SEMI-GROUPS AND COLLECTIVELY COMPACT SETS OF LINEAR OPERATORS

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A set of linear operators from one Banach space to another is collectively compact if and only if the union of the images of the unit ball has compact closure. Semi-groups \( S = \{ T(t) : t \geq 0 \} \) of bounded linear operators on a complex Banach space into itself and in which every operator \( T(t) \), \( t > 0 \) is compact are considered. Since \( T(t_1 + t_2) = T(t_1)T(t_2) \) for each operator in the semi-group, it would be expected that the theory of collectively compact sets of linear operators could be profitably applied to semi-groups.

1. Introduction. Let \( X \) be a complex Banach space with unit ball \( X \) and let \( [X, X] \) denote the space of all bounded linear operators on \( X \) equipped with the uniform operator topology. The semi-group definitions and terminology used are those of Hille and Phillips [6]. Let \( S \) be a semi-group of vector-valued functions \( T : [0, \infty) \to [X, X] \). It is assumed that \( T(t) \) is strongly continuous for \( t \geq 0 \). If \( \lim_{t \to t_0} \| T(t)x - T(t_0)x \| = 0 \) for each \( t_0 \geq 0 \), \( x \in X \) and if there is a constant \( M \) such that the \( \| T(t) \| \leq M \) for each \( t \geq 0 \), then \( S = \{ T(t) : t \geq 0 \} \) is called an equicontinuous semi-group of class \( C_0 \). The infinitesimal generator \( A \) of the semi-group \( S \) is defined by

\[
Ax = \lim_{s \to 0} \frac{1}{s} [T(s)x - x]
\]

whenever the limit exists. The domain \( D(A) \) of \( A \) is a dense subset of \( X \) consisting of just those elements \( x \) for which this limit exists. \( A \) is a closed linear operator having resolvents \( R(\lambda) \) which, for each complex number \( \lambda \) with the real part of \( \lambda \) greater than zero, are given by the absolutely summable Riemann-Stieltjes integral

\[
R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x dt, \quad x \in X.
\]

It follows from (1) that

\[
\| R(\lambda) \| \leq \frac{M}{\text{re}(\lambda)}, \quad \text{re}(\lambda) > 0.
\]

In particular, sets of the type \( \{ R(\lambda) : \text{re}(\lambda) \geq \alpha > 0 \} \) are equicontinuous subsets of \( [X, X] \).

Results yielding the collective compactness of the resolvents of
2. Semi-groups of compact operators. First, note that (1) states that the resolvents of $A$ are Laplace transforms of the semi-group $S$. Consequently, there are many other important integral expressions involving the elements of the semi-group and the resolvents. In order to take advantage of these, we prove the following lemma, in which $|v|$ denotes the total variation of a complex measure $v$.

**Lemma 2.1.** Let $\Omega$ be a topological space and $\mathcal{M}$ a collection of complex-valued Borel measures on $\Omega$. Suppose there exists a constant $\alpha$ for which $|v| \leq \alpha$ for each $v \in \mathcal{M}$. Let $\mathcal{K} : \Omega \to [X, X]$ be an operator-valued function defined on $\Omega$ which is strongly measurable with respect to each $v \in \mathcal{M}$ and $x \in X$, let $F_v(x) = \int_{\Omega} K(w)x \, dv$, where the integral exists in the Bochner sense since $\int_{\Omega} \|K(w)x\| \, dv < \infty$ [6, page 80]. Let $\mathcal{F} = \{F_v : v \in \mathcal{M}\}$. Whenever $\mathcal{K}(\mathcal{K}^*)$ is collectively compact, $\mathcal{F}(\mathcal{F}^*)$ is also collectively compact.

**Proof.** Assume that $\mathcal{K}$ is collectively compact. Let $B = \{K(w)x : w \in \Omega, \|x\| \leq 1\}$ and let $C$ denote the balanced convex hull of $B$. Both $B$ and $C$ are totally bounded subsets of $X$. It suffices to show that $F_v(x) \in \alpha C$ for any $F_v \in \mathcal{F}$ and $x$ with $\|x\| \leq 1$. Let $\varepsilon > 0$ and choose $\{K(w_i)x_i, \ldots, K(w_n)x_n\}$, an $\varepsilon/\alpha$-net for $B$. For $i = 1, \ldots, n$, let $\Omega_i = \{w : \|K(w)x - K(w_i)x_i\| \leq \varepsilon/\alpha\}$ and let $\Omega_i = \bigcup_{i=1}^{n} \Omega_i$ be a decomposition of the $\Omega_i$ into pairwise disjoint sets. Then

$$\left\|F_v(x) - \sum_{i=1}^{n} K(w_i)x_i v(\Omega_i)\right\| \leq \sum_{i=1}^{n} \int_{\Omega_i} \|K(w)x - K(w_i)x_i\| \, d|v|(w) \leq (\varepsilon/\alpha) \int_{\Omega} |v| \, v(\Omega) \leq \varepsilon.$$  

Since $\sum_{i=1}^{n} v(\Omega_i) \leq \alpha$, $\sum_{i=1}^{n} K(w_i)x_i v(\Omega_i)$ is an element of $\alpha C$. It follows that $F_v(x) \in \alpha C$ and so $\mathcal{F}$ is also collectively compact.

Now assume that $\mathcal{K}^*$ is collectively compact. Let $V$ be any neighborhood of $0$ in the norm topology of $X$. There exists an $\varepsilon > 0$ such that $U = \{x : \|x\| \leq \varepsilon\} \subseteq V$. Since $\mathcal{K}^*$ is collectively compact, [2, Theorem 2.11, part (c)] implies that there exists a weak neighborhood $W$ of the origin with $\mathcal{K}(W \cap X_i) \subseteq (1/\alpha)U$. For $F_v \in \mathcal{F}$ and $x \in W \cap X_i$, $\|F_v(x)\| \leq \int_{\Omega} \|K(w)x\| \, d|v| \leq (\varepsilon/\alpha) \int_{\Omega} |v| \, v(\Omega) \leq \varepsilon$.
ε. So \( \mathcal{F}(W \cap X,) \subseteq V \). Again using [2, Theorem 2.1, part (c)], we see that \( \mathcal{F}^* \) is also collectively compact.

The following is essentially a result of P. Lax [6, page 304]. Rephrased in the terminology of collectively compact sets of operators, it becomes quite transparent.

**Theorem 2.2.** Suppose that some \( T(t_0), t_0 > 0, \) is a compact operator. Then \( \mathcal{H} = \{ T(t) : t \geq t_0 \} \) is a totally bounded, collectively compact subset of \([X, X]\). Consequently, \( T(t) \) is continuous in the uniform operator topology for \( t \geq t_0 \).

**Proof.** Since \( T(t) = T(t - t_0)T(t_0) = T(t_0)T(t - t_0) \) for \( t \geq t_0, \) it follows that \( \mathcal{H} = T(t_0)\mathcal{I} = \mathcal{F}T(t_0) \). \( T(t_0) \) is a compact operator and the collection \( \mathcal{I} \) is equicontinuous. By Lemmas 2.1 and 2.3 of [2], both \( \mathcal{H} \) and \( \mathcal{H}^* \) are collectively compact. [2, Corollary 2.6] implies that \( \mathcal{H} \) is a totally bounded subset of \([X, X]\). Since \( T(t) \) is continuous in the strong operator topology, \( T(t) \) is continuous in the uniform operator topology for \( t \geq t_0 \).

**Corollary 2.3.** Suppose every \( T(t), t > 0, \) is a compact operator. Let \( \mathcal{F} = \{ R(\lambda) : \Re(\lambda) \geq 1 \} \) be the collection of the resolvents of the infinitesimal generator \( A \) corresponding to the half-plane \( \{ \lambda \in \mathbb{C} : \Re(\lambda) \geq 1 \} \). Then \( \mathcal{F} \) is a totally bounded, collectively compact set of operators.

It should be noted that for any \( \alpha > 0, \) the following arguments can be applied to \( \{ R(\lambda) : \Re(\lambda) \geq \alpha \} \). One particular half-plane is chosen simply to keep the notation as uncomplicated as possible.

**Proof.** It will suffice to show that for each \( \varepsilon > 0, \) there exists a totally bounded, collectively compact set of operators \( \mathcal{H} \) such that for any \( R(\lambda) \in \mathcal{F}, \) there exists a \( K \in \mathcal{H} \) with \( ||R(\lambda) - K|| \leq \varepsilon. \) For this \( \varepsilon, \) choose \( \delta > 0 \) with \( \int_0^\delta e^{-t}dt < \varepsilon/M, \) where \( M \) is such that \( ||T(\lambda)|| \leq M \) for \( t > 0. \) Let \( \lambda \) be any complex number with \( \Re(\lambda) \geq 1 \) and \( x \in X. \) Since \( R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)xdt, \) \( ||R(\lambda)x - \int_0^\infty e^{-\lambda t}T(t)xdt|| \leq \int_0^\delta e^{-\lambda t}||T(t)x||dt \leq \int_0^\delta e^{-\lambda t}dtM||x|| \leq \varepsilon||x||. \) Consequently, \( ||R(\lambda) - \int_0^\infty e^{-\lambda t}T(t)dt|| \leq \varepsilon. \) Now \( \mathcal{H} = \{ \int_0^\infty e^{-\lambda t}T(t)dt : \Re(\lambda) \geq 1 \} \) is a totally bounded, collectively compact set of operators. To see this, note that \( \sup \left\{ \int_0^\infty |e^{-\lambda t}|dt : \Re(\lambda) \geq 1 \right\} \leq 1 \) and that both \( \{ T(t) : t \geq \delta \} \) and \( \{ T^*(t) : t \geq \delta \} \) are collectively compact. Lemma 2.1 implies that both
\( \mathcal{K} \) and \( \mathcal{K}^* \) are collectively compact. As before, [2, Corollary 2.6] implies that \( \mathcal{K} \) is a totally bounded subset of \([X, X]\).

The following lemma will be useful in the next section. Since a quotable reference cannot be found, a brief proof is included.

**Lemma 2.4.** Let \( \mathcal{S} \) be an equicontinuous semi-group of class \( C_0 \). Then \( R(\lambda) \) converges to zero in the strong operator topology as \( |\lambda| \to \infty, \ re(\lambda) \geq 1 \). Whenever \( \{R(\lambda): re(\lambda) \geq 1\} \) is a totally bounded subset of \([X, X]\), the \( R(\lambda) \) converge to zero in the uniform operator topology as \( |\lambda| \to \infty, \ re(\lambda) \geq 1 \).

**Proof.** The second assertion follows immediately from the first.

Let \( x \in D(A) \), the domain of the infinitesimal generator \( A \). Since \( R(\lambda)(\lambda - A)x = x \), we have the identity

\[
R(\lambda)x = \frac{1}{\lambda}[x + R(\lambda)Ax].
\]

By (2) of § 1, \( \{R(\lambda)Ax: re(\lambda) \geq 1\} \) is a bounded subset of \( X \). It follows that \( ||R(\lambda)x|| \to 0 \) as \( |\lambda| \to \infty, \ re(\lambda) \geq 1 \), for each \( x \in D(A) \). Since \( D(A) \) is dense in \( X \), the Banach-Steinhaus theorem implies that this type of convergence holds for each \( x \in X \). We see that the first assertion of this lemma holds also.

3. Semi-groups with compact resolvents. Suppose that the domain of the infinitesimal generator of a semi-group can be given a topology \( \tau \) such that the topological space \( \langle D(A), \tau \rangle \) is a Banach space and the natural injection \( i: \langle D(A), \tau \rangle \to X \) is a compact operator. In such cases, it might be possible to prove that certain sets of the resolvents of \( A \) are equicontinuous subsets of \([X, \langle D(A), \tau \rangle]\), i.e., collectively compact subsets of \([X, X]\). A specific example is the case in which \( X \) is some \( L^p \) space and \( A \) is the negative of a uniformly strongly elliptic differential operator defined on a Sobolev space \( H = \langle D(A), \tau \rangle \). The so-called “a priori inequalities” [4, Theorems 18.2 and 19.2, pages 69 and 77] imply that, after a suitable translation, \( \{R(\lambda): re(\lambda) \geq 1\} \) is an equicontinuous subset of \([L^p, H]\). Since the injection \( i: H \to L^p \) is a compact operator [4, Theorem 11.2, page 31], \( \{R(\lambda): re(\lambda) \geq 1\} \) is a collectively compact subset of \([L^p, L^p]\). The obvious question is what are the implications of such assumptions for a general semi-group \( \mathcal{S} \).

We first consider the case in which \( A \) has one compact resolvent. Of course, the first resolvent equation,
then implies that all resolvents of $A$ are compact operators.

**Lemma 3.1.** Suppose $A$ has one compact resolvent. Let $\Omega$ be a compact subset of $\{\lambda: \text{re}(\lambda) > 0\}$. Then $\{R(\lambda): \lambda \in \Omega\}$ is collectively compact.

**Proof.** Since $R(\lambda)$ is a holomorphic function in the right half-plane, $\{R(\lambda): \lambda \in \Omega\}$ is a totally bounded subset of $[X, X]$. Each element in this collection is a compact operator. So [2, Corollary 2.7] implies that $\{R(\lambda): \lambda \in \Omega\}$ is collectively compact.

The following is a partial converse of Theorem 2.2.

**Proposition 3.2.** Suppose $A$ has compact resolvents. Let $t > 0$. If $T(t)$ is continuous in the uniform operator topology for $t \in [t_0, \infty)$, then $T(t_0)$ is a compact operator.

**Proof.** Since the resolvents are Laplace transforms of $\{T(t): t \geq 0\}$, we may use the formula based upon fractional integration of order two [6, page 220] which states that

$$\int_0^s (s - t)T(t)dt = \frac{1}{2\pi i} \int_{1 - i\infty}^{1 + i\infty} e^{is}\frac{R(\lambda)}{\lambda}d\lambda, \quad s > 0.$$ 

For $\varepsilon > 0$, choose $N$ such that

$$\int_{1 - iN}^{1 + i\infty} + \int_{1 - i\infty}^{1 + iN} \frac{1}{|\lambda^2|} ||e^{is}\frac{R(\lambda)}{\lambda}||d\lambda < \varepsilon.$$ 

Then

$$\left| \int_0^s (s - t)T(t)dt - \frac{1}{2\pi i} \int_{1 - iN}^{1 + i\infty} e^{is}\frac{R(\lambda)}{\lambda}d\lambda \right| < \varepsilon.$$ 

By Lemmas 3.1 and 2.1, the integral of $(e^{is}/\lambda^2)R(\lambda)$ over the finite segment of the vertical line is a compact operator. It follows that for each $s \geq 0$, $\int_0^s (s - t)T(t)dt$ is a compact operator.

Consider the function

$$F(s) = \int_0^s (s - t)T(t)dt, \quad s \geq 0.$$ 

Each value of $F$ is a compact operator. Elementary calculations show that $F$ is differentiable in the uniform operator topology. Consequently, each
\[ F'(s) = \int_0^s T(t) dt, \quad s \geq 0 , \]

is the limit in the uniform operator topology of a sequence of compact operators. Hence, each \( F'(s), \ s \geq 0, \) is a compact operator. In taking derivatives again, we see that for \( h > 0, \)

\[ \left| \frac{1}{h} \int_{t_0}^{t_0 + h} T(t) dt - T(t_0) \right| \leq \sup \{ \| T(t_0 + \alpha) - T(t_0) \| : 0 \leq \alpha \leq h \} . \]

If \( T(t_0 + \alpha) \) is continuous in the uniform operator topology for \( \alpha \geq 0, \) then

\[ T(t_0) = \text{uniform limit} \left( \frac{1}{h} \int_{t_0}^{t_0 + h} T(t) dt \right) . \]

It follows that \( T(t_0) \) is a compact operator.

See [6, page 537] for a discussion of the following example.

**Example 3.3.** Consider the semi-group \( \mathcal{S} \) of left translations on the space \( C_0[0, 1] \) consisting of continuous functions \( x(u) \) vanishing at 1, where the norm \( \| x \| = \sup \{ |x(u)| : 0 \leq u \leq 1 \} \). Let \( [T(t)x](u) = x(u + t), \) for \( 0 \leq u \leq \max \{0, 1 - t\}, \) and 0 for \( \max \{0, 1 - t\} \leq u \leq 1 \). The infinitesimal generator of \( \mathcal{S} \) is the operator of differentiation \( d/(du) \) with domain

\[ D\left( \frac{d}{du} \right) = \{ x : x' \in C_0[0, 1] \} . \]

The compact resolvents are given by

\[ [R(\lambda)x](u) = \int_0^{1-u} e^{-\lambda t} x(u + t) dt, \ \lambda \in \mathbb{C} . \]

For \( t \geq 1, \) \( T(t) \) is the compact operator 0 while for \( t, s < 1, \)

\( \| T(t) - T(s) \| = 2. \) This can easily be seen by evaluating \( T(t) - T(s) \) at a function \( x \in C_0[0, 1] \) with \( \| x \| \leq 1 \) and \( x(t) = 1, \ x(s) = -1. \)

So \( T(t) \) is continuous in the uniform operator topology only for \( t \geq 1. \)

Choose a monotonically increasing sequence of positive functions \( \{ y_n \} \subseteq C_0[0, 1] \) such that \( \lim_n y_n(u) = 1 \) for each \( u < 1. \) For \( t < 1, \)

\( \{ T(t)y_n \} \) is a sequence of functions having no subsequence which can converge uniformly. So \( T(t), \ t < 1, \) is not a compact operator.

For \( \lambda = \sigma + i\tau, \) let \( x_n(u) = e^{i\tau u} y_n(u) \) in the definition of \( R(\lambda). \)

We see that

\[ [R(\lambda)x_n](0) = \int_0^1 e^{-\sigma t} y_n(t) dt . \]
Since $\|x_n\| = 1$ for each $n$,

$$\| R(\lambda) \| \geq \sup_n |[R(\lambda)x_n](0)| = \int_0^1 e^{-\sigma t} dt.$$ 

It follows immediately from the definition of $R(\lambda)$ that the reverse inequality holds also. Consequently, $\| R(\lambda) \| = \int_0^1 e^{-\sigma t} dt$. In particular, $\lim_{t \to -\infty} ||R(\sigma + i\tau)|| = 0$. This serves to distinguish this differential operator from the class of infinitesimal generators which we consider next.

**Lemma 3.4.** Suppose $\mathcal{S}$ is a semi-group such that the set of resolvents $\{R(\lambda); \text{re}(\lambda) = 1\}$ corresponding to the vertical line $\text{re}(\lambda) = 1$ is collectively compact. Then $\{R(\lambda); \text{re}(\lambda) \geq 1\}$ is also collectively compact.

**Proof.** For each $x \in X$, $R(\lambda)x$ is a holomorphic and bounded function of $\lambda$, $\text{re}(\lambda) > 1/2$. So $R(\lambda)x$ admits Poisson's integral representation [6, page 229]

$$R(\sigma + i\tau)x = \frac{\sigma - 1}{\pi} \int_{-\infty}^{\infty} \frac{R(1 + i\beta)x}{(\sigma - 1)^2 + (\tau - \beta)^2} d\beta$$

for $\sigma > 1$, $x \in X$. Since $\{R(1 + i\beta); -\infty < \beta < \infty\}$ is collectively compact and the integral of the Poisson kernel over $-\infty < \beta < \infty$ is identically one, Lemma 2.1 implies that $\{R(\lambda); \text{re}(\lambda) > 1\}$ is collectively compact. Taking the union of this set and $\{R(\lambda); \text{re}(\lambda) = 1\}$, one obtains the desired result.

For $x \in X$ and $x^* \in X^*$,

$$\langle x^*, R(\sigma + i\tau)x \rangle = \int_0^\infty e^{-\sigma t} e^{-i\tau \langle x^*, T(t)x \rangle} dt.$$ 

This is this Fourier transform of the absolutely summable function $e^{-\sigma t} \langle x^*, T(t)x \rangle$, $t \geq 0$. The convergence of

$$\| R(\sigma + i\tau) \| = \sup \{ |\langle x^*, R(\sigma + i\tau)x \rangle| : \|x\|, \|x^*\| \leq 1 \}$$

to 0 as $|\sigma|$ and $|\tau|$ approach infinity can be viewed as a "uniform" Riemann-Lebesgue lemma.

**Theorem 3.5.** If $\mathcal{S} = \{R(\lambda); \text{re}(\lambda) \geq 1\}$ is collectively compact, then $\| R(\lambda) \|$ converges to 0 as $|\lambda|$ approaches $\infty$, $\text{re}(\lambda) \geq 1$.

**Proof.** Throughout the following proof, we assume that $\text{re}(\lambda) \geq 1$. 

Let $\varepsilon > 0$ be given and choose real $\beta$ so large that $1 + \beta \geq M/\varepsilon$, where $M$ is the constant in §1 which bounds the operator norms of elements of $\mathcal{S}$. By (2),

$$\| R(\lambda + \beta) \| \leq \frac{M}{\text{re}(\lambda) + \beta} \leq \frac{M}{1 + \beta} \leq \varepsilon .$$

In view of Lemma 2.4, $\mathcal{F}$ is an equicontinuous collection with $R(\lambda)$ converging to zero as $|\lambda| \to \infty$ pointwise on the relatively compact set $\mathcal{F}(X_i)$. Therefore, $\| R(\lambda)F \| \to 0$ as $|\lambda| \to \infty$ uniformly for $F \in \mathcal{F}$. Choose $N$ such that $|\lambda| \geq N$ implies that

$$\| R(\lambda)R(\lambda + \beta) \| \leq \varepsilon/\beta .$$

The first resolvent equation states that

$$R(\lambda) - R(\lambda + \beta) = (\lambda + \beta - \lambda)R(\lambda)R(\lambda + \beta) .$$

So, for $|\lambda| \geq N$,

$$\| R(\lambda) \| \leq \| \beta R(\lambda)R(\lambda + \beta) \| + \| R(\lambda + \beta) \| \leq 2\varepsilon .$$

Note that we have used the fact that $\mathcal{F}$ contains those resolvents $R(\lambda)$ with $\text{re}(\lambda)$ arbitrarily large in an essential way.

**Corollary 3.6.** Let $\mathcal{S}$ be any semi-group whose infinitesimal generator $A$ has compact resolvents, i.e., each $R(\lambda)$, $\text{re}(\lambda) > 0$, is a compact operator. Then $\mathcal{F} = \{ R(\lambda): \text{re}(\lambda) \geq 1 \}$ is collectively compact if and only if $\| R(\lambda) \| \to 0$ as $|\lambda| \to \infty$, $\text{re}(\lambda) \geq 1$.

**Proof.** The assumption that $\| R(\lambda) \| \to 0$ as $|\lambda| \to \infty$, $\text{re}(\lambda) \geq 1$, simply implies that $R(\lambda)$ can be extended to a continuous function on the compactification of the half-plane $\{ \lambda: \text{re}(\lambda) \geq 1 \}$. Consequently, if $A$ has compact resolvents, $\mathcal{F}$ is a totally bounded set of compact operators. [2, Corollary 2.7] implies that $\mathcal{F}$ is collectively compact.

The converse is simply Theorem 3.5.

The behavior of the holomorphic function $R(\lambda)$ on the vertical line $\text{re}(\lambda) = 1$ is of fundamental importance. For example, if $d(\lambda)$ denotes the distance of the complex number $\lambda$ from the spectrum of $A$, then [3, page 566]

$$d(1 + i\tau) \geq \frac{1}{\| R(1 + i\tau) \|} .$$

We see that the spectrum of $A$ must be bounded on the right by the curve
\[ \gamma(\tau) = 1 - \frac{1}{\| R(1 + i\tau) \|} + i\tau, \quad -\infty < \tau < \infty. \]

In particular, it follows from Theorem 3.5 and Lemma 3.4 that when \( \{R(\lambda): re(\lambda) = 1\} \) is collectively compact, the spectrum of \(A\) is severely restricted.

The usual methods of inverting Fourier transforms can be typified by the use of \((C, 1)\) means. In [5, page 350], it is shown that for each \( t > 0 \)

\[ T(t) = \lim_{w \to \infty} \frac{1}{2\pi} \int_{-w}^{w} \left( 1 - \frac{|\tau|}{w} \right)e^{(1+i\tau)t}R(1 + i\tau)d\tau. \]

However, the measures involved no longer satisfy the requirements of Lemma 2.1. As this situation is typical, we are not able to prove that if \( \{R(\lambda): re(\lambda) = 1\} \) is collectively compact, then each \( T(t) \in \mathcal{S}; t > 0 \), is a compact operator.

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Received April 4, 1973.

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