SEMI-GROUPS AND COLLECTIVELY COMPACT SETS OF LINEAR OPERATORS

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A set of linear operators from one Banach space to another is collectively compact if and only if the union of the images of the unit ball has compact closure. Semi-groups $S = \{T(t): t \geq 0\}$ of bounded linear operators on a complex Banach space into itself and in which every operator $T(t)$, $t > 0$ is compact are considered. Since $T(t_1 + t_2) = T(t_1)T(t_2)$ for each operator in the semi-group, it would be expected that the theory of collectively compact sets of linear operators could be profitably applied to semi-groups.

1. Introduction. Let $X$ be a complex Banach space with unit ball $X$, and let $[X, X]$ denote the space of all bounded linear operators on $X$ equipped with the uniform operator topology. The semi-group definitions and terminology used are those of Hille and Phillips [6]. Let $S$ be a semi-group of vector-valued functions $T: [0, \infty) \rightarrow [X, X]$. It is assumed that $T(t)$ is strongly continuous for $t \geq 0$. If $\lim_{t \to t_0} \|T(t)x - T(t_0)x\| = 0$ for each $t_0 \geq 0$, $x \in X$ and if there is a constant $M$ such that the $\|T(t)\| \leq M$ for each $t \geq 0$, then $S = \{T(t): t \geq 0\}$ is called an equicontinuous semi-group of class $C_0$. The infinitesimal generator $A$ of the semi-group $S$ is defined by

$$Ax = \lim_{s \to 0} \frac{1}{S}[T(s)x - x]$$

whenever the limit exists. The domain $D(A)$ of $A$ is a dense subset of $X$ consisting of just those elements $x$ for which this limit exists. $A$ is a closed linear operator having resolvents $R(\lambda)$ which, for each complex number $\lambda$ with the real part of $\lambda$ greater than zero, are given by the absolutely summable Riemann-Stieltjes integral

$$(1) \quad R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)xdt, \ x \in X.$$  

It follows from (1) that

$$(2) \quad \|R(\lambda)\| \leq \frac{M}{re(\lambda)}, \ re(\lambda) > 0.$$  

In particular, sets of the type $\{R(\lambda): re(\lambda) \geq \alpha > 0\}$ are equicontinuous subsets of $[X, X]$.

Results yielding the collective compactness of the resolvents of
2. Semi-groups of compact operators. First, note that (1) states that the resolvents of \( A \) are Laplace transforms of the semi-group \( S \). Consequently, there are many other important integral expressions involving the elements of the semi-group and the resolvents. In order to take advantage of these, we prove the following lemma, in which \( |v| \) denotes the total variation of a complex measure \( v \).

**Lemma 2.1.** Let \( \Omega \) be a topological space and \( \mathcal{M} \) a collection of complex-valued Borel measures on \( \Omega \). Suppose there exists a constant \( \alpha \) for which \( |v| \Omega \leq \alpha \) for each \( v \in \mathcal{M} \). Let \( \mathcal{X} : \Omega \rightarrow [X, X] \) be an operator-valued function defined on \( \Omega \) which is strongly measurable with respect to each \( v \in \mathcal{M} \) [6, page 74] and suppose \( \mathcal{K} = \{K(w) : w \in \Omega \} \) is a bounded subset of \([X, X]\). For each \( v \in \mathcal{M} \) and \( x \in X \), let \( F_v(x) = \int_{\Omega} |K(w)x| \, dv \) where the integral exists in the Bochner sense since \( \int_{\Omega} |K(w)x| \, dv < \infty \) [6, page 80]. Let \( \mathcal{F} = \{F_v : v \in \mathcal{M}\} \). Whenever \( \mathcal{X}(\mathcal{M}) \) is collectively compact, \( \mathcal{F}(\mathcal{M}) \) is also collectively compact.

**Proof.** Assume that \( \mathcal{X} \) is collectively compact. Let \( B = \{K(w)x : w \in \Omega, \|x\| \leq 1\} \) and let \( C \) denote the balanced convex hull of \( B \). Both \( B \) and \( C \) are totally bounded subsets of \( X \). It suffices to show that \( F_v(x) \in \alpha C \) for any \( F_v \in \mathcal{F} \) and \( x \) with \( \|x\| \leq 1 \). Let \( \varepsilon > 0 \) and choose \( \{K(w_i)x_i, \cdots, K(w_n)x_n\} \), an \( \varepsilon/\alpha \)-net for \( B \). For \( i = 1, \cdots, n \), let \( \Omega_i = \{w : \|K(w)x - K(w_i)x_i\| \leq \varepsilon/\alpha\} \) and let \( \Omega_i = \Omega_i \cup \bigcup_{j=1}^n \Omega_j \) be a decomposition of the \( \Omega_i \) into pairwise disjoint sets. Then

\[
\left\| F_v(x) - \sum_{i=1}^n K(w_i)x_i v(\Omega_i) \right\| \leq \sum_{i=1}^n \int_{\Omega_i} \|K(w)x - K(w_i)x_i\| \, dv(w) \\
\leq (\varepsilon/\alpha) |v|(\Omega) \leq \varepsilon.
\]

Since \( \sum_{i=1}^n \varepsilon(\Omega_i) \leq \alpha \), \( \sum_{i=1}^n K(w_i)x_i v(\Omega_i) \) is an element of \( \alpha C \). It follows that \( F_v(x) \in \alpha C \) and so \( \mathcal{F} \) is also collectively compact.

Now assume that \( \mathcal{X}^\ast \) is collectively compact. Let \( V \) be any neighborhood of 0 in the norm topology of \( X \). There exists an \( \varepsilon > 0 \) such that \( U = \{x : \|x\| \leq \varepsilon\} \subseteq V \). Since \( \mathcal{X}^\ast \) is collectively compact, [2, Theorem 2.11, part (c)] implies that there exists a weak neighborhood \( W \) of the origin with \( \mathcal{X}(W \cap X) \subseteq (1/\alpha)U \). For \( F_v \in \mathcal{F} \) and \( x \in W \cap X \), \( \|F_v(x)\| \leq \int_{\Omega} \|K(w)x\| \, dv \leq (\varepsilon/\alpha) |v| \subseteq (1/\alpha)U \).
ε. So $\mathcal{F}(W \cap X) \subseteq V$. Again using [2, Theorem 2.1, part (c)], we see that $\mathcal{F}^*$ is also collectively compact.

The following is essentially a result of P. Lax [6, page 304]. Rephrased in the terminology of collectively compact sets of operators, it becomes quite transparent.

**Theorem 2.2.** Suppose that some $T(t_0), t_0 > 0$, is a compact operator. Then $\mathcal{K} = \{T(t): t \geq t_0\}$ is a totally bounded, collectively compact subset of $[X, X]$. Consequently, $T(t)$ is continuous in the uniform operator topology for $t \geq t_0$.

**Proof.** Since $T(t) = T(t - t_0)T(t_0) = T(t_0)T(t - t_0)$ for $t \geq t_0$, it follows that $\mathcal{K} = T(t_0)\mathcal{F} = \mathcal{F}T(t_0)$. $T(t_0)$ is a compact operator and the collection $\mathcal{F}$ is equicontinuous. By Lemmas 2.1 and 2.3 of [2], both $\mathcal{K}$ and $\mathcal{K}^*$ are collectively compact. [2, Corollary 2.6] implies that $\mathcal{K}$ is a totally bounded subset of $[X, X]$. Since $T(t)$ is continuous in the strong operator topology, $T(t)$ is continuous in the uniform operator topology for $t \geq t_0$.

**Corollary 2.3.** Suppose every $T(t), t > 0$, is a compact operator. Let $\mathcal{F} = \{R(\lambda): \text{re}(\lambda) \geq 1\}$ be the collection of the resolvents of the infinitesimal generator $A$ corresponding to the half-plane $\{\lambda \in \mathbb{C}: \text{re}(\lambda) \geq 1\}$. Then $\mathcal{F}$ is a totally bounded, collectively compact set of operators.

It should be noted that for any $\alpha > 0$, the following arguments can be applied to $\{R(\lambda): \text{re}(\lambda) \geq \alpha\}$. One particular half-plane is chosen simply to keep the notation as uncomplicated as possible.

**Proof.** It will suffice to show that for each $\varepsilon > 0$, there exists a totally bounded, collectively compact set of operators $\mathcal{K}$ such that for any $R(\lambda) \in \mathcal{F}$, there exists a $K \in \mathcal{K}$ with $||R(\lambda) - K|| \leq \varepsilon$. For this $\varepsilon$, choose $\delta > 0$ with $\int_0^\infty e^{-\lambda t} dt < \varepsilon/M$, where $M$ is such that $||T(\lambda)|| \leq M$ for $t > 0$. Let $\lambda$ be any complex number with $\text{re}(\lambda) \geq 1$ and $x \in X$. Since $R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t) x dt$, $||R(\lambda)x - \int_0^\infty e^{-\lambda t} T(t) x dt|| \leq \int_0^\infty e^{-\lambda t} ||T(t)x|| dt \leq \int_0^\infty e^{-\lambda t} dt M ||x|| \leq \varepsilon ||x||$. Consequently, $||R(\lambda) - \int_0^\infty e^{-\lambda t} T(t) dt \leq \varepsilon$. Now $\mathcal{K} = \{\int_0^\infty e^{-\lambda t} T(t) dt: \text{re}(\lambda) \geq 1\}$ is a totally bounded, collectively compact set of operators. To see this, note that $\sup \int_0^\infty |e^{-\lambda t}| dt: \text{re}(\lambda) \geq 1 \leq 1$ and that both $\{T(t): t \geq \delta\}$ and $\{T^*(t): t \geq \delta\}$ are collectively compact. Lemma 2.1 implies that both
and \( \mathcal{K}^* \) are collectively compact. As before, [2, Corollary 2.6] implies that \( \mathcal{K} \) is a totally bounded subset of \([X, X]\).

The following lemma will be useful in the next section. Since a quotable reference cannot be found, a brief proof is included.

**Lemma 2.4.** Let \( \mathcal{S} \) be an equicontinuous semi-group of class \( C_0 \). Then \( R(\lambda) \) converges to zero in the strong operator topology as \( |\lambda| \to \infty, \text{re}(\lambda) \geq 1 \). Whenever \( \{R(\lambda); \text{re}(\lambda) \geq 1\} \) is a totally bounded subset of \([X, X]\), the \( R(\lambda) \) converge to zero in the uniform operator topology as \( |\lambda| \to \infty, \text{re}(\lambda) \geq 1 \).

**Proof.** The second assertion follows immediately from the first.

Let \( x \in D(A) \), the domain of the infinitesimal generator \( A \). Since \( R(\lambda)(\lambda - A)x = x \), we have the identity

\[
R(\lambda)x = \frac{1}{\lambda}[x + R(\lambda)Ax].
\]

By (2) of § 1, \( \{R(\lambda)Ax; \text{re}(\lambda) \geq 1\} \) is a bounded subset of \( X \). It follows that \( \|R(\lambda)x\| \to 0 \) as \( |\lambda| \to \infty, \text{re}(\lambda) \geq 1 \), for each \( x \in D(A) \). Since \( D(A) \) is dense in \( X \), the Banach-Steinhaus theorem implies that this type of convergence holds for each \( x \in X \). We see that the first assertion of this lemma holds also.

3. Semi-groups with compact resolvents. Suppose that the domain of the infinitesimal generator of a semi-group can be given a topology \( \tau \) such that the topological space \( \langle D(A), \tau \rangle \) is a Banach space and the natural injection \( i: \langle D(A), \tau \rangle \to X \) is a compact operator. In such cases, it might be possible to prove that certain sets of the resolvents of \( A \) are equicontinuous subsets of \([X, \langle D(A), \tau \rangle]\), i.e., collectively compact subsets of \([X, X]\). A specific example is the case in which \( X \) is some \( L^p \) space and \( A \) is the negative of a uniformly strongly elliptic differential operator defined on a Sobolev space \( H = \langle D(A), \tau \rangle \). The so-called "a priori inequalities" [4, Theorems 18.2 and 19.2, pages 69 and 77] imply that, after a suitable translation, \( \{R(\lambda); \text{re}(\lambda) \geq 1\} \) is an equicontinuous subset of \([L^p, H]\). Since the injection \( i: H \to L^p \) is a compact operator [4, Theorem 11.2, page 31], \( \{R(\lambda); \text{re}(\lambda) \geq 1\} \) is a collectively compact subset of \([L^p, L^p]\). The obvious question is what are the implications of such assumptions for a general semi-group \( \mathcal{S} \).

We first consider the case in which \( A \) has one compact resolvent. Of course, the first resolvent equation,
then implies that all resolvents of $A$ are compact operators.

**Lemma 3.1.** Suppose $A$ has one compact resolvent. Let $\Omega$ be a compact subset of $\{ \lambda : \text{re}(\lambda) > 0 \}$. Then $\{R(\lambda) : \lambda \in \Omega \}$ is collectively compact.

**Proof.** Since $R(\lambda)$ is a holomorphic function in the right half-plane, $\{R(\lambda) : \lambda \in \Omega \}$ is a totally bounded subset of $[X, X]$. Each element in this collection is a compact operator. So [2, Corollary 2.7] implies that $\{R(\lambda) : \lambda \in \Omega \}$ is collectively compact.

The following is a partial converse of Theorem 2.2.

**Proposition 3.2.** Suppose $A$ has compact resolvents. Let $t_{0} > 0$. If $T(t)$ is continuous in the uniform operator topology for $t \in [t_{0}, \infty)$, then $T(t_{0})$ is a compact operator.

**Proof.** Since the resolvents are Laplace transforms of $\{T(t) : t \geq 0\}$, we may use the formula based upon fractional integration of order two [6, page 220] which states that

$$
\int_{0}^{s} (s - t)T(t)dt = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{is} R(\lambda) d\lambda, \quad s > 0.
$$

For $\varepsilon > 0$, choose $N$ such that

$$
\int_{1-i\infty}^{1+i\infty} + \int_{1-i\infty}^{1+i\infty} \frac{1}{|\lambda|^2} ||e^{is}R(\lambda)|| d|\lambda| < \varepsilon.
$$

Then

$$
\left| \int_{0}^{s} (s - t)T(t)dt - \int_{1-i\infty}^{1+i\infty} \frac{e^{is}R(\lambda)}{\lambda^2} d\lambda \right| < \varepsilon.
$$

By Lemmas 3.1 and 2.1, the integral of $(e^{is}/\lambda^2)R(\lambda)$ over the finite segment of the vertical line is a compact operator. It follows that for each $s \geq 0$, $\int_{0}^{s} (s - t)T(t)dt$ is a compact operator.

Consider the function

$$
F(s) = \int_{0}^{s} (s - t)T(t)dt, \quad s \geq 0.
$$

Each value of $F$ is a compact operator. Elementary calculations show that $F$ is differentiable in the uniform operator topology. Consequently, each
\[ F'(s) = \int_0^s T(t)\,dt, \quad s \geq 0, \]

is the limit in the uniform operator topology of a sequence of compact operators. Hence, each \( F'(s), \ s \geq 0, \) is a compact operator. In taking derivatives again, we see that for \( h > 0, \)

\[ \left\| \frac{1}{h} \int_{t_0}^{t_0+h} T(t)\,dt - T(t_0) \right\| \leq \sup \{|T(t_0 + \alpha) - T(t_0)|: 0 \leq \alpha \leq h\}. \]

If \( T(t_0 + \alpha) \) is continuous in the uniform operator topology for \( \alpha \geq 0, \) then

\[ T(t_0) = \text{uniform lim} \frac{1}{h} \int_{t_0}^{t_0+h} T(t)\,dt. \]

It follows that \( T(t_0) \) is a compact operator.

See [6, page 537] for a discussion of the following example.

**Example 3.3.** Consider the semi-group \( S \) of left translations on the space \( C_0[0, 1] \) consisting of continuous functions \( x(u) \) vanishing at 1, where the norm \( \|x\| = \sup \{|x(u)|: 0 \leq u \leq 1\} \). Let \( [T(t)x](u) = x(u + t), \) for \( 0 \leq u \leq \max \{0, 1 - t\}, \) and 0 for \( \max \{0, 1 - t\} \leq u \leq 1. \) The infinitesimal generator of \( S \) is the operator of differentiation \( d/(du) \) with domain

\[ D\left(\frac{d}{du}\right) = \{x: x' \in C_0[0, 1]\}. \]

The compact resolvents are given by

\[ [R(\lambda)x](u) = \int_0^{1-u} e^{-\lambda t} x(u + t)\,dt, \ \lambda \in \mathbb{C}. \]

For \( t \geq 1, \) \( T(t) \) is the compact operator 0 while for \( t, s < 1, \)

\[ \|T(t) - T(s)\| = 2. \]

This can easily be seen by evaluating \( T(t) - T(s) \) at a function \( x \in C_0[0, 1] \) with \( \|x\| \leq 1 \) and \( x(t) = 1, \ x(s) = -1. \)

So \( T(t) \) is continuous in the uniform operator topology only for \( t \geq 1. \)

Choose a monotonically increasing sequence of positive functions \( \{y_n\} \subseteq C_0[0, 1] \) such that \( \lim_n y_n(u) = 1 \) for each \( u < 1. \) For \( t < 1, \)

\( \{T(t)y_n\} \) is a sequence of functions having no subsequence which can converge uniformly. So \( T(t), t < 1, \) is not a compact operator.

For \( \lambda = \sigma + i\tau, \) let \( x_n(u) = e^{i\tau s} y_n(u) \) in the definition of \( R(\lambda). \)

We see that

\[ [R(\lambda)x_n](0) = \int_0^1 e^{-\sigma t} y_n(t)\,dt. \]
Since \( ||x_n|| = 1 \) for each \( n \),

\[
|| R(\lambda) || \geq \sup_n |(R(\lambda)x_n)(0)| = \int_0^1 e^{-\sigma t} dt .
\]

It follows immediately from the definition of \( R(\lambda) \) that the reverse inequality holds also. Consequently, \( || R(\lambda) || = \int_0^1 e^{-\sigma t} dt \). In particular, \( \lim_{|t| \to \infty} ||R(\sigma + i\tau)|| \neq 0 \). This serves to distinguish this differential operator from the class of infinitesimal generators which we consider next.

**Lemma 3.4.** Suppose \( \mathcal{S} \) is a semi-group such that the set of resolvents \( \{R(\lambda): \text{re}(\lambda) = 1\} \) corresponding to the vertical line \( \text{re}(\lambda) = 1 \) is collectively compact. Then \( \{R(\lambda): \text{re}(\lambda) \geq 1\} \) is also collectively compact.

**Proof.** For each \( x \in X \), \( R(\lambda)x \) is a holomorphic and bounded function of \( \lambda, \, \text{re}(\lambda) > 1/2 \). So \( R(\lambda)x \) admits Poisson's integral representation [6, page 229]

\[
R(\sigma + i\tau)x = \frac{\sigma - 1}{\pi} \int_{-\infty}^{\infty} \frac{R(1 + i\beta)x}{(\sigma - 1)^2 + (\tau - \beta)^2} d\beta
\]

for \( \sigma > 1, \, x \in X \). Since \( \{R(1 + i\beta): -\infty < \beta < \infty\} \) is collectively compact and the integral of the Poisson kernel over \( -\infty < \beta < \infty \) is identically one, Lemma 2.1 implies that \( \{R(\lambda): \text{re}(\lambda) > 1\} \) is collectively compact. Taking the union of this set and \( \{R(\lambda): \text{re}(\lambda) = 1\} \), one obtains the desired result.

For \( x \in X \) and \( x^* \in X^* \),

\[
\langle x^*, R(\sigma + i\tau)x \rangle = \int_0^\infty e^{-\sigma t}(e^{-\sigma t}\langle x^*, T(t)x \rangle)dt .
\]

This is this Fourier transform of the absolutely summable function \( e^{-\sigma t}\langle x^*, T(t)x \rangle, \, t \geq 0 \). The convergence of

\[
|| R(\sigma + i\tau) || = \sup \{ || \langle x^*, R(\sigma + i\tau)x \rangle : || x ||, || x^* || \leq 1 \}
\]

to 0 as \( |\sigma| \) and \( |\tau| \) approach infinity can be viewed as a "uniform" Riemann-Lebesgue lemma.

**Theorem 3.5.** If \( \mathcal{S} = \{R(\lambda): \text{re}(\lambda) \geq 1\} \) is collectively compact, then \( || R(\lambda) || \) converges to 0 as \( |\lambda| \) approaches \( \infty \), \( \text{re}(\lambda) \geq 1 \).

**Proof.** Throughout the following proof, we assume that \( \text{re}(\lambda) \geq 1 \).
Let $\varepsilon > 0$ be given and choose real $\beta$ so large that $1 + \beta \geq M/\varepsilon$, where $M$ is the constant in §1 which bounds the operator norms of elements of $\mathcal{S}$. By (2),

$$|| R(\lambda + \beta) || \leq \frac{M}{\text{re}(\lambda) + \beta} \leq \frac{M}{1 + \beta} \leq \varepsilon.$$ 

In view of Lemma 2.4, $\mathcal{F}$ is an equicontinuous collection with $R(\lambda)$ converging to zero as $|\lambda| \to \infty$ pointwise on the relatively compact set $\mathcal{F}(x)$. Therefore, $|| R(\lambda)F || \to 0$ as $|\lambda| \to \infty$ uniformly for $F \in \mathcal{F}$. Choose $N$ such that $|\lambda| \geq N$ implies that

$$|| R(\lambda)R(\lambda + \beta) || \leq \varepsilon/\beta.$$ 

The first resolvent equation states that

$$R(\lambda) - R(\lambda + \beta) = (\lambda + \beta - \lambda)R(\lambda)R(\lambda + \beta).$$

So, for $|\lambda| \geq N$,

$$|| R(\lambda) || \leq || \beta R(\lambda)R(\lambda + \beta) || + || R(\lambda + \beta) || \leq 2\varepsilon.$$

Note that we have used the fact that $\mathcal{F}$ contains those resolvents $R(\lambda)$ with $\text{re}(\lambda)$ arbitrarily large in an essential way.

**Corollary 3.6.** Let $\mathcal{S}$ be any semi-group whose infinitesimal generator $A$ has compact resolvents, i.e., each $R(\lambda)$, $\text{re}(\lambda) > 0$, is a compact operator. Then $\mathcal{F} = \{R(\lambda): \text{re}(\lambda) \geq 1\}$ is collectively compact if and only if $|| R(\lambda) || \to 0$ as $|\lambda| \to \infty$, $\text{re}(\lambda) \geq 1$.

**Proof.** The assumption that $|| R(\lambda) || \to 0$ as $|\lambda| \to \infty$, $\text{re}(\lambda) \geq 1$, simply implies that $R(\lambda)$ can be extended to a continuous function on the compactification of the half-plane $\{\lambda: \text{re}(\lambda) \geq 1\}$. Consequently, if $A$ has compact resolvents, $\mathcal{F}$ is a totally bounded set of compact operators. [2, Corollary 2.7] implies that $\mathcal{F}$ is collectively compact.

The converse is simply Theorem 3.5.

The behavior of the holomorphic function $R(\lambda)$ on the vertical line $\text{re}(\lambda) = 1$ is of fundamental importance. For example, if $d(\lambda)$ denotes the distance of the complex number $\lambda$ from the spectrum of $A$, then [3, page 566]

$$d(1 + i\tau) \geq \frac{1}{|| R(1 + i\tau) ||}.$$

We see that the spectrum of $A$ must be bounded on the right by the curve.
\[ \gamma(\tau) = 1 - \frac{1}{||R(1 + i\tau)||} + i\tau, \quad -\infty < \tau < \infty. \]

In particular, it follows from Theorem 3.5 and Lemma 3.4 that when \( \{R(\lambda): \text{re}(\lambda) = 1\} \) is collectively compact, the spectrum of \( A \) is severely restricted.

The usual methods of inverting Fourier transforms can be typified by the use of \((C, 1)\) means. In [5, page 350], it is shown that for each \( t > 0 \)

\[ T(t) = \lim_{w \to \infty} \frac{1}{2\pi} \int_{-w}^{w} \left(1 - \frac{|\tau|}{w}\right)e^{-(1+i\tau)t}R(1 + i\tau)d\tau. \]

However, the measures involved no longer satisfy the requirements of Lemma 2.1. As this situation is typical, we are not able to prove that if \( \{R(\lambda): \text{re}(\lambda) = 1\} \) is collectively compact, then each \( T(t) \in \mathcal{S}, \ t > 0 \), is a compact operator.

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