ON THE IRRATIONALITY OF CERTAIN SERIES

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A criterion is established for the rationality of series of the form \( \sum b_n/(a_1, \cdots, a_n) \) where \( a_n, b_n \) are integers, \( a_n \geq 2 \) and \( \lim b_n/(a_{n-1}a_n) = 0 \). This criterion is applied to prove irrationality and rational independence of certain special series of the above type.

1. Introduction. In an earlier paper [2] we proved the following result:

**Theorem 1.1.** If \( \{a_n\} \) is a monotonic sequence of positive integers with \( a_n \geq n^{1/12} \) for all large \( n \), then the series

\[
\sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1a_2 \cdots a_n} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\sigma(n)}{a_1a_2 \cdots a_n}
\]

are irrational.

We conjectured that the series (1.2) are irrational under the single assumption that \( \{a_n\} \) is monotonic and we observed that some such condition is needed in view of the possible choices \( a_n = \varphi(n) + 1 \) or \( a_n = \sigma(n) + 1 \). These particular choices do not satisfy the hypothesis \( \lim \inf a_{n+1}/a_n > 0 \) but we do not know whether that hypothesis which is weaker than that of the monotonicity of \( a_n \) would suffice.

In this note we obtain various improvements and generalizations of Theorem 1.1, in particular by relaxing the growth conditions on the \( a_n \) and using more precise results in the distribution of primes.

In § 2 we obtain some general conditions for the rationality of series of the form \( \sum b_n/(a_1, \cdots, a_n) \) which are modifications of [2, Lemma 2.29]. In § 3 we use a result of A. Selberg [3] on the regularity of primes in intervals to obtain improvements and generalizations of Theorem 1.1.

2. Criteria for rationality.

**Theorem 2.1.** Let \( \{b_n\} \) be a sequence of integers and \( \{a_n\} \) a sequence of positive integers with \( a_n > 1 \) for all large \( n \) and

\[
\lim_{n=1} \frac{|b_n|}{a_{n-1}a_n} = 0.
\]

Then the series
is rational if and only if there exists a positive integer $B$ and a sequence of integers $\{c_n\}$ so that for all large $n$ we have

(2.4) \[ Bb_n = c_n a_n - c_{n+1}, \quad |c_{n+1}| < a_n/2. \]

**Proof.** Assume that (2.4) holds beyond $N$. Then

\[
B a_1 \cdots a_{n-1} \sum_{n=N}^\infty \frac{b_n}{a_1 \cdots a_n} = \text{integer} + \sum_{n=N}^\infty \frac{c_n a_n - c_{n+1}}{a_n \cdots a_1} \n = \text{integer} + c_N = \text{integer}.
\]

Thus condition (2.4) is sufficient for the rationality of the series (2.3).

To prove the necessity of (2.4) assume that the series (2.3) equals $A/B$ and that $N$ is so large that $a_n \geq 2$ and $|b_n/(a_{n-1}a_n)| < 1/(4B)$ for all $n \geq N$. Then

(2.5) \[ A a_1 \cdots a_{n-1} = Ba_1 \cdots a_{n-1} \sum_{n=1}^\infty \frac{b_n}{a_1 \cdots a_n} \n = \text{integer} + \frac{Bb_N}{a_N} + \sum_{n=N+1}^\infty \frac{Bb_n}{a_N \cdots a_n}. \]

If we call the last sum $R_N$ we get

(2.6) \[ |R_N| \leq \max_{n>N} \left| \frac{Bb_n}{a_n \cdots a_{n-1}} \right| \sum_{n=N+1}^\infty \frac{1}{a_n \cdots a_{n-1}} \n = \frac{1}{4} \sum_{k=0}^\infty \frac{1}{2^k} = \frac{1}{2}. \]

Thus, if we choose $c_N$ to be the integer nearest to $Bb_N/a_N$, and write $Bb_N = c_N a_N - c_{N+1}$ then (2.5) yields that $-c_{N+1}/a_N + R_N$ is an integer of absolute value less than 1 and hence 0, so that

(2.7) \[ \frac{c_{N+1}}{a_N} = R_N = \frac{Bb_{N+1}}{a_N a_{N+1}} + \frac{1}{a_N} R_{N+1} \]

or

(2.8) \[ \frac{Bb_{N+1}}{a_{N+1}} = c_{N+1} - R_{N+1}. \]

From (2.8) it follows that $c_{N+1}$ is the integer nearest to $Bb_{N+1}/a_{N+1}$ and if we write $Bb_{N+1} = c_{N+1} a_{N+1} - c_{N+2}$ we get

(2.9) \[ \frac{Bb_{N+2}}{a_{N+2}} = c_{N+2} - R_{N+2}. \]
Proceeding in this manner we get the desired sequence \( \{c_n\} \).

**Remark.** Since (2.2) implies \( R_n \to 0 \) it follows that for rational values of the series (2.3) we get \( c_{n+1}/a_n \to 0 \). Thus either \( a_n \to \infty \) or \( c_n = 0 \) and hence \( b_n = 0 \) for all large \( n \).

**Corollary 2.10.** Let \( \{a_n\}, \{b_n\} \) satisfy the hypotheses of Theorem 2.1 and in addition the conditions that for all large \( n \) we have \( b_n > 0, a_{n+1} \geq a_n, \lim (b_{n+1} - b_n)/a_l \leq 0 \) and \( \lim \inf a_n/b_n = 0 \). Then the series (2.3) is irrational.

**Proof.** According to Theorem 2.1 the rationality of (2.3) implies the existence of a positive integer \( B \) and a sequence of integers \( \{c_n\} \) so that

\[
Bb_n = c_n a_n - a_{n+1}
\]

for all large \( n \) where \( c_{n+1}/a_n \to 0 \). Thus

\[
\frac{b_{n+1}}{b_n} = \frac{c_{n+1}a_{n+1} - c_{n+2}}{c_n a_n - c_{n+1}} > \frac{(c_{n+1} - \varepsilon)}{c_n a_n} \geq \frac{c_{n+1} - \varepsilon}{c_n}
\]

for all \( \varepsilon > 0 \) and sufficiently large \( n \). Thus \( c_{n+1} > c_n \) would lead to

\[
(2.11) \quad b_{n+1} > \left( 1 + \frac{1 - \varepsilon}{c_n} \right)b_n > b_n + (1 - \varepsilon) \left( a_n - \frac{c_{n+1}}{c_n} \right)/B
\]

\[
> b_n + (1 - \varepsilon)a_n/B.
\]

This contradicts our hypothesis for sufficiently large \( n \). Thus we get \( 0 < c_{n+1} \leq c_n \) for all large \( n \) and hence \( b_n/a_n \) is bounded contrary to the hypothesis that \( \lim \inf a_n/b_n = 0 \).

In fact, if we omit the hypothesis \( \lim \inf a_n/b_n = 0 \) then we get rational values for the series (2.3) only when \( Bb_n = C(a_n - 1) \) with positive integers \( B, C \) for all large \( n \).

3. Some special sequences.

**Theorem 3.1.** Let \( p_n \) be the \( n \)th prime and let \( \{a_n\} \) be a monotonic sequence of positive integers satisfying \( \lim p_n/a_n = 0 \) and \( \lim \inf a_n/p_n = 0 \). Then the series

\[
\sum_{n=1}^{\infty} \frac{p_n}{a_1 \cdots a_n}
\]

is irrational.

**Proof.** Since the series (3.2) satisfies the hypotheses of Theorem
2.1 it follows that there is a sequence \( \{c_n\} \) and an integer \( B \) so that for all large \( n \) we have

\[
Bp_n = c_na_n - c_{n+1}.
\]

For large \( n \) an equality \( c_n = c_{n+1} \) would imply \( c_n \mid B \) and \( a_n > p_n \). Since \( \{c_n\} \) is unbounded there must exist an index \( m \geq n \) so that \( c_m \leq c_n < c_{m+1} \). But this implies by an argument analogous to (2.11) that

\[
p_{m+1} > p_m + a_m/(2B) > (1 + \frac{1}{2B})p_m
\]

which is impossible for large \( m \). Thus we may assume that \( c_n \neq c_{n+1} \) for all large \( n \). Now consider an interval \( N \leq n \leq 2N \). If \( c_{n+1} > c_n \) then as in (3.4) we get

\[
p_{n+1} > p_n + a_n/(2B) > p_n + \sqrt{p_n}
\]

which therefore happens for fewer than \( (p_{2N} - p_N)/\sqrt{p_N} < N^{1/2+\epsilon} \) values in the interval \( (N, 2N) \). If \( c_{n+1} < c_n \) then we get

\[
1 > \frac{c_na_n - c_{n+1}}{c_{n+1}a_{n+1} - c_{n+2}} > \frac{c_n(a_n - 1)}{c_{n+1}a_{n+1}} > \left(1 + \frac{1}{c_{n+1}}\right)\frac{a_n - 1}{a_{n+1}}
\]

so that

\[
a_{n+1} > a_n + \frac{a_n - 1}{c_{n+1}} > a_n + 1.
\]

Since case (3.5) holds for more than \( N/2 \) values of \( n \) in \( (N, 2N) \) we get \( a_{2N} > N/2 \) and thus for all large \( n \) we have \( a_n > n/4 \), \( c_n < p_n/a_n + 1 < \sqrt{n}/4 \). Substituting these values in (3.5) we get

\[
a_{n+1} > a_n + \sqrt{n}
\]

when \( c_{n+1} < c_n \), \( n \) large;

so that \( a_{2N} > N^{3/2}/2 \), contradicting the hypothesis that \( \lim \inf a_n/p_n = 0 \).

**Theorem 3.7.** Let \( \{a_n\} \) be a monotonic sequence of positive integers with \( a_n > n^{1/2 + \delta} \) for some positive \( \delta > 0 \) and all large \( n \). Then the numbers \( 1, x, y, z \) are rationally independent. Here

\[
x = \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \cdots a_n}, \quad y = \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}
\]

and

\[
z = \sum_{n=1}^{\infty} \frac{d_n}{a_1 \cdots a_n}
\]
where \( \{d_n\} \) is any sequence of integers satisfying \( |d_n| < n^{1/2-\delta} \) for all large \( n \) and infinitely many \( d_n \neq 0 \).

**Proof.** Assume that there exist integers \( A, B, C \) not all 0 so that setting \( b_n = A\varphi(n) + B\sigma(n) + Cd_n \) we get that \( S = \sum_{n=1}^{\infty} b_n/(a_1, \ldots, a_n) \) is an integer.

From Theorem 2.1 it follows directly that \( z \) is irrational and thus not both \( A \) and \( B \) can be zero. We consider first the case \( A + B \neq 0 \) so that without loss of generality we may assume \( A + B = D > 0 \). Since \( S \) satisfies the hypotheses of Theorem 2.1 there exist integers \( \{c_n\} \) so that

\[
b_n = c_n a_n - c_{n+1} \quad \text{for all large} \quad n.
\]

Since \( |b_n| < n^{1-\delta/2} \) for all large \( n \) we get

\[
|c_n| < n^{(1-\delta)/2} \quad \text{for all large} \quad n.
\]

Let \( p_n \) be the \( n \)th prime and set

\[
a'_n = a_n, \quad b'_n = b_n, \quad c'_n = c_n, \quad c''_n = c_{n+1},
\]

then

\[
b'_n = A(p_n - 1) + B(p_n + 1) + Cd_{p_n} = Dp_n + d'_n
\]

where

\[
d'_n = Cd_{p_n} - A + B \quad \text{with} \quad |d'_n| < n^{(1-\delta)/2} \quad \text{for all large} \quad n.
\]

Now

\[
b'_{n+1} = c'_n a'_n - c''_n
\]

so that from

\[
\frac{b'_{n+1}}{b'_n} = \frac{Dp_{n+1} + d'_{n+1}}{Dp_n + d'_n} = \frac{p_{n+1}}{p_n} \left(1 + o(n^{-(1+\delta)/2})\right)
\]

we get

\[
\frac{p_{n+1}}{p_n} = \frac{c'_n a'_n - c''_n}{c'_n a'_n - c''_n} \left(1 + o(n^{-(1+\delta)/2})\right)
\]

or

\[
(3.8) \quad \frac{c'_{n+1}}{c'_n} = \frac{1 - c''_{n+1}/(a'_{n+1}c'_{n+1})}{1 - c''_n/(a'_n c'_n)} \left(1 + o(n^{-(1+\delta)/2})\right)
\]

\[
= \frac{c'_{n+1}}{c'_n} \left(1 + o(n^{-(1+\delta)/2})\right).
\]
Here the last inequality follows from the fact that
\[
\frac{c_{n+1}}{c_n} = \frac{(b_{n+1} + c_{n+1})/a_{n+1}}{(b_n + c_n)/a_n} = \frac{|A\phi(n+1) + B\sigma(n+1)| + O(n^{(1-\delta)/2})}{|A\phi(n) + B\sigma(n)| + O(n^{(1-\delta)/2})} = o(n^{1/2}).
\]

From (3.8) we get that \(c_{n+1} > c_n\) implies
\[
(3.9) \quad p_{n+1} > p_n + p_n^{1/2-1/4} > p_n + \frac{1}{2}p_n^{1/2+\delta}
\]
for all large \(n\).

We now use the following result of A. Selberg [3, Theorem 4].

**Theorem 3.10.** Let \(\phi(x)\) be positive and increasing and \(\phi(x)/x\) decreasing for \(x > 0\), further suppose
\[
\phi(x)/x \to 0 \quad \text{and} \quad \lim \inf \frac{\log \phi(x) \log x}{19/77} > 1 \quad \text{for} \quad x \to \infty.
\]
Then for almost all \(x > 0\),
\[
\pi(x + \phi(x)) - \pi(x) \sim \frac{\phi(x)}{\log x}.
\]

We now apply this theorem with the choice \(\phi(x) = x^{1/2+\delta}\) to inequality (3.9) and consider the primes \(N \leq p_m < p_{m+1} < \cdots < p_n < 2N\) in an interval \((N, 2N)\) with \(N\) large. According to Theorem 3.10 the union of the set of intervals \((p_i, p_{i+1})\) where \(p_i, p_{i+1}\) satisfy (3.9) and \(m \leq i < n\), form a set of total length \(\varepsilon N\) where \(\varepsilon > 0\) is arbitrarily small. Also the number of indices \(i\) for which (3.9) holds is \(o(\sqrt{N})\). Thus by (3.8) and (3.9) we have
\[
\frac{c'_n}{c'_m} = \frac{\prod_{s=m}^{n-1} c'_{i+1}}{\prod_{s=m}^{n-1} c'_{i}} < \frac{N + \varepsilon N}{N}(1 + o(N^{-(1/3)})^{1/2}) < 1 + 2\varepsilon < 2^{\varepsilon}.
\]

From the monotonicity of \(a_n\) it now follows that for any \(\varepsilon > 0\) we have
\[
(3.11) \quad |c_n| < n^\varepsilon \quad \text{for all large} \quad n.
\]

Substituting this inequality in (3.9) we get that \(c_{n+1} > c_n\) would imply
\[
(3.12) \quad p_{n+1} > p_n + p_n^{1/2+1/4} > p_n + \frac{1}{2}p_n^{1-\varepsilon/4}
\]
which is impossible for large \(n\) when \(\varepsilon < 5/12\). Thus \(\{c'_n\}\) becomes nonincreasing for large \(n\) and hence constant, \(c'_n = c\), for large \(n\).
This implies \( a_p > \frac{p}{c+1} \) for large primes \( p \) and by the monotonicity of \( a_n \) we get

\[
\frac{a_n}{n} > \frac{a_p}{2p} > \frac{1}{4c}
\]

where \( p \) is the largest prime \( \leq n \).

Now consider the successive equations

\[
\begin{align*}
    b_p &= ca_p - c_{p+1} \\
    b_{p+1} &= c_{p+1}a_{p+1} - c_{p+2}
\end{align*}
\]

Thus

\[
\begin{align*}
    A\phi(p+1) + B\sigma(p+1) + O(p^{1/2-\varepsilon}) &= c_{p+1}a_{p+1} \\
    Dp + O(p^{1/2-\varepsilon}) &= ca_p
\end{align*}
\]

for all large primes \( p \). This leads to

\[
(3.13) \quad \left| \frac{A\phi(p+1)}{D} + \frac{B\sigma(p+1)}{D} - \frac{c_{p+1}}{c} \right| < p^{-1/2},
\]

and hence to the conclusion that the only limit points of the sequence

\[
\left\{ \frac{A\phi(p+1)}{D} + \frac{B\sigma(p+1)}{D} \right\}_{p = \text{prime}}
\]

are rational numbers with denominator \( c \). To see that this is not the case, consider first the case \( B \neq 0 \). Then by Dirichlet’s theorem about primes in arithmetic progressions we see that \( \sigma(p+1)/(p+1) \) is everywhere dense in \((1, \infty)\). Thus we can choose \( p \) so that the distance of \( B\sigma(p+1)/D(p+1) \) to the nearest fraction with denominator \( c \) is greater than \( 1/(3c) \) while at the same time \( \sigma(p+1)/(p+1) \) is so large that \( |A\phi(p+1)/D(p+1)| < 1/(3c) \), contradicting (3.13). If \( B = 0 \) we use the fact that \( \phi(p+1)/(p+1) \) is dense in \((0, 1)\) to get the same contradiction.

Finally we must consider the case \( A + B = 0 \). Here we can go through the same argument as before except that we consider the subsequence

\[
b_{2p} = A\phi(2p) + B\sigma(2p) + Cd_{2p} = 2Bp + (3B + Cd_{2p}) = 2Bp + O(p^{1/2-\varepsilon}).
\]

As before we get

\[
b_{2p} = ca_{2p} - c_{2p+1} \quad \text{for all large primes} \quad p
\]

which leads to the wrong conclusion that

\[
\left\{ \frac{\sigma(2p+1)}{2p+1} - \frac{\varphi(2p+1)}{2p+1} \right\}_{p = \text{prime}}
\]

has rational numbers with denominator \( c \) as its only limit points.
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