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ON THE IRRATIONALITY OF CERTAIN SERIES

PAUL ERDŐS AND ERNST GABOR STRAUS

ON THE IRRATIONALITY OF CERTAIN SERIES

P. ERDÖS AND E. G. STRAUS

A criterion is established for the rationality of series of the form $\sum b_n/(a_1, \dots, a_n)$ where a_n, b_n are integers, $a_n \geq 2$ and $\lim b_n/(a_{n-1}a_n) = 0$. This criterion is applied to prove irrationality and rational independence of certain special series of the above type.

1. **Introduction.** In an earlier paper [2] we proved the following result:

THEOREM 1.1. *If $\{a_n\}$ is a monotonic sequence of positive integers with $a_n \geq n^{1/2}$ for all large n , then the series*

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 a_2 \cdots a_n} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\sigma(n)}{a_1 a_2 \cdots a_n}$$

are irrational.

We conjectured that the series (1.2) are irrational under the single assumption that $\{a_n\}$ is monotonic and we observed that some such condition is needed in view of the possible choices $a_n = \varphi(n) + 1$ or $a_n = \sigma(n) + 1$. These particular choices do not satisfy the hypothesis $\liminf a_{n+1}/a_n > 0$ but we do not know whether that hypothesis which is weaker than that of the monotonicity of a_n would suffice.

In this note we obtain various improvements and generalizations of Theorem 1.1, in particular by relaxing the growth conditions on the a_n and using more precise results in the distribution of primes.

In §2 we obtain some general conditions for the rationality of series of the form $\sum b_n/(a_1, \dots, a_n)$ which are modifications of [2, Lemma 2.29]. In §3 we use a result of A. Selberg [3] on the regularity of primes in intervals to obtain improvements and generalizations of Theorem 1.1.

2. Criteria for rationality.

THEOREM 2.1. *Let $\{b_n\}$ be a sequence of integers and $\{a_n\}$ a sequence of positive integers with $a_n > 1$ for all large n and*

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{|b_n|}{a_{n-1}a_n} = 0.$$

Then the series

$$(2.3) \quad \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$$

is rational if and only if there exists a positive integer B and a sequence of integers $\{c_n\}$ so that for all large n we have

$$(2.4) \quad Bb_n = c_n a_n - c_{n+1}, \quad |c_{n+1}| < a_n/2.$$

Proof. Assume that (2.4) holds beyond N . Then

$$\begin{aligned} Ba_1 \cdots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} &= \text{integer} + \sum_{n=N}^{\infty} \frac{c_n a_n - c_{n+1}}{a_N \cdots a_n} \\ &= \text{integer} + c_N = \text{integer}. \end{aligned}$$

Thus condition (2.4) is sufficient for the rationality of the series (2.3).

To prove the necessity of (2.4) assume that the series (2.3) equals A/B and that N is so large that $a_n \geq 2$ and $|b_n/(a_{n-1}a_n)| < 1/(4B)$ for all $n \geq N$. Then

$$(2.5) \quad \begin{aligned} Aa_1 \cdots a_{N-1} &= Ba_1 \cdots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \\ &= \text{integer} + \frac{Bb_N}{a_N} + \sum_{n=N+1}^{\infty} \frac{Bb_n}{a_N \cdots a_n}. \end{aligned}$$

If we call the last sum R_N we get

$$(2.6) \quad \begin{aligned} |R_N| &\leq \max_{n>N} \frac{|Bb_n|}{a_{n-1}a_n} \sum_{n=N+1}^{\infty} \frac{1}{a_N \cdots a_{n-2}} \\ &< \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2}. \end{aligned}$$

Thus, if we choose c_N to be the integer nearest to Bb_N/a_N and write $Bb_N = c_N a_N - c_{N+1}$ then (2.5) yields that $-c_{N+1}/a_N + R_N$ is an integer of absolute value less than 1 and hence 0, so that

$$(2.7) \quad \frac{c_{N+1}}{a_N} = R_N = \frac{Bb_{N+1}}{a_N a_{N+1}} + \frac{1}{a_N} R_{N+1}$$

or

$$(2.8) \quad \frac{Bb_{N+1}}{a_{N+1}} = c_{N+1} - R_{N+1}.$$

From (2.8) it follows that c_{N+1} is the integer nearest to Bb_{N+1}/a_{N+1} and if we write $Bb_{N+1} = c_{N+1} a_{N+1} - c_{N+2}$ we get

$$(2.9) \quad \frac{Bb_{N+2}}{a_{N+2}} = c_{N+2} - R_{N+2}.$$

Proceeding in this manner we get the desired sequence $\{c_n\}$.

REMARK. Since (2.2) implies $R_n \rightarrow 0$ it follows that for rational values of the series (2.3) we get $c_{n+1}/a_n \rightarrow 0$. Thus either $a_n \rightarrow \infty$ or $c_n = 0$ and hence $b_n = 0$ for all large n .

COROLLARY 2.10. *Let $\{a_n\}$, $\{b_n\}$ satisfy the hypotheses of Theorem 2.1 and in addition the conditions that for all large n we have $b_n > 0$, $a_{n+1} \geq a_n$, $\lim (b_{n+1} - b_n)/a_n \leq 0$ and $\liminf a_n/b_n = 0$. Then the series (2.3) is irrational.*

Proof. According to Theorem 2.1 the rationality of (2.3) implies the existence of a positive integer B and a sequence of integers $\{c_n\}$ so that

$$Bb_n = c_n a_n - a_{n+1}$$

for all large n where $c_{n+1}/a_n \rightarrow 0$. Thus

$$\frac{b_{n+1}}{b_n} = \frac{c_{n+1}a_{n+1} - c_{n+2}}{c_n a_n - c_{n+1}} > \frac{(c_{n+1} - \varepsilon)}{c_n a_n} \geq \frac{c_{n+1} - \varepsilon}{c_n}$$

for all $\varepsilon > 0$ and sufficiently large n . Thus $c_{n+1} > c_n$ would lead to

$$(2.11) \quad b_{n+1} > \left(1 + \frac{1 - \varepsilon}{c_n}\right)b_n > b_n + (1 - \varepsilon)\left(a_n - \frac{c_{n+1}}{c_n}\right)/B \\ > b_n + (1 - \varepsilon)^2 a_n / B.$$

This contradicts our hypothesis for sufficiently large n . Thus we get $0 < c_{n+1} \leq c_n$ for all large n and hence b_n/a_n is bounded contrary to the hypothesis that $\liminf a_n/b_n = 0$.

In fact, if we omit the hypothesis $\liminf a_n/b_n = 0$ then we get rational values for the series (2.3) only when $Bb_n = C(a_n - 1)$ with positive integers B, C for all large n .

3. Some special sequences.

THEOREM 3.1. *Let p_n be the n th prime and let $\{a_n\}$ be a monotonic sequence of positive integers satisfying $\lim p_n/a_n^2 = 0$ and $\liminf a_n/p_n = 0$. Then the series*

$$(3.2) \quad \sum_{n=1}^{\infty} \frac{p_n}{a_1 \cdots a_n}$$

is irrational.

Proof. Since the series (3.2) satisfies the hypotheses of Theorem

2.1 it follows that there is a sequence $\{c_n\}$ and an integers B so that for all large n we have

$$(3.3) \quad Bp_n = c_n a_n - c_{n+1}.$$

For large n an equality $c_n = c_{n+1}$ would imply $c_n \mid B$ and $a_n > p_n$. Since $\{c_n\}$ is unbounded there must exist an index $m \geq n$ so that $c_m \leq c_n < c_{m+1}$. But this implies by an argument analogous to (2.11) that

$$(3.4) \quad p_{m+1} > p_m + a_m/(2B) > \left(1 + \frac{1}{2B}\right)p_m$$

which is impossible for large m . Thus we may assume that $c_n \neq c_{n+1}$ for all large n . Now consider an interval $N \leq n \leq 2N$. If $c_{n+1} > c_n$ then as in (3.4) we get

$$p_{n+1} > p_n + a_n/(2B) > p_n + \sqrt{p_n}$$

which therefore happens for fewer than $(p_{2N} - p_N)/\sqrt{p_N} < N^{1/2+\varepsilon}$ values in the interval $(N, 2N)$. If $c_{n+1} < c_n$ then we get

$$1 > \frac{c_n a_n - c_{n+1}}{c_{n+1} a_{n+1} - c_{n+2}} > \frac{c_n(a_n - 1)}{c_{n+1} a_{n+1}} > \left(1 + \frac{1}{c_{n+1}}\right) \frac{a_n - 1}{a_{n+1}}$$

so that

$$(3.5) \quad a_{n+1} > a_n + \frac{a_n - 1}{c_{n+1}} > a_n + 1.$$

Since case (3.5) holds for more than $N/2$ values of n in $(N, 2N)$ we get $a_{2N} > N/2$ and thus for all large n we have $a_n > n/4$, $c_n < p_n/a_n + 1 < \sqrt{n}/4$. Substituting these values in (3.5) we get

$$(3.6) \quad a_{n+1} > a_n + \sqrt{n} \quad \text{when } c_{n+1} < c_n, n \text{ large};$$

so that $a_{2N} > N^{3/2}/2$, contradicting the hypothesis that $\liminf a_n/p_n = 0$.

THEOREM 3.7. *Let $\{a_n\}$ be a monotonic sequence of positive integers with $a_n > n^{1/2+\delta}$ for some positive $\delta > 0$ and all large n . Then the numbers $1, x, y, z$ are rationally independent. Here*

$$x = \sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \cdots a_n}, \quad y = \sum_{n=1}^{\infty} \frac{\sigma(n)}{a_1 \cdots a_n}$$

and

$$z = \sum_{n=1}^{\infty} \frac{d_n}{a_1 \cdots a_n}$$

where $\{d_n\}$ is any sequence of integers satisfying $|d_n| < n^{1/2-\delta}$ for all large n and infinitely many $d_n \neq 0$.

Proof. Assume that there exist integers A, B, C not all 0 so that setting $b_n = A\varphi(n) + B\sigma(n) + Cd_n$ we get that $S = \sum_{n=1}^{\infty} b_n/(a_1, \dots, a_n)$ is an integer.

From Theorem 2.1 it follows directly that z is irrational and thus not both A and B can be zero. We consider first the case $A + B \neq 0$ so that without loss of generality we may assume $A + B = D > 0$. Since S satisfies the hypotheses of Theorem 2.1 there exist integers $\{c_n\}$ so that

$$b_n = c_n a_n - c_{n+1} \quad \text{for all large } n.$$

Since $|b_n| < n^{1+\delta/2}$ for all large n we get

$$|c_n| < n^{(1-\delta)/2} \quad \text{for all large } n.$$

Let p_n be the n th prime and set

$$a'_n = a_{p_n}, \quad b'_n = b_{p_n}, \quad c'_n = c_{p_n}, \quad c''_n = c_{p_{n+1}},$$

then

$$b'_n = A(p_n - 1) + B(p_n + 1) + Cd_{p_n} = Dp_n + d'_n$$

where

$$d'_n = Cd_{p_n} - A + B \quad \text{with} \quad |d'_n| < n^{(1-\delta)/2} \quad \text{for all large } n.$$

Now

$$\begin{aligned} b'_n &= c'_n a'_n - c''_n \\ b'_{n+1} &= c'_{n+1} a'_{n+1} - c''_{n+1} \end{aligned}$$

so that from

$$\begin{aligned} \frac{b'_{n+1}}{b'_n} &= \frac{Dp_{n+1} + d'_{n+1}}{Dp_n + d'_n} = \frac{p_{n+1}}{p_n} \frac{1 + d'_{n+1}/(Dp_{n+1})}{1 + d'_n/(Dp_n)} \\ &= \frac{p_{n+1}}{p_n} (1 + o(n^{-(1+\delta)/2})) \end{aligned}$$

we get

$$\begin{aligned} \frac{p_{n+1}}{p_n} &= \frac{c'_{n+1} a'_{n+1} - c''_{n+1}}{c'_n a'_n - c''_n} (1 + o(n^{-(1+\delta)/2})) \\ (3.8) \quad &= \frac{c'_{n+1}}{c'_n} \frac{1 - c''_{n+1}/(a'_{n+1} c'_{n+1})}{1 - c''_n/(a'_n c'_n)} (1 + o(n^{-(1+\delta)/2})) \\ &= \frac{c'_{n+1}}{c'_n} (1 + o(n^{-(1+\delta)/2})). \end{aligned}$$

Here the last inequality follows from the fact that

$$\begin{aligned} \left| \frac{c_{n+1}}{c_n} \right| &= \left| \frac{(b_{n+1} + c_{n+2})/a_{n+1}}{(b_n + c_{n+1})/a_n} \right| = \frac{|A\varphi(n+1) + B\sigma(n+1)| + O(n^{(1-\delta)/2})}{|A\varphi(n) + B\sigma(n)| + O(n^{(1-\delta)/2})} \\ &= o(n^{\delta/2}). \end{aligned}$$

From (3.8) we get that $c'_{n+1} > c'_n$ implies

$$(3.9) \quad p_{n+1} > p_n + \frac{p_n}{c'_n} - p_n^{1/2-\delta/4} > p_n + \frac{1}{2} p_n^{1/2+\delta}$$

for all large n .

We now use the following result of A. Selberg [3, Theorem 4].

THEOREM 3.10. *Let $\Phi(x)$ be positive and increasing and $\Phi(x)/x$ decreasing for $x > 0$, further suppose*

$$\Phi(x)/x \rightarrow 0 \quad \text{and} \quad \liminf \log \Phi(x)/\log x > 19/77 \quad \text{for} \quad x \rightarrow \infty.$$

Then for almost all $x > 0$,

$$\pi(x + \Phi(x)) - \pi(x) \sim \frac{\Phi(x)}{\log x}.$$

We now apply this theorem with the choice $\Phi(x) = x^{1/2+\delta}$ to inequality (3.9) and consider the primes $N \leq p_m < p_{m+1} < \dots < p_n < 2N$ in an interval $(N, 2N)$ with N large. According to Theorem 3.10 the union of the set of intervals (p_i, p_{i+1}) where p_i, p_{i+1} satisfy (3.9) and $m \leq i < n$, form a set of total length $< \varepsilon N$ where $\varepsilon > 0$ is arbitrarily small. Also the number of indices i for which (3.9) holds is $o(\sqrt{N})$. Thus by (3.8) and (3.9) we have

$$\begin{aligned} \frac{c'_n}{c'_m} &= \prod_{i=m}^{n-1} \frac{c'_{i+1}}{c'_i} = \prod_{\substack{i=m \\ c'_{i+1}c'_i}}^{n-1} \frac{c'_{i+1}}{c'_i} < \frac{N + \varepsilon N}{N} (1 + o(N^{-(+\delta)/2}))^{\sqrt{N}} \\ &< 1 + 2\varepsilon < 2^{2\varepsilon}. \end{aligned}$$

From the monotonicity of a_n it now follows that for any $\varepsilon > 0$ we have

$$(3.11) \quad |c_n| < n^\varepsilon \quad \text{for all large } n.$$

Substituting this inequality in (3.9) we get that $c'_{n+1} > c'_n$ would imply

$$(3.12) \quad p_{n+1} > p_n + \frac{p_n}{c'_n} - p_n^{1/2+\delta/4} > p_n + \frac{1}{2} p_n^{1-\varepsilon}$$

which is impossible for large n when $\varepsilon < 5/12$. Thus $\{c'_n\}$ becomes nonincreasing for large n and hence constant, $c'_n = c$, for large n .

This implies $a_p > p/(c + 1)$ for large primes p and by the monotonicity of a_n we get

$$\frac{a_n}{n} > \frac{a_p}{2p} > \frac{1}{4c}$$

where p is the largest prime $\leq n$.

Now consider the successive equations

$$\begin{aligned} b_p &= ca_p - c_{p+1} \\ b_{p+1} &= c_{p+1}a_{p+1} - c_{p+2} . \end{aligned}$$

Thus

$$\begin{aligned} A\varphi(p + 1) + B\sigma(p + 1) + O(p^{1/2-\delta}) &= c_{p+1}a_{p+1} \\ Dp + O(p^{1/2-\delta}) &= ca_p \end{aligned}$$

for all large primes p . This leads to

$$(3.13) \quad \left| \frac{A}{D} \frac{\varphi(p + 1)}{p + 1} + \frac{B}{D} \frac{\sigma(p + 1)}{p + 1} - \frac{c_{p+1}}{c} \right| < p^{-1/2} ,$$

and hence to the conclusion that the only limit points of the sequence

$$\left\{ \frac{A}{D} \frac{\varphi(p + 1)}{p + 1} + \frac{B}{D} \frac{\sigma(p + 1)}{p + 1} \mid p = \text{prime} \right\}$$

are rational numbers with denominator c . To see that this is not the case, consider first the case $B \neq 0$. Then by Dirichlet's theorem about primes in arithmetic progressions we see that $\sigma(p + 1)/(p + 1)$ is everywhere dense in $(1, \infty)$. Thus we can choose p so that the distance of $B\sigma(p + 1)/D(p + 1)$ to the nearest fraction with denominator c is greater than $1/(3c)$ while at the same time $\sigma(p + 1)/(p + 1)$ is so large that $|A\varphi(p + 1)/D(p + 1)| < 1/(3c)$, contradicting (3.13). If $B = 0$ we use the fact that $\varphi(p + 1)/(p + 1)$ is dense in $(0, 1)$ to get the same contradiction.

Finally we must consider the case $A + B = 0$. Here we can go through the same argument as before except that we consider the subsequence $b_{2p} = A\varphi(2p) + B\sigma(2p) + Cd_{2p} = 2Bp + (3B + Cd_{2p}) = 2Bp + O(p^{1/2-\delta})$. As before we get

$$b_{2p} = ca_{2p} - c_{2p+1} \quad \text{for all large primes } p$$

which leads to the wrong conclusion that

$$\left\{ \frac{\sigma(2p + 1)}{2p + 1} - \frac{\varphi(2p + 1)}{2p + 1} \mid p = \text{prime} \right\}$$

has rational numbers with denominator c as its only limit points.

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