THE ISOMETRIES OF $L^p (X, K)$

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Let $(X, \Sigma, \mu)$ be a finite measure space, and denote by $L^p(X, K)$ the Banach space of measurable functions $F$ defined on $X$ and taking values in a separable Hilbert space $K$, such that $\| F(x) \|_p$ is integrable. In this article a characterization is given of the linear isometries of $L^p(X, K)$ onto itself, for $1 \leq p < \infty$, $p \neq 2$. It is shown that $T$ is such an isometry iff $T$ is of the form $(T(F))(x) = U(x)h(x)(\phi(F))(x)$, where $\phi$ is a set isomorphism of $\Sigma$ onto itself, $U$ is a weakly measurable operator-valued function such that $U(x)$ is a.e. an isometry of $K$ onto itself, and $h$ is a scalar function which is related to $\phi$ via a formula involving Radon-Nikodym derivatives.

Throughout this paper the letter $K$ will represent a separable Hilbert space which may be either real or complex. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $K$, and by $S$ the one-dimensional Hilbert space which is the scalar field associated with $K$.

A function $F$ from $X$ to $K$ will be called measurable if the scalar function $\langle F, e \rangle$ is measurable for each $e \in K$. Then for $1 \leq p < \infty$, we denote by $L^p(X, K)$ the Banach space of (equivalence classes of) measurable functions $F$ from $X$ to $K$ for which the norm

$$\| F \|_p = \left\{ \int \| F(x) \|_p^p d\mu \right\}^{1/p}, \quad p < \infty,$$

$$\| F \|_\infty = \text{ess sup } \| F(x) \|$$

is finite. (Here $\| \cdot \|_p$ denotes the norm in $L^p(X, K)$ and $L^p(X, S)$, and $\| \cdot \|$ that in $K$.) If $F \in L^p(X, K)$, we define the support of $F$ to be the set $\{ x \in X : F(x) \neq 0 \}$.

Let $\{ e_1, e_2, \cdots \}$ be some orthonormal basis for $K$. For $F \in L^p(X, K)$, we define the measurable coordinate functions $f_n$ by $f_n(x) = \langle F(x), e_n \rangle$. Then almost everywhere we have $\sum_n |f_n(x)|^2 < \infty$, and $F(x) = \sum_n f_n(x)e_n$. Moreover, it is easily seen that each $f_n$ belongs to $L^p(X, S)$.

Here we investigate the isometries of $L^p(X, K)$, for $1 \leq p < \infty$, $p \neq 2$. For the case in which $X$ is the unit interval, $\mu$ Lebesgue measure, and $K = S$, the isometries were determined by Banach in [1, p. 178]. In [4], Lamperti obtained a complete description of the isometries of $L^p(X, S)$ for an arbitrary finite measure space $(X, \Sigma, \mu)$.

Following Lamperti’s terminology, we will call a mapping $\Phi$ of $\Sigma$ onto itself, defined modulo null sets, a regular set isomorphism if it satisfies the properties.
\[ \Phi(A') = [\Phi(A)]', \]
\[ \Phi\left( \bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} \Phi(A_n), \]
and
\[ \mu[\Phi(A)] = 0 \quad \text{if, and only if,} \quad \mu(A) = 0, \]
for all sets \( A, A_n \) in \( \Sigma \). (Throughout, \( A' \) will denote the complement of \( A \).) A regular set isomorphism induces a linear transformation, also denoted by \( \Phi \), on the space of measurable scalar functions defined on \( X \), which is characterized by \( \Phi(\chi_A) = \chi_{\Phi(A)} \), where \( \chi_A \) is the characteristic function of the measurable set \( A \). This process is described in [3, pp. 453–454]. The induced transformation, moreover, has the property that it preserves a.e. convergence:

(1) if \( \lim_{n} f_n(x) = f(x) \) a.e., then \( \lim_{n} (\Phi(f_n))(x) = (\Phi(f))(x) \) a.e.

Now given a regular set isomorphism \( \Phi \) of \( \Sigma \) onto itself, and \( F = \sum_{n} f_n e_n \in L^p(X, K) \), we define \( \Phi(F) \) by the equation

(2) \[ (\Phi(F))(x) = \sum_{n} (\Phi(f_n))(x) e_n. \]

For the case in which \( K \) is infinite dimensional, one must, of course, verify that the series on the right in (2) is indeed convergent in \( K \) for almost all \( x \). But, for all scalar simple functions, we have \( (\Phi(|f|^2))(x) = |\Phi(f)|^2(x) \) and hence, by (1), this identity holds for all measurable scalar functions. Thus, as \( \|F(x)\|^2 = \sum_{n} |f_n(x)|^2 = \lim_N \sum_{n=1}^{N} |f_n(x)|^2 \), again using (1), we have

(3) \[ |\Phi(\|F\|)|^2(x) = (\Phi(\|F\|)))(x) = \lim_{N} \left( \Phi\left( \sum_{n=1}^{N} |f_n|^2 \right) \right)(x) \]
\[ = \lim_{N} \sum_{n=1}^{N} (\Phi(f_n))(x)^2 = \sum_{n} (\Phi(f_n))(x)^2 = \|\Phi(F)(x)\|^2. \]

Moreover, it is readily verified that the definition of \( \Phi(F) \) is independent of the choice of orthonormal basis for \( K \).

For the case in which \( K \) is one-dimensional, Lamperti has shown that if \( T \) is an isometry of \( L^p(X, S) \) onto itself, \( 1 \leq p < \infty, \ p \neq 2 \), then there exists a regular set isomorphism \( \Phi \), and a measurable scalar function \( h(x) \) such that for \( f \in L^p(X, S) \)

(4) \[ (T(f))(x) = h(x)(\Phi(f))(x). \]

Moreover, if the measure \( \nu \) is defined by \( \nu(A) = \mu[\Phi^{-1}(A)], \ A \in \Sigma \), then

(5) \[ |h(x)|^p = d\nu/d\mu \quad \text{a.e. on} \quad X. \]
Conversely, given any regular set isomorphism $\Phi$ of $\Sigma$ onto itself, and a function $h(x)$ satisfying (5), the operator $T$ defined by (4) is an isometry of $L^p(X, S)$ onto itself. Here we establish that the isometries of $L^p(X, K)$, for any separable Hilbert space $K$, closely resemble those of $L^p(X, S)$, except for the emergence of a measurable operator-valued function.

2. The isometries. We begin with a lemma whose proof exactly parallels that of Lemma 14, [5, p. 331], with the real numbers $\xi$ and $\eta$ in that lemma replaced by vectors in $K$.

**Lemma 1.** Let $\varphi$ and $\psi$ be two elements of $K$. If $1 \leq p \leq 2$, then

$$||\varphi + \psi||^p + ||\varphi - \psi||^p \leq 2(||\varphi||^p + ||\psi||^p),$$

and if $2 \leq p < \infty$,

$$||\varphi + \psi||^p + ||\varphi - \psi||^p \geq 2(||\varphi||^p + ||\psi||^p).$$

If $p \neq 2$, equality can hold only if $\varphi$ or $\psi$ is zero.

By integration, we then obtain the following:

**Lemma 2.** If $1 \leq p < \infty$ and $p \neq 2$, and if $F$ and $G$ are in $L^p(X, K)$, then

$$||F + G||^p + ||F - G||^p = 2||F||^p + 2||G||^p$$

if and only if $F$ and $G$ have a.e. disjoint supports.

Throughout the remainder of this article we assume that $p$ is a given real number with $1 \leq p < \infty$, $p \neq 2$. We define $q$ to be that extended real number such that $1/p + 1/q = 1$. (The usual conventions are in effect.) $T$ will denote a fixed isometry of $L^p(X, K)$ onto itself.

We will repeatedly use the map $T^{-1}$ defined on $L^q(X, K)$ by

$$\int \langle F(x), (T^{-1}(G))(x) \rangle d\mu = \int \langle (T^{-1}(F))(x), G(x) \rangle d\mu,$$

for $F \in L^p(X, K), G \in L^q(X, K)$, which is, almost, the Banach space adjoint of $T^{-1}$. For the dual space of $L^p(X, K)$ is $L^q(X, K^*)$, where $K^*$ is the dual of $K$, [2, p. 282]. And if $\sigma$ is the usual conjugate-linear isometry of $K^*$ onto $K$, $\sigma$ induces a conjugate-linear isometric mapping of $L^q(X, K^*)$ onto $L^p(X, K)$, which we shall also denote by $\sigma$, and which is determined by $(\sigma(G^*)(x)) = \sigma(G^*(x))$, $G^* \in L^q(X, K^*)$. Our map $T^{-1}$ is then actually $\sigma \circ T^{-1} \circ \sigma^{-1}$, where $T^{-1}$ is the true Banach space adjoint.
For any element \( e \in K \), we denote by \( e \) that element of \( L^p(X, K) \) which is constantly equal to \( e \). If \( e \neq 0 \), it is an easy consequence of (6), and of the fact that \( T \) is onto, that the support of \( T(e) \) must be equal to \( X \) a.e.

**Lemma 3.** Let \( e \) be any vector in \( K \). If \( A \) is any measurable subset of \( X \), then \( T(\chi_A e) \) is equal to \( T(e) \) on the support of \( T(\chi_A e) \).

*Proof.* The functions \( \chi_A e \) and \( \chi_A^\prime e \) have disjoint supports, and thus (6) holds if \( F \) and \( G \) are replaced, respectively, by \( \chi_A e \) and \( \chi_A^\prime e \). Since \( T \) is isometric, it follows that (6) also holds for \( T(\chi_A e) \) and \( T(\chi_A^\prime e) \), and hence that these latter two functions have disjoint supports. Since \( T(e) = T(\chi_A e) + T(\chi_A^\prime e) \), the desired conclusion follows.

**Lemma 4.** Let \( e \) be an element of \( K \) with \( ||e|| = 1 \), and let \( F = T(e) \). If \( E \) is the vector function defined a.e. by \( E(x) = F(x)/||F(x)|| \), then \( T^*\mathbf{1}(e) \) is that element of \( L^q(X, K) \) determined by \( (T^*\mathbf{1}(e))(x) = ||F(x)||^{p-1}E(x) \) for almost all \( x \in X \).

*Proof.* We have \( ||F||_p = ||e||_p = [\mu(X)]^{1/p} \). Moreover, as \( T^*\mathbf{1} \) is an isometry of \( L^q(X, K) \) onto itself, we also have \( ||T^*\mathbf{1}(e)||_q = [\mu(X)]^{1/q} \), this latter equality holding even in the limiting case \( q = \infty \), since \( ||e||_\infty = 1 \).

Let \( G = T^*\mathbf{1}(e) \), and define the vector function \( H \) by \( H(x) = G(x)/||G(x)|| \) if \( x \) belongs to the support of \( G \), and \( H(x) = 0 \) otherwise. (If \( q = \infty \), we do not yet know that the support of \( G \) is equal to \( X \) a.e., although this fact can readily be established by a separate argument involving extreme points.) We then have

\[
\mu(X) = \int \langle e, e \rangle d\mu = \int \langle (T(e))(x), (T^*\mathbf{1}(e))(x) \rangle d\mu
\]

\[
= \int \langle F(x), G(x) \rangle d\mu
\]

\[
= \int ||F(x)|| ||G(x)|| \langle E(x), H(x) \rangle d\mu
\]

\[
\leq \int ||F(x)|| ||G(x)|| d\mu \leq ||F||_p ||G||_q = \mu(X) .
\]

Hence we must have equality throughout in (7). Thus, by a known result for scalar functions, [5, p. 113], for \( p > 1 \) the equality \( \int ||F(x)|| ||G(x)|| d\mu = ||F||_p ||G||_q \) implies that

\[
||G(x)||^q = ||G||_q ||F(x)||^p/||F||_p^p = ||F(x)||^p
\]
a.e., so that \( ||G(x)|| = ||F(x)||^{p-1} \) a.e. If \( p = 1 \), the equality
\[ \int ||F(x)|| ||G(x)|| \, d\mu = \mu(X) = ||F|| \text{ implies that } ||G(x)|| = 1 = ||F(x)||^{p-1} \]
a.e. in this case too. Finally, the equality
\[ \int ||F(x)|| ||G(x)|| \langle E(x), H(x) \rangle \, d\mu = \int ||F(x)|| ||G(x)|| \, d\mu \]
yields the fact that \( H(x) = E(x) \) a.e., which completes the proof of the lemma.

**Lemma 5.** Let \( e \) and \( \varphi \) be two orthogonal elements of \( K \), each
with norm one, and let \( F_e = T(e) \) and \( F_\varphi = T(\varphi) \). If \( E_e \) and \( E_\varphi \)
are the vector functions defined a.e. by \( E_e(x) = F_e(x)/||F_e(x)|| \) and \( E_\varphi(x) = F_\varphi(x)/||F_\varphi(x)|| \), then \( \langle E_e(x), E_\varphi(x) \rangle = 0 \) a.e.

**Proof.** Let \( A \) be any measurable subset of \( X \). Then \( F_e = \chi_A F_e + \chi_A^c F_e \),
and since the two functions on the right have disjoint supports, (6) holds when \( F \) and \( G \) are replaced, respectively, by \( \chi_A F_e \)
and \( \chi_A^c F_e \). Hence (6) also holds for \( T^{-1}(\chi_A F_e) \) and \( T^{-1}(\chi_A^c F_e) \), and
these latter functions thus have disjoint supports. Since \( e = T^{-1}(\chi_A F_e) + T^{-1}(\chi_A^c F_e) \), if we let \( B \) denote the support of \( T^{-1}(\chi_A F_e) \),
it follows that \( T(\chi_B e) = \chi_A F_e \).
We then have, using Lemma 4,
\[ 0 = \int \langle \chi_B e, \varphi \rangle \, d\mu = \int \langle (T(\chi_B e))(x), (T^{*-1}(\varphi))(x) \rangle \, d\mu \]
\[ = \int \langle \chi_A ||F_e(x)|| E_e(x), ||F_\varphi(x)||^{p-1} E_\varphi(x) \rangle \, d\mu \]
\[ = \int_A ||F_e(x)|| ||F_\varphi(x)||^{p-1} \langle E_e(x), E_\varphi(x) \rangle \, d\mu . \]
Since \( ||F_e(x)|| ||F_\varphi(x)||^{p-1} \) is an a.e. positive element of \( L^p(X, S) \), and \( A \)
is an arbitrary measurable subset of \( X \), we must have \( \langle E_e(x), E_\varphi(x) \rangle = 0 \) a.e. on \( X \).

**Lemma 6.** For any element \( e \) of \( K \) with norm one, let \( F_e \) and \( E_e \)
be defined as in the previous lemma. Then for \( f \in L^p(X, S) \),
\( (T(f e))(x) = \tilde{f}(x) E_e(x) \) for some scalar function \( \tilde{f} \), and the mapping
\( f(x) \to \langle (T(f e))(x), E_e(x) \rangle \) is an isometry of \( L^p(X, S) \) onto itself.

**Proof.** If \( A \) is any measurable subset of \( X \), we know from
Lemma 3 that \( (T(\chi_A e))(x) \) is equal to \( ||F_e(x)|| E_e(x) \) on the support
of \( T(\chi_A e) \). It thus follows that for any simple function \( f \in L^p(X, S) \),
\( (T(f e))(x) = \tilde{f}(x) E_e(x) \), where \( \tilde{f} \) is a function in \( L^p(X, S) \) with
the same norm as \( f \). For arbitrary \( f \in L^p(X, S) \), let \( \{f_k\} \) be a sequence
of simple functions converging to \( f \) in the norm of \( L^p(X, S) \). Then
\[
\lim_k \| (T(f_k e))(x) - (T(f e))(x) \|_p \, d\mu = 0 .
\]

Hence \( \| (T(f_k e))(x) - (T(f e))(x) \|_p \) tends to zero in measure, and so a subsequence tends to zero a.e. That is, \((T(f_k e))(x)\) tends to \((T(f e))(x)\) almost everywhere.

Now, for almost all \( x \), the elements of \( K \) given by \((T(f_k e))(x), j = 1, 2, \cdots \) lie in the one-dimensional (hence closed) subspace of \( K \) spanned by \( E_e(x) \), and thus \((T(f e))(x)\) must lie in this subspace. That is, \((T(f e))(x) = \tilde{f}(x)E_e(x)\), for some \( \tilde{f} \in L^p(X, S) \) with \( \| \tilde{f} \|_p = \| f \|_p \), and the given mapping is an isometry of \( L^p(X, S) \) into itself.

It is readily seen that the map is, in fact, onto \( L^p(X, S) \). For suppose we are given a function of the form \( f(x)E_e(x) \), where \( f \in L^p(X, S) \). Incorporate \( e \) into an orthonormal basis for \( K \) — say \( e = e_1 \), where \( \{e_n: n = 1, 2, \cdots \} \) is such a basis. Let \( F(x) = \sum_n f_n(x)e_n \) be the element of \( L^p(X, K) \) which maps onto \( \tilde{f}(x)E_e(x) \) under \( T \).

Now \( F_0(x) = \sum_{n \geq 2} f_n(x)e_n \) belongs to \( L^p(X, \tilde{K}) \), where \( \tilde{K} \) is the Hilbert space which is the closed linear span of \( \{e_n: n \geq 2\} \), and vector-valued simple functions of the form \( G = \sum_{j=1}^{\infty} \chi_{\Delta_j} f_j \), \( f_j \in \tilde{K} \), are dense in \( L^p(X, \tilde{K}) \). By Lemmas 3 and 5, for all such \( G \), \( \langle (T(G))(x), E_e(x) \rangle = 0 \) a.e., from which it follows that \( \langle (T(F_0))(x), E_e(x) \rangle = 0 \) a.e. Thus as \( \tilde{f}(x)E_e(x) = (T(f e))(x) + (T(F_0))(x) \), with \((T(f e))(x)\) pointwise a scalar multiple of \( E_e(x) \) and \((T(F_0))(x)\) a.e. orthogonal to \( E_e(x) \), we conclude that \( T(F_0) \), and hence \( F_0 \), are both equal to the zero element of \( L^p(X, K) \). It follows that the mapping given by the lemma is indeed onto \( L^p(X, S) \).

**Lemma 7.** Let \( \{e_n: n = 1, 2, \cdots \} \) be some fixed orthonormal basis for \( K \), and for each \( n \) define \( F_n, E_n \) by \( F_n = T(e_n), E_n(x) = F_n(x)/\| F_n(x) \| \). Then there exists a regular set isomorphism \( \Phi \) and a fixed scalar function \( h(x) \) defined on \( X \) and satisfying (5), such that for all \( n = 1, 2, \cdots \) and for all \( f \in L^p(X, S) \), \( (T(fe_n))(x) = h(x)(\Phi(f))(x)E_n(x) \).

**Proof.** By Lemma 6 and Lamperti's result for scalar functions, we know that if \( e_m \) and \( e_n \) are two elements of the given orthonormal basis and if \( f \in L^p(X, S) \), then \( (T(fe_m))(x) = h_m(x)(\Phi_m(f))(x)E_m(x) \) and \( (T(fe_n))(x) = h_n(x)(\Phi_n(f))(x)E_n(x) \), where \( h_m(x) \) and \( h_n(x) \) are scalar functions defined on \( X \), and \( \Phi_m, \Phi_n \) are linear transformations induced by regular set isomorphisms. We wish to show that \( h_m = h_n \) and \( \Phi_m = \Phi_n \) modulo sets of measure zero.

If \( A \) is any measurable subset of \( X \), we have

\[
(8) \quad (T(\chi_A e_m))(x) = h_m(x)\chi_{s_m(A)}(x)E_m(x) ,
\]
and

\[(9) \quad (T(\chi_A e_n)(x)) = \frac{h_n(x)}{\sqrt{2}} \varphi_{\phi_{\chi_A n}}(x) E_n(x).\]

Consider \(\chi_A(e_m + e_n)\). If we let \(F_{m,n} = T[(e_m + e_n)/\sqrt{2}]\), and define \(E_{m,n}\) by \(E_{m,n}(x) = F_{m,n}(x)/||F_{m,n}(x)||\), again by using Lemma 6 and Lamperti's result, we conclude that there exists a scalar function \(h_{m,n}\) and a regular set isomorphism \(\phi_{m,n}\) such that

\[(10) \quad (T[\chi_A(e_m + e_n)/\sqrt{2}](x)) = \frac{h_{m,n}(x)}{\sqrt{2}} \varphi_{\phi_{\chi_A n}}(x) E_{m,n}(x).\]

Now, using the linearity of \(T\), we have

\[
E_{m,n}(x) = F_{m,n}(x)/||F_{m,n}(x)||
= (F_m(x) + F_n(x))/||F_m(x) + F_n(x)||
= (||F_m(x)|| E_m(x) + ||F_n(x)|| E_n(x))/||F_m(x) + F_n(x)||.
\]

And, combining (11) with Lemma 4, we have

\[
(T^{-1}[(e_m + e_n)/\sqrt{2}]) (x) = ||(F_m(x) + F_n(x))/\sqrt{2}||^{p-1} E_{m,n}(x)
\]

\[
= ||(F_m(x) + F_n(x))/\sqrt{2}||^{p-1} (||F_m(x)|| E_m(x)
+ ||F_n(x)|| E_n(x))/||F_m(x) + F_n(x)||.
\]

Also, using Lemma 4 and the linearity of \(T^{-1}\), we find that

\[
(T^{-1}[(e_m + e_n)/\sqrt{2}]) (x) = ||F_m(x)||^{p-1} E_m(x)/\sqrt{2}
+ ||F_n(x)||^{p-1} E_n(x)/\sqrt{2}.
\]

Since Lemma 5 shows that \(E_m(x)\) and \(E_n(x)\) are a.e. linearly independent, we conclude from (12) and (13) that

\[
2^{-(p-1)/2} ||F_m(x) + F_n(x)||^{p-2} ||F_m(x)|| = ||F_m(x)||^{p-1}/\sqrt{2}, \quad \text{a.e.,}
\]

from which it follows that \(||F_m(x) + F_n(x)|| = \sqrt{2} ||F_m(x)||\) a.e. Similarly, \(||F_m(x) + F_n(x)|| = \sqrt{2} ||F_n(x)||\) a.e., so that (11) then gives

\(E_{m,n}(x) = E_m(x)/\sqrt{2} + E_n(x)/\sqrt{2}\).

Thus from (10) we conclude that \((T[\chi_A(e_m + e_n)/\sqrt{2}](x)) = h_{m,n}(x)\varphi_{\phi_{\chi_A n}}(x) E_m(x)/\sqrt{2} + h_{m,n}(x)\varphi_{\phi_{\chi_A n}}(x) E_n(x)/\sqrt{2}\). But the linearity of \(T\), together with (8) and (9), implies that \((T[\chi_A(e_m + e_n)/\sqrt{2}](x)) = h_{m}(x)\varphi_{\phi_{\chi_A n}}(x) E_m(x)/\sqrt{2} + h_{n}(x)\varphi_{\phi_{\chi_A n}}(x) E_n(x)/\sqrt{2}\). Hence, once again employing the a.e. linear independence of \(E_m(x)\) and \(E_n(x)\), we find that \(h_{m}(x)\varphi_{\phi_{\chi_A n}}(x) = h_{m,n}(x)\varphi_{\phi_{\chi_A n}}(x) = h_{n}(x)\varphi_{\phi_{\chi_A n}}(x)\) a.e. Since this equality holds for every measurable set \(A\), we can conclude that \(h_{n} = h_{m}\) and \(\phi_{n} = \phi_{m}\), modulo sets of measure zero.

Thus, if we let \(\phi = \phi_{1}\) and \(h = h_{1}\), then for all \(f \in L^{p}(X, S)\) and all \(n\), we have \((T(f e_n))(x) = h(x)(\phi(f))(x) E_n(x)\) a.e., and \(h = h_{1}\) satisfies (5) by Lemma 6. This concludes the proof of lemma.
A function $U$ defined on $X$ and taking values in the space of bounded operators on $K$ is called weakly measurable if $\langle U(x)e, \varphi \rangle$ is measurable for all $e, \varphi \in K$.

**THEOREM.** Let $T$ be an isometry of $L^p(X, K)$ onto itself, and let $\{e_n: n = 1, 2, \ldots\}$ be some fixed orthonormal basis for $K$. Then there exists a regular set isomorphism $\Phi$ of the $\sigma$-algebra $\Sigma$ of measurable sets onto itself (defined modulo null sets), a scalar function $h$ defined on $X$ satisfying (5), and a weakly measurable operator-valued function $U$ defined on $X$, where $U(x)$ is an isometry of $K$ onto itself for almost all $x \in X$, such that for $F \in L^p(X, K)$,

$$(T(F))(x) = U(x)h(x)(\Phi(F))(x),$$

where $\Phi(F)$ is defined by (2). Conversely, every map $T$ of this form is an isometry of $L^p(X, K)$ onto itself.

**Proof.** If $T$ is of this form, then it follows from (3) and the fact that $U(x)$ is almost everywhere an isometry, that

$$||U(x)h(x)(\Phi(F))(x)|| = |h(x)||\Phi(||F||)|(x),$$

for $F \in L^p(X, K)$, so that $T$ is norm-preserving by Lamperti's result for the scalar case. The fact that $T$ maps $L^p(X, K)$ onto itself can readily be established, for example, by noting that since $\Phi$ is onto, and $U(x)$ is a.e. an isometry of $K$ onto $K$, no nonzero element of $L^p(X, K)$ can annihilate the range of $T$.

Now suppose that $T$ is any isometry of $L^p(X, K)$ onto itself. We define $U(x)$ on the basis vectors $e_n$ of $K$ by $U(x)e_n = E_n(x)$, where the $E_n$ are determined as in Lemma 7, and then extend $U(x)$ linearly to $K$. Since by Lemma 5, $\{E_n(x): n = 1, 2, \ldots\}$ is almost everywhere an orthonormal set in $K$, $U(x)$ is an isometry of $K$ into itself a.e., and if $K$ is of finite dimension, the remaining assertions of the theorem then follow immediately from Lemma 7.

Thus we may as well assume that $K$ is infinite dimensional. Let $F(x) = \sum_n f_n(x)e_n$ belong to $L^p(X, K)$. Then the sequence $\{F_N\}$, where $F_N(x) = \sum_{n=1}^N f_n(x)e_n$, converges a.e. to $F$ and is dominated by $||F||$. Hence by the dominated convergence theorem, $||F_N - F||_p \to 0$. We thus have $T(F) = \lim_N T(F_N)$ in $L^p(X, K)$, and so at least a subsequence of the $T(F_N)$ converges a.e. to $T(F)$. But we know from (3) and the fact that $U(x)$ is almost everywhere norm-preserving that $U(x)h(x)(\Phi(F))(x) = \lim_N U(x)h(x)(\Phi(F_N))(x) = \lim_N (T(F_N))(x)$ exists in $K$ for almost all $x \in X$, and thus it follows that $(T(F))(x) = U(x)h(x)(\Phi(F))(x)$, as claimed. Finally, since the elements of $T(L^p(X, K))$ take their values a.e. in the range of $U(x)$, and since $T$ is onto, $U(x)$ must map $K$ onto $K$ for almost all $x \in X$. 
3. Remarks and problems. (i) Throughout we have assumed that the measure space is finite, but the theorem is also valid for \( \sigma \)-finite measure spaces, and the generalization to this latter case is largely straightforward. We say "largely" only because there are a few modifications (other than the obvious ones) of statements and proofs necessary for the \( \sigma \)-finite case, whose necessity might easily be overlooked. For example, if the space is \( \sigma \)-finite, a suitable reformulation of Lemma 4 is the following:

Let \( A \) be a measurable subset of \( X \) with finite positive measure and let \( e \) be an element of \( K \) with \( \| e \| = 1 \). If \( T(\chi_A e) = F \), and if \( E \) is that vector function defined by \( E(x) = F(x)/\| F(x) \| \) if \( x \) belongs to the support of \( F \), and \( E(x) = 0 \) otherwise, then \( T^{*-1}(\chi_A e) \) is determined by \( (T^{*-1}(\chi_A e))(x) = \| F(x) \|^{\frac{1}{p}} E(x) \), for almost all \( x \in X \).

The proof of this fact is analogous to that given for Lemma 4, provided \( p > 1 \). However, in the case \( p = 1 \), additional arguments, unnecessary if \( \mu(X) \) is finite, have to be introduced.

(ii) For a certain class of measure spaces, the set isomorphism \( \Phi \) may, of course, be replaced by a measurable point mapping [5, Chap. 15].

(iii) In [4], Lamperti provides a description of all isometries of \( L^p(X, S) \) into itself, not just the surjective ones. One may ask if such a description is attainable in the vector case. The type of argument needed would presumably differ substantially from that used here, since we often rely on the existence of the mapping \( T^{*-1} \) from \( L^p(X, K) \) to itself.

(iv) Can a reasonable description of the isometries be obtained if the Hilbert space \( K \) is replaced by a suitable class of Banach spaces? In particular, it might be of interest to see if \( K \) can be replaced by an arbitrary finite dimensional Banach space.

References


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