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**RAMSEY THEORY AND CHROMATIC NUMBERS**

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## RAMSEY THEORY AND CHROMATIC NUMBERS

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Let  $\chi(G)$  denote the chromatic number of a graph  $G$ . For positive integers  $n_1, n_2, \dots, n_k$  ( $k \geq 1$ ) the chromatic Ramsey number  $\chi(n_1, n_2, \dots, n_k)$  is defined as the least positive integer  $p$  such that for any factorization  $K_p = \bigcup_{i=1}^k G_i$ ,  $\chi(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ . It is shown that  $\chi(n_1, n_2, \dots, n_k) = 1 + \prod_{i=1}^k (n_i - 1)$ . The vertex-arboricity  $a(G)$  of a graph  $G$  is the fewest number of subsets into which the vertex set of  $G$  can be partitioned so that each subset induces an acyclic graph. For positive integers  $n_1, n_2, \dots, n_k$  ( $k \geq 1$ ) the vertex-arboricity Ramsey number  $a(n_1, n_2, \dots, n_k)$  is defined as the least positive integer  $p$  such that for any factorization  $K_p = \bigcup_{i=1}^k G_i$ ,  $a(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ . It is shown that  $a(n_1, n_2, \dots, n_k) = 1 + 2k \prod_{i=1}^k (n_i - 1)$ .

**Introduction.** The classical Ramsey number  $r(m, n)$ , for positive integers  $m$  and  $n$ , is the least positive integer  $p$  such that for any graph  $G$  of order  $p$ , either  $G$  contains the complete graph  $K_m$  of order  $m$  as a subgraph or the complement  $\bar{G}$  of  $G$  contains  $K_n$  as a subgraph. More generally, for  $k(\geq 1)$  positive integers  $n_1, n_2, \dots, n_k$ , the Ramsey number  $r(n_1, n_2, \dots, n_k)$  is defined as the least positive integer  $p$  such that for any factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$  (i.e., the  $G_i$  are spanning, pairwise edge-disjoint, possibly empty subgraphs of  $K_p$  such that the union of the edge sets of the  $G_i$  equals the edge set of  $K_p$ ),  $G_i$  contains  $K_{n_i}$  as a subgraph for at least one  $i$ ,  $1 \leq i \leq k$ . It is known (see [5]) that all such Ramsey numbers exist; however, the actual values of  $r(n_1, n_2, \dots, n_k)$ ,  $k \geq 1$ , are known in only seven cases (see [2, 3]) for which  $\min\{n_1, n_2, \dots, n_k\} \geq 3$ .

A clique in a graph  $G$  is a maximal complete subgraph of  $G$ . The clique number  $\omega(G)$  is the maximum order among the cliques of  $G$ . The Ramsey number  $r(n_1, n_2, \dots, n_k)$  may be alternatively defined as the least positive integer  $p$  such that for any factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$ ,  $\omega(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ .

The foregoing observation suggests the following definition. Let  $f$  be a graphical parameter, and let  $n_1, n_2, \dots, n_k$ ,  $k \geq 1$  be positive integers. The  $f$ -Ramsey number  $f(n_1, n_2, \dots, n_k)$  is the least positive integer  $p$  such that for any factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$ ,  $f(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ . Hence,  $\omega(n_1, n_2, \dots, n_k) = r(n_1, n_2, \dots, n_k)$ , i.e., the  $\omega$ -Ramsey number is the Ramsey number.

The object of this paper is to investigate  $f$ -Ramsey numbers for two graphical parameters  $f$ , namely chromatic number and vertex-arboricity.

**Chromatic Ramsey numbers.** The *chromatic number*  $\chi(G)$  of a graph  $G$  is the fewest number of colors which may be assigned to the vertices of  $G$  so that adjacent vertices are assigned different colors. For positive integers  $n_1, n_2, \dots, n_k$ , the *chromatic Ramsey number*  $\chi(n_1, n_2, \dots, n_k)$  is the least positive integer  $p$  such that for any factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$ ,  $\chi(G_i) \geq n_i$  for some  $i$ ,  $1 \leq i \leq k$ . The existence of the numbers  $\chi(n_1, n_2, \dots, n_k)$  is guaranteed by the fact that  $\chi(n_1, n_2, \dots, n_k) \leq r(n_1, n_2, \dots, n_k)$ . We are now prepared to present a formula for  $\chi(n_1, n_2, \dots, n_k)$ . We begin with a lemma.

LEMMA. *If  $G = G_1 \cup G_2 \cup \dots \cup G_k$ , then*

$$\chi(G) \leq \sum_{i=1}^k \chi(G_i).$$

*Proof.* For  $i = 1, 2, \dots, k$ , let a  $\chi(G_i)$ -coloring be given for  $G_i$ . We assign to a vertex  $v$  of  $G$  the color  $(c_1, c_2, \dots, c_k)$ , where  $c_i$  is the color assigned to  $v$  in  $G_i$ . This produces a coloring of  $G$  using at most  $\prod_{i=1}^k \chi(G_i)$  colors; hence,  $\chi(G) \leq \prod_{i=1}^k \chi(G_i)$ .

THEOREM 1. *For positive integers  $n_1, n_2, \dots, n_k$ ,*

$$\chi(n_1, n_2, \dots, n_k) = 1 + \prod_{i=1}^k (n_i - 1).$$

*Proof.* The result is immediate if  $n_i = 1$  for some  $i$ ; hence, we assume that  $n_i \geq 2$  for all  $i$ ,  $1 \leq i \leq k$ . First, we verify that

$$\chi(n_1, n_2, \dots, n_k) \leq 1 + \prod_{i=1}^k (n_i - 1).$$

Let  $p = 1 + \prod_{i=1}^k (n_i - 1)$ , and assume there exists a factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$  such that  $\chi(G_i) \leq n_i - 1$  for each  $i = 1, 2, \dots, k$ . Then by the Lemma, it follows that

$$1 + \prod_{i=1}^k (n_i - 1) = \chi(K_p) \leq \prod_{i=1}^k \chi(G_i) \leq \prod_{i=1}^k (n_i - 1),$$

which produces a contradiction. Thus, in any factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$  for  $p = 1 + \prod_{i=1}^k (n_i - 1)$ , we have  $\chi(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ .

In order to show that

$$\chi(n_1, n_2, \dots, n_k) \geq 1 + \prod_{i=1}^k (n_i - 1),$$

we exhibit a factorization  $K_{N_k} = G_1 \cup G_2 \cup \dots \cup G_k$ , where  $N_k =$

$\prod_{i=1}^k (n_i - 1)$  and  $\chi(G_i) \leq n_i - 1$  for  $i = 1, 2, \dots, k$ . The factorization is accomplished by employing induction on  $k$ . For  $k = 1$ , we simply observe that  $\chi(K_{N_1}) = \chi(K_{n_1-1}) = n_1 - 1$ . Assume there exists a factorization  $K_{N_{k-1}} = H_1 \cup H_2 \cup \dots \cup H_{k-1}$  such that  $\chi(H_i) \leq n_i - 1$  for  $i = 1, 2, \dots, k - 1$ . Let  $\bar{F}$  denote  $n_k - 1$  (pairwise disjoint) copies of  $K_{N_{k-1}}$  and define  $G_k$  by  $G_k = \bar{F}$ . Thus,  $\bar{G}_k$  contains  $n_k - 1$  pairwise disjoint copies of  $H_i$  for  $i = 1, 2, \dots, k - 1$ , which we denote by  $G_i$ . Hence,  $K_{N_k} = G_1 \cup G_2 \cup \dots \cup G_k$ , where  $\chi(G_i) \leq n_i - 1$  for each  $i$ ,  $1 \leq i \leq k$ , which produces the desired result.

**Vertex-arboricity Ramsey numbers.** The *vertex-arboricity*  $a(G)$  of a graph  $G$  is the minimum number of subsets into which the vertex set of  $G$  may be partitioned so that each subset induces an acyclic subgraph. As with the chromatic number, the vertex-arboricity may be considered a coloring number since  $a(G)$  is the least number of colors which may be assigned to the vertices of  $G$  so that no cycle of  $G$  has all of its vertices assigned the same color.

Our next result will establish a formula for the *vertex-arboricity Ramsey number*  $a(n_1, n_2, \dots, n_k)$ , defined as the least positive integer  $p$  such that for every factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$ ,  $a(G_i) \geq n_i$  for some  $i$ ,  $1 \leq i \leq k$ . Since  $a(K_n) = \{n/2\}$ , it follows that  $a(n_1, n_2, \dots, n_k) \leq r(2n_1 - 1, 2n_2 - 1, \dots, 2n_k - 1)$ . In the proof of the following result, we shall make use of the (*edge*) *arboricity*  $a_1(G)$  of a graph, which is the minimum number of subsets into which the edge set of  $G$  may be partitioned so that the subgraph induced by each subset is acyclic. It is known (see [1, 4]) that  $a_1(K_n) = \{n/2\}$ .

**THEOREM 2.** For positive integers  $n_1, n_2, \dots, n_k$ ,

$$a(n_1, n_2, \dots, n_k) = 1 + 2k \prod_{i=1}^k (n_i - 1) .$$

*Proof.* In order to show that

$$a(n_1, n_2, \dots, n_k) \leq 1 + 2k \prod_{i=1}^k (n_i - 1) ,$$

we let  $p = 1 + 2k \prod_{i=1}^k (n_i - 1)$  and assume there exists a factorization  $K_p = G_1 \cup G_2 \cup \dots \cup G_k$  such that  $a(G_i) \leq n_i - 1$  for each  $i = 1, 2, \dots, k$ . For each  $i = 1, 2, \dots, k$ , there is a partition  $\{U_{i,1}, U_{i,2}, \dots, U_{i,n_i-1}\}$  of the vertex set  $V(G_i)$  of  $G_i$  such that the subgraph  $\langle U_{i,j} \rangle$  of  $G_i$  induced by  $U_{i,j}$  is acyclic,  $j = 1, 2, \dots, n_i - 1$ . At least one of the sets  $U_{1,1}, U_{1,2}, \dots, U_{1,n_1-1}$ , say  $U_{1,m_1}$ , contains at least  $1 + 2k \prod_{i=2}^k (n_i - 1)$  vertices. Thus, at least one of the sets  $U_{2,1}, U_{2,2}, \dots,$

$U_{2, n_2-1}$ , say  $U_{2, m_2}$ , contains at least  $1 + 2k \prod_{i=2}^k (n_i - 1)$  vertices of  $U_{1, m_1}$ . Proceeding inductively, we arrive at subsets  $U_{1, m_1}, U_{2, m_2}, \dots, U_{k, m_k}$  such that  $\bigcap_{i=1}^t U_{i, m_i}$  contains at least  $1 + 2k \prod_{i=t+1}^k (n_i - 1)$  vertices,  $1 \leq t \leq k-1$ . In particular,  $\bigcap_{i=1}^k U_{i, m_i}$ , contains a set  $U$  having  $1 + 2k$  vertices. For each  $i = 1, 2, \dots, k$ ,  $\langle U \rangle$  is an acyclic subgraph of the graph  $\langle U_{i, m_i} \rangle$ . This implies that  $a_1(K_{1+2k}) \leq k$ , which is contradictory. Therefore,  $a(G_i) \geq n_i$  for at least one  $i$ ,  $1 \leq i \leq k$ .

The proof will be complete once we have verified that

$$a(n_1, n_2, \dots, n_k) \geq 1 + 2k \prod_{i=1}^k (n_i - 1).$$

Let  $r = \prod_{i=1}^k (n_i - 1)$ . We shall exhibit a factorization  $K_{2kr} = G_1 \cup G_2 \cup \dots \cup G_k$  such that  $a(G_i) \leq n_i - 1$  for  $i = 1, 2, \dots, k$ . We begin with  $r$  pairwise disjoint copies of  $K_{2k}$ , labeled  $K_{2k}^1, K_{2k}^2, \dots, K_{2k}^r$ . Since  $a_1(K_{2k}) = k$ , it follows that  $K_{2k} = \bigcup_{i=1}^k F_i$ , where each  $F_i$  is an acyclic graph. We introduce the notation  $F_{il}$  to denote the  $F_i$  contained in  $K_{2k}^l$ ,  $l = 1, 2, \dots, r$  and  $i = 1, 2, \dots, k$ . With each of the  $r$   $k$ -tuples  $(c_1, c_2, \dots, c_k)$ ,  $c_j = 1, 2, \dots, n_j - 1$  and  $j = 1, 2, \dots, k$ , we identify a complete graph  $K_{2k}^l$ ,  $l = 1, 2, \dots, r$ , in such a way that the identification is one-to-one. Then, for each  $i = 1, 2, \dots, k$  and  $l = 1, 2, \dots, r$ , we associate with  $F_{il}$  the  $k$ -tuple identified with  $K_{2k}^l$ . Define the graph  $G_i$ ,  $i = 1, 2, \dots, k$ , to consist of the graphs  $F_{i1}, F_{i2}, \dots, F_{ir}$ ; in addition, each vertex of  $F_{is}$  is adjacent to each vertex of  $F_{it}$ ,  $s, t = 1, 2, \dots, r$ , provided the  $i$ th coordinate is the first coordinate in which their associated  $k$ -tuples differ (otherwise, there are no edges between  $F_{is}$  and  $F_{it}$ ). It is then seen that  $K_{2kr} = \bigcup_{i=1}^k G_i$ . For each  $i = 1, 2, \dots, k$ , define  $V_{i,j}$  to be the set of all vertices  $v$  such that  $v$  is a vertex of an  $F_{il}$  whose associated  $k$ -tuple  $(c_1, c_2, \dots, c_k)$  has  $c_i = j$ ;  $j = 1, 2, \dots, n_i - 1$ . Then  $\{V_{i,1}, V_{i,2}, \dots, V_{i, n_i-1}\}$  is a partition of  $V(G_i)$  for which the subgraph  $\langle V_{i,j} \rangle$  consists of  $r/(n_i - 1)$  pairwise disjoint copies of  $F_i$ ,  $j = 1, 2, \dots, n_i - 1$ . Thus,  $\langle V_{i,j} \rangle$  is an acyclic graph for each such  $j$ . Hence,  $a(G_i) \leq n_i - 1$ ,  $i = 1, 2, \dots, k$ .

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