No. 1 September 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

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Let (X, Σ, μ) be a finite measure space, and denote by $L^p(X, K)$ the Banach space of measurable functions F defined on X and taking values in a separable Hilbert space K, such that $\|F(x)\|^p$ is integrable. In this article a characterization is given of the linear isometries of $L^p(X, K)$ onto itself, for $1 \leqq p < \infty$, $p \neq 2$. It is shown that T is such an isometry iff T is of the form $(T(F))(x) = U(x)h(x)(\Phi(F))(x)$, where Φ is a set isomorphism of Σ onto itself, U is a weakly measurable operator-valued function such that $U(x)$ is a.e. an isometry of K onto itself, and h is a scalar function which is related to Φ via a formula involving Radon-Nikodym derivatives.

Throughout this paper the letter K will represent a separable Hilbert space which may be either real or complex. We denote by $\langle \cdot, \cdot \rangle$ the inner product in K, and by S the one-dimensional Hilbert space which is the scalar field associated with K.

A function F from X to K will be called measurable if the scalar function $\langle F, e\rangle$ is measurable for each $e \in K$. Then for $1 \leqq p < \infty$, we denote by $L^p(X, K)$ the Banach space of (equivalence classes of) measurable functions F from X to K for which the norm

$$\|F\|_p = \left\{\int \|F(x)\|^p d\mu\right\}^{1/p}, \qquad p < \infty,$$
$$\|F\|_\infty = \operatorname{ess\,sup} \|F(x)\|$$

is finite. (Here $\|\cdot\|_p$ denotes the norm in $L^p(X, K)$ and $L^p(X, S)$, and $\|\cdot\|$ that in K.) If $F \in L^p(X, K)$, we define the support of F to be the set $\{x \in X: F(x) \neq 0\}$.

Let $\{e_1, e_2, \cdots\}$ be some orthonormal basis for K. For $F \in L^p(X, K)$, we define the measurable coordinate functions f_n by $f_n(x) = \langle F(x), e_n\rangle$. Then almost everywhere we have $\sum_n |f_n(x)|^2 < \infty$, and $F(x) = \sum_n f_n(x)e_n$. Moreover, it is easily seen that each f_n belongs to $L^p(X, S)$.

Here we investigate the isometries of $L^p(X, K)$, for $1 \leqq p < \infty$, $p \neq 2$. For the case in which X is the unit interval, μ Lebesgue measure, and $K = S$, the isometries were determined by Banach in [1, p. 178]. In [4], Lamperti obtained a complete description of the isometries of $L^p(X, S)$ for an arbitrary finite measure space (X, Σ, μ).

Following Lamperti's terminology, we will call a mapping Φ of Σ onto itself, defined modulo null sets, a *regular set isomorphism* if it satisfies the properties

$$\Phi(A') = [\Phi(A)]' ,$$
$$\Phi\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} \Phi(A_n) ,$$

and

$$\mu[\Phi(A)] = 0 \quad \text{if, and only if ,} \quad \mu(A) = 0 ,$$

for all sets A, A_n in Σ. (Throughout, A' will denote the complement of A.) A regular set isomorphism induces a linear transformation, also denoted by Φ, on the space of measurable scalar functions defined on X, which is characterized by $\Phi(\chi_A) = \chi_{\Phi(A)}$, where χ_A is the characteristic function of the measurable set A. This process is described in [3, pp. 453–454]. The induced transformation, moreover, has the property that it preserves a.e. convergence:

$$(1) \qquad \text{if } \lim_n f_n(x) = f(x) \text{ a.e., then } \lim_n (\Phi(f_n))(x) = (\Phi(f))(x) \text{ a.e.}$$

Now given a regular set isomorphism Φ of Σ onto itself, and $F = \sum_n f_n e_n \in L^p(X, K)$, we define $\Phi(F)$ by the equation

$$(2) \qquad (\Phi(F))(x) = \sum_n (\Phi(f_n))(x) e_n .$$

For the case in which K is infinite dimensional, one must, of course, verify that the series on the right in (2) is indeed convergent in K for almost all x. But, for all scalar simple functions, we have $(\Phi(|f|^2))(x) = |\Phi(f)|^2(x)$ and hence, by (1), this identity holds for all measurable scalar functions. Thus, as $\| F(x) \|^2 = \sum_n |f_n(x)|^2 = \lim_N \sum_{n=1}^N |f_n(x)|^2$, again using (1), we have

$$(3) \qquad \begin{aligned} |\Phi(\| F \|)|^2(x) &= (\Phi(\| F \|^2))(x) = \lim_N \Big(\Phi\Big(\sum_{n=1}^N |f_n|^2\Big)\Big)(x) \\ &= \lim_N \sum_{n=1}^N |(\Phi(f_n))(x)|^2 = \sum_n |(\Phi(f_n))(x)|^2 = \| (\Phi(F))(x) \|^2 . \end{aligned}$$

Moreover, it is readily verified that the definition of $\Phi(F)$ is independent of the choice of orthonormal basis for K.

For the case in which K is one-dimensional, Lamperti has shown that if T is an isometry of $L^p(X, S)$ onto itself, $1 \leq p < \infty$, $p \neq 2$, then there exists a regular set isomorphism Φ, and a measurable scalar function $h(x)$ such that for $f \in L^p(X, S)$

$$(4) \qquad (T(f))(x) = h(x)(\Phi(f))(x) .$$

Moreover, if the measure ν is defined by $\nu(A) = \mu[\Phi^{-1}(A)]$, $A \in \Sigma$, then

$$(5) \qquad |h(x)|^p = d\nu/d\mu \quad \text{a.e. on} \quad X .$$

Conversely, given any regular set isomorphism Φ of Σ onto itself, and a function $h(x)$ satisfying (5), the operator T defined by (4) is an isometry of $L^p(X, S)$ onto itself. Here we establish that the isometries of $L^p(X, K)$, for any separable Hilbert space K, closely resemble those of $L^p(X, S)$, except for the emergence of a measurable operator-valued function.

2. **The isometries.** We begin with a lemma whose proof exactly parallels that of Lemma 14, [5, p. 331], with the real numbers ξ and η in that lemma replaced by vectors in K.

Lemma 1. *Let φ and ψ be two elements of K. If $1 \leqq p \leqq 2$, then*

$$\|\varphi + \psi\|^p + \|\varphi - \psi\|^p \leqq 2(\|\varphi\|^p + \|\psi\|^p) ,$$

and if $2 \leqq p < \infty$,

$$\|\varphi + \psi\|^p + \|\varphi - \psi\|^p \geqq 2(\|\varphi\|^p + \|\psi\|^p) .$$

If $p \neq 2$, equality can hold only if φ or ψ is zero.

By integration, we then obtain the following:

Lemma 2. *If $1 \leqq p < \infty$ and $p \neq 2$, and if F and G are in $L^p(X, K)$, then*

$$\|F + G\|_p^p + \|F - G\|_p^p = 2\|F\|_p^p + 2\|G\|_p^p \tag{6}$$

if and only if F and G have a.e. *disjoint supports.*

Throughout the remainder of this article we assume that p is a given real number with $1 \leqq p < \infty$, $p \neq 2$. We define q to be that extended real number such that $1/p + 1/q = 1$. (The usual conventions are in effect.) T will denote a fixed isometry of $L^p(X, K)$ onto itself.

We will repeatedly use the map T^{*-1} defined on $L^q(X, K)$ by

$$\int\langle F(x), (T^{*-1}(G))(x)\rangle d\mu = \int\langle (T^{-1}(F))(x), G(x)\rangle d\mu ,$$

for $F \in L^p(X, K)$, $G \in L^q(X, K)$, which is, almost, the Banach space adjoint of T^{-1}. For the dual space of $L^p(X, K)$ is $L^q(X, K^*)$, where K^* is the dual of K, [2, p. 282]. And if σ is the usual conjugate-linear isometry of K^* onto K, σ induces a conjugate-linear isometric mapping of $L^q(X, K^*)$ onto $L^q(X, K)$, which we shall also denote by σ, and which is determined by $(\sigma(G^*))(x) = \sigma(G^*(x))$, $G^* \in L^q(X, K^*)$. Our map T^{*-1} is then actually $\sigma \circ T^{\#-1} \circ \sigma^{-1}$, where $T^{\#-1}$ is the true Banach space adjoint.

For any element $e \in K$, we denote by $\mathbf{e}$ that element of $L^p(X, K)$ which is constantly equal to e. If $e \neq 0$, it is an easy consequence of (6), and of the fact that T is onto, that the support of $T(\mathbf{e})$ must be equal to X a.e.

LEMMA 3. *Let e be any vector in K. If A is any measurable subset of X, then $T(\chi_A e)$ is equal to $T(\mathbf{e})$ on the support of $T(\chi_A e)$.*

Proof. The functions $\chi_A e$ and $\chi_{A'} e$ have disjoint supports, and thus (6) holds if F and G are replaced, respectively, by $\chi_A e$ and $\chi_{A'} e$. Since T is isometric, it follows that (6) also holds for $T(\chi_A e)$ and $T(\chi_{A'} e)$, and hence that these latter two functions have disjoint supports. Since $T(\mathbf{e}) = T(\chi_A e) + T(\chi_{A'} e)$, the desired conclusion follows.

LEMMA 4. *Let e be an element of K with $\|e\| = 1$, and let $F = T(\mathbf{e})$. If E is the vector function defined* a.e. *by $E(x) = F(x)/\|F(x)\|$, then $T^{*-1}(\mathbf{e})$ is that element of $L^q(X, K)$ determined by $(T^{*-1}(\mathbf{e}))(x) = \|F(x)\|^{p-1}E(x)$ for almost all $x \in X$.*

Proof. We have $\|F\|_p = \|\mathbf{e}\|_p = [\mu(X)]^{1/p}$. Moreover, as T^{*-1} is an isometry of $L^q(X, K)$ onto itself, we also have $\|T^{*-1}(\mathbf{e})\|_q = [\mu(X)]^{1/q}$, this latter equality holding even in the limiting case $q = \infty$, since $\|\mathbf{e}\|_\infty = 1$.

Let $G = T^{*-1}(\mathbf{e})$, and define the vector function H by $H(x) = G(x)/\|G(x)\|$ if x belongs to the support of G, and $H(x) = 0$ otherwise. (If $q = \infty$, we do not yet know that the support of G is equal to X a.e., although this fact can readily be established by a separate argument involving extreme points.) We then have

$$\begin{aligned} \mu(X) &= \int \langle \mathbf{e}, \mathbf{e} \rangle d\mu = \int \langle (T(\mathbf{e}))(x), (T^{*-1}(\mathbf{e}))(x) \rangle d\mu \\ &= \int \langle F(x), G(x) \rangle d\mu \\ &= \int \|F(x)\| \, \|G(x)\| \, \langle E(x), H(x) \rangle d\mu \\ &\leqq \int \|F(x)\| \, \|G(x)\| \, d\mu \leqq \|F\|_p \, \|G\|_q = \mu(X) \, . \end{aligned} \tag{7}$$

Hence we must have equality throughout in (7). Thus, by a known result for scalar functions, [5, p. 113], for $p > 1$ the equality $\int \|F(x)\| \, \|G(x)\| \, d\mu = \|F\|_p \, \|G\|_q$ implies that

$$\|G(x)\|^q = \|G\|_q^q \, \|F(x)\|^p / \|F\|_p^p = \|F(x)\|^p$$

a.e., so that $\|G(x)\| = \|F(x)\|^{p-1}$ a.e. If $p = 1$, the equality

$\int \| F(x) \| \, \| G(x) \| \, d\mu = \mu(X) = \| F \|_1$ implies that $\| G(x) \| = 1 = \| F(x) \|^{p-1}$ a.e. in this case too. Finally, the equality

$$\int \| F(x) \| \, \| G(x) \| \, \langle E(x), H(x) \rangle d\mu = \int \| F(x) \| \, \| G(x) \| \, d\mu$$

yields the fact that $H(x) = E(x)$ a.e., which completes the proof of the lemma.

LEMMA 5. *Let e and φ be two orthogonal elements of K, each with norm one, and let $F_e = T(\mathbf{e})$ and $F_\varphi = T(\varphi)$. If E_e and E_φ are the vector functions defined* a.e. *by $E_e(x) = F_e(x)/\| F_e(x) \|$ and $E_\varphi(x) = F_\varphi(x)/\| F_\varphi(x) \|$, then $\langle E_e(x), E_\varphi(x) \rangle = 0$* a.e.

Proof. Let A be any measurable subset of X. Then $F_e = \chi_A F_e + \chi_{A'} F_e$, and since the two functions on the right have disjoint supports, (6) holds when F and G are replaced, respectively, by $\chi_A F_e$ and $\chi_{A'} F_e$. Hence (6) also holds for $T^{-1}(\chi_A F_e)$ and $T^{-1}(\chi_{A'} F_e)$, and these latter functions thus have disjoint supports. Since $\mathbf{e} = T^{-1}(\chi_A F_e) + T^{-1}(\chi_{A'} F_e)$, if we let B denote the support of $T^{-1}(\chi_A F_e)$, it follows that $T(\chi_B e) = \chi_A F_e$.

We then have, using Lemma 4,

$$\begin{aligned} 0 &= \int \langle \chi_B e, \varphi \rangle d\mu = \int \langle (T(\chi_B e))(x), (T^{*-1}(\varphi))(x) \rangle d\mu \\ &= \int \langle \chi_A \| F_e(x) \| \, E_e(x), \| F_\varphi(x) \|^{p-1} E_\varphi(x) \rangle d\mu \\ &= \int_A \| F_e(x) \| \, \| F_\varphi(x) \|^{p-1} \langle E_e(x), E_\varphi(x) \rangle d\mu \, . \end{aligned}$$

Since $\| F_e(x) \| \, \| F_\varphi(x) \|^{p-1}$ is an a.e. positive element of $L^1(X, S)$, and A is an arbitrary measurable subset of X, we must have $\langle E_e(x), E_\varphi(x) \rangle = 0$ a.e. on X.

LEMMA 6. *For any element e of K with norm one, let F_e and E_e be defined as in the previous lemma. Then for $f \in L^p(X, S)$, $(T(fe))(x) = \tilde{f}(x) E_e(x)$ for some scalar function $\tilde{f}$, and the mapping $f(x) \longrightarrow \langle (T(fe))(x), E_e(x) \rangle$ is an isometry of $L^p(X, S)$ onto itself.*

Proof. If A is any measurable subset of X, we know from Lemma 3 that $(T(\chi_A e))(x)$ is equal to $\| F_e(x) \| \, E_e(x)$ on the support of $T(\chi_A e)$. It thus follows that for any simple function $f \in L^p(X, S)$, $(T(fe))(x) = \tilde{f}(x) E_e(x)$, where $\tilde{f}$ is a function in $L^p(X, S)$ with the same norm as f. For arbitrary $f \in L^p(X, S)$, let $\{f_k\}$ be a sequence of simple functions converging to f in the norm of $L^p(X, S)$. Then

$$\lim_k \int \| (T(f_k e))(x) - (T(fe))(x) \|^p \, d\mu = 0 \, .$$

Hence $\| (T(f_k e))(x) - (T(fe))(x) \|^p$ tends to zero in measure, and so a subsequence tends to zero a.e. That is, $(T(f_{k_j} e))(x)$ tends to $(T(fe))(x)$ almost everywhere.

Now, for almost all x, the elements of K given by $(T(f_{k_j} e))(x)$, $j = 1, 2, \cdots$ lie in the one-dimensional (hence closed) subspace of K spanned by $E_e(x)$, and thus $(T(fe))(x)$ must lie in this subspace. That is, $(T(fe))(x) = \tilde{f}(x)E_e(x)$, for some $\tilde{f} \in L^p(X, S)$ with $\| \tilde{f} \|_p = \| f \|_p$, and the given mapping is an isometry of $L^p(X, S)$ into itself.

It is readily seen that the map is, in fact, onto $L^p(X, S)$. For suppose we are given a function of the form $\tilde{f}(x)E_e(x)$, where $\tilde{f} \in L^p(X, S)$. Incorporate e into an orthonormal basis for K – say $e = e_1$, where $\{e_n : n = 1, 2, \cdots\}$ is such a basis. Let $F(x) = \sum_n f_n(x)e_n$ be the element of $L^p(X, K)$ which maps onto $\tilde{f}(x)E_e(x)$ under T.

Now $F_0(x) = \sum_{n \geqq 2} f_n(x)e_n$ belongs to $L^p(X, \hat{K})$, where $\hat{K}$ is the Hilbert space which is the closed linear span of $\{e_n : n \geqq 2\}$, and vector-valued simple functions of the form $G = \sum_{j=1}^r \chi_{A_j}\varphi_j$, $\varphi_j \in \hat{K}$, are dense in $L^p(X, \hat{K})$. By Lemmas 3 and 5, for all such G, $\langle (T(G))(x), E_e(x) \rangle = 0$ a.e., from which it follows that $\langle (T(F_0))(x), E_e(x) \rangle = 0$ a.e. Thus as $\tilde{f}(x)E_e(x) = (T(f_1 e))(x) + (T(F_0))(x)$, with $(T(f_1 e))(x)$ pointwise a scalar multiple of $E_e(x)$ and $(T(F_0))(x)$ a.e. orthogonal to $E_e(x)$, we conclude that $T(F_0)$, and hence F_0, are both equal to the zero element of $L^p(X, K)$. It follows that the mapping given by the lemma is indeed onto $L^p(X, S)$.

LEMMA 7. *Let* $\{e_n : n = 1, 2, \cdots\}$ *be some fixed orthonormal basis for* K, *and for each* n *define* F_n, E_n *by* $F_n = T(\mathbf{e}_n)$, $E_n(x) = F_n(x)/\| F_n(x) \|$. *Then there exists a regular set isomorphism* Φ *and a fixed scalar function* $h(x)$ *defined on* X *and satisfying* (5), *such that for all* $n = 1, 2, \cdots$ *and for all* $f \in L^p(X, S)$, $(T(fe_n))(x) = h(x)(\Phi(f))(x)E_n(x)$.

Proof. By Lemma 6 and Lamperti's result for scalar functions, we know that if e_m and e_n are two elements of the given orthonormal basis and if $f \in L^p(X, S)$, then $(T(fe_m))(x) = h_m(x)(\Phi_m(f))(x)E_m(x)$ and $(T(fe_n))(x) = h_n(x)(\Phi_n(f))(x)E_n(x)$, where $h_m(x)$ and $h_n(x)$ are scalar functions defined on X, and Φ_m, Φ_n are linear transformations induced by regular set isomorphisms. We wish to show that $h_m = h_n$ and $\Phi_m = \Phi_n$ modulo sets of measure zero.

If A is any measurable subset of X, we have

$$(T(\chi_A e_m))(x) = h_m(x)\chi_{\Phi_m(A)}(x)E_m(x) \, , \tag{8}$$

and

$$(T(\chi_A e_n))(x) = h_n(x)\chi_{\Phi_n(A)}(x)E_n(x) . \tag{9}$$

Consider $\chi_A(e_m + e_n)/\sqrt{2}$. If we let $F_{m,n} = T[(e_m + e_n)/\sqrt{2}]$, and define $E_{m,n}$ by $E_{m,n}(x) = F_{m,n}(x)/\|F_{m,n}(x)\|$, again by using Lemma 6 and Lamperti's result, we conclude that there exists a scalar function $h_{m,n}$ and a regular set isomorphism $\Phi_{m,n}$ such that

$$(T[\chi_A(e_m + e_n)/\sqrt{2}])(x) = h_{m,n}(x)\chi_{\Phi_{m,n}(A)}(x)E_{m,n}(x) . \tag{10}$$

Now, using the linearity of T, we have

$$\begin{aligned} E_{m,n}(x) &= F_{m,n}(x)/\|F_{m,n}(x)\| \\ &= (F_m(x) + F_n(x))/\|F_m(x) + F_n(x)\| \\ &= (\|F_m(x)\|\,E_m(x) + \|F_n(x)\|\,E_n(x))/\|F_m(x) + F_n(x)\| . \end{aligned} \tag{11}$$

And, combining (11) with Lemma 4, we have

$$\begin{aligned} &(T^{*-1}[(e_m + e_n)/\sqrt{2}])(x) = \|(F_m(x) + F_n(x))/\sqrt{2}\|^{p-1} E_{m,n}(x) \\ &= \|(F_m(x) + F_n(x))/\sqrt{2}\|^{p-1} (\|F_m(x)\|\,E_m(x) \\ &\quad + \|F_n(x)\|\,E_n(x))/\|F_m(x) + F_n(x)\| . \end{aligned} \tag{12}$$

Also, using Lemma 4 and the linearity of T^{*-1}, we find that

$$\begin{aligned} (T^{*-1}[(e_m + e_n)/\sqrt{2}])(x) = \|F_m(x)\|^{p-1} E_m(x)/\sqrt{2} \\ + \|F_n(x)\|^{p-1} E_n(x)/\sqrt{2} . \end{aligned} \tag{13}$$

Since Lemma 5 shows that $E_m(x)$ and $E_n(x)$ are a.e. linearly independent, we conclude from (12) and (13) that

$$2^{(1-p)/2}\|F_m(x) + F_n(x)\|^{p-2}\|F_m(x)\| = \|F_m(x)\|^{p-1}/\sqrt{2} \text{ , a.e.,}$$

from which it follows that $\|F_m(x) + F_n(x)\| = \sqrt{2}\,\|F_m(x)\|$ a.e. Similarly, $\|F_m(x) + F_n(x)\| = \sqrt{2}\,\|F_n(x)\|$ a.e., so that (11) then gives $E_{m,n}(x) = E_m(x)/\sqrt{2} + E_n(x)/\sqrt{2}$.

Thus from (10) we conclude that $(T[\chi_A(e_m + e_n)/\sqrt{2}])(x) = h_{m,n}(x)\chi_{\Phi_{m,n}(A)}(x)E_m(x)/\sqrt{2} + h_{m,n}(x)\chi_{\Phi_{m,n}(A)}(x)E_n(x)/\sqrt{2}$. But the linearity of T, together with (8) and (9), implies that $(T[\chi_A(e_m+e_n)/\sqrt{2}])(x) = h_m(x)\chi_{\Phi_m(A)}(x)E_m(x)/\sqrt{2} + h_n(x)\chi_{\Phi_n(A)}(x)E_n(x)/\sqrt{2}$. Hence, once again employing the a.e. linear independence of $E_m(x)$ and $E_n(x)$, we find that $h_m(x)\chi_{\Phi_m(A)}(x) = h_{m,n}(x)\chi_{\Phi_{m,n}(A)}(x) = h_n(x)\chi_{\Phi_n(A)}(x)$ a.e. Since this equality holds for every measurable set A, we can conclude that $h_n = h_m$ and $\Phi_n = \Phi_m$, modulo sets of measure zero.

Thus, if we let $\Phi = \Phi_1$ and $h = h_1$, then for all $f \in L^p(X, S)$ and all n, we have $(T(fe_n))(x) = h(x)(\Phi(f))(x)E_n(x)$ a.e., and $h = h_1$ satisfies (5) by Lemma 6. This concludes the proof of lemma.

A function U defined on X and taking values in the space of bounded operators on K is called weakly measurable if $\langle U(x)e, \varphi\rangle$ is measurable for all $e, \varphi \in K$.

THEOREM. *Let T be an isometry of $L^p(X, K)$ onto itself, and let $\{e_n: n = 1, 2, \cdots\}$ be some fixed orthonormal basis for K. Then there exists a regular set isomorphism Φ of the σ-algebra Σ of measurable sets onto itself (defined modulo null sets), a scalar function h defined on X satisfying* (5), *and a weakly measurable operator-valued function U defined on X, where $U(x)$ is an isometry of K onto itself for almost all $x \in X$, such that for $F \in L^p(X, K)$,*

$$(T(F))(x) = U(x)h(x)(\Phi(F))(x) \ ,$$

where $\Phi(F)$ is defined by (2). *Conversely, every map T of this form is an isometry of $L^p(X, K)$ onto itself.*

Proof. If T is of this form, then it follows from (3) and the fact that $U(x)$ is almost everywhere an isometry, that

$$\| U(x)h(x)(\Phi(F))(x) \| = | h(x) | \, | \Phi(\| F \|) | (x) \ , \quad \text{for} \quad F \in L^p(X, K) \ ,$$

so that T is norm-preserving by Lamperti's result for the scalar case. The fact that T maps $L^p(X, K)$ onto itself can readily be established, for example, by noting that since Φ is onto, and $U(x)$ is a.e. an isometry of K onto K, no nonzero element of $L^q(X, K)$ can annihilate the range of T.

Now suppose that T is any isometry of $L^p(X, K)$ onto itself. We define $U(x)$ on the basis vectors e_n of K by $U(x)e_n = E_n(x)$, where the E_n are determined as in Lemma 7, and then extend $U(x)$ linearly to K. Since by Lemma 5, $\{E_n(x): n = 1, 2, \cdots\}$ is almost everywhere an orthonormal set in K, $U(x)$ is an isometry of K into itself a.e., and if K is of finite dimension, the remaining assertions of the theorem then follow immediately from Lemma 7.

Thus we may as well assume that K is infinite dimensional. Let $F(x) = \sum_n f_n(x)e_n$ belong to $L^p(X, K)$. Then the sequence $\{F_N\}$, where $F_N(x) = \sum_{n=1}^N f_n(x)e_n$, converges a.e. to F and is dominated by $\| F \|$. Hence by the dominated convergence theorem, $\| F_N - F \|_p \rightarrow 0$. We thus have $T(F) = \lim_N T(F_N)$ in $L^p(X, K)$, and so at least a subsequence of the $T(F_N)$ converges a.e. to $T(F)$. But we know from (3) and the fact that $U(x)$ is almost everywhere norm-preserving that $U(x)h(x)(\Phi(F))(x) = \lim_N U(x)h(x)(\Phi(F_N))(x) = \lim_N (T(F_N))(x)$ exists in K for almost all $x \in X$, and thus it follows that $(T(F))(x) = U(x)h(x)(\Phi(F))(x)$, as claimed. Finally, since the elements of $T(L^p(X, K))$ take their values a.e. in the range of $U(x)$, and since T is onto, $U(x)$ must map K onto K for almost all $x \in X$.

3. Remarks and problems. (i) Throughout we have assumed that the measure space is finite, but the theorem is also valid for σ-finite measure spaces, and the generalization to this latter case is largely straightforward. We say "largely" only because there are a few modifications (other than the obvious ones) of statements and proofs necessary for the σ-finite case, whose necessity might easily be overlooked. For example, if the space is σ-finite, a suitable reformulation of Lemma 4 is the following:

Let A be a measurable subset of X with finite positive measure and let e be an element of K with $\|e\| = 1$. If $T(\chi_A e) = F$, and if E is that vector function defined by $E(x) = F(x)/\|F(x)\|$ if x belongs to the support of F, and $E(x) = 0$ otherwise, then $T^{*-1}(\chi_A e)$ is determined by $(T^{*-1}(\chi_A e))(x) = \|F(x)\|^{p-1}E(x)$, for almost all $x \in X$.

The proof of this fact is analogous to that given for Lemma 4, provided $p > 1$. However, in the case $p = 1$, additional arguments, unnecessary if $\mu(X)$ is finite, have to be introduced.

(ii) For a certain class of measure spaces, the set isomorphism Φ may, of course, be repleaced by a measurable point mapping [5, Chap. 15].

(iii) In [4], Lamperti provides a description of all isometries of $L^p(X, S)$ *into* itself, not just the surjective ones. One may ask if such a description is attainable in the vector case. The type of argument needed would presumably differ substantially from that used here, since we often rely on the existence of the mapping T^{*-1} from $L^q(X, K)$ to itself.

(iv) Can a reasonable description of the isometries be obtained if the Hilbert space K is replaced by a suitable class of Banach spaces? In particular, it might be of interest to see if K can be replaced by an arbitrary finite dimensional Banach space.

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Received July 3, 1973.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 55, No. 1 September, 1974