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**BEST APPROXIMATION BY A SATURATION CLASS OF
POLYNOMIAL OPERATORS**

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The problem of determining a saturation class has been considered by Zamanski, Sunouchi and Watari and others. Zamanski has considered the Cesaro means of order 1 and Sunouchi and Watari have studied the Riesz means of type n . The object of the present paper is to extend these results by considering Nörlund means which include the above-mentioned results as particular cases.

1. Let $\{p_n\}$ be a sequence of positive constants such that

$$P_n = p_0 + \dots + p_n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty .$$

A given series $\sum_{n=0}^{\infty} d_n$ with the sequence of partial sums $\{S_n\}$ is said to summable (N, p_n) to d , provided that

$$(1.1) \quad \begin{aligned} N_n \left[\sum_{l=0}^{\infty} d_l \right] &= \frac{1}{P_n} \sum_{k=0}^n P_{n-k} d_k \\ &= \frac{1}{P_n} \sum_{k=0}^n p_{n-k} S_k \longrightarrow d, \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

and N_n are called the Nörlund operators.

Let

$$(1.2) \quad \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x)$$

be the Fourier series associated with a continuous periodic function $f(x)$, with period 2π .

We define

$$(1.3) \quad N_n(x) \equiv N_n(f; x) \equiv \frac{1}{P_n} \sum_{k=0}^n P_{n-k} A_k(x)$$

and the norm

$$\|f(x) - N_n(x)\| \equiv \max_{0 \leq x \leq 2\pi} |f(x) - N_n(x)| .$$

If there exists positive nonincreasing function $\phi(n)$ and a class of functions K , with the following properties:

- (I) $\|f(x) - N_n(x)\| = o(\phi(n)) \implies f(x)$ is constant,
- (II) $\|f(x) - N_n(x)\| = O(\phi(n)) \implies f(x) \in K$

and

$$(III) \quad f(x) \in K \implies \|f(x) - N_n(x)\| = O(\phi(n)),$$

then the Nörlund operators are saturated with the order $\phi(n)$ and the class K .

In this paper we prove that the above method of summations is saturated with the order p_n/P_n and that the class K consists of all continuous functions f such that $\tilde{f} \in Lip 1$, where \tilde{f} is the conjugate function of f . By definition

$$\tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi [f(x+t) - f(x-t)] \cot \frac{t}{2} dt,$$

if the integral converges absolutely for all x and if

$$\int_0^\pi |f(x+t) - f(x-t)| \cot \frac{t}{2} dt$$

is an integrable function.

The problem of determining a saturation class by considering $(C, 1)$ means of the Fourier series of $f(x)$ has been considered by Zamanski [6]. Sunouchi and Watari [4] have considered the problem by taking (R, λ, k) means of the Fourier series. Some of these results were later extended by Sunouchi [3] and others [2, 5].

2. We shall prove the following theorem.

THEOREM. *Let $\{p_n\}$ be a sequence of positive constants satisfying the following conditions,*

$$(2.1) \quad \frac{p_{n-k}}{p_n} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty \quad \text{for a fixed } k \leq n,$$

and

$$(2.2) \quad \sum_{k=0}^n |p_{n-k} - p_{n-k-1}| = O(p_n) \quad \text{where } [p_{-1} = 0].$$

Then the operators N_n are saturated with order p_n/P_n and the class of all continuous functions f for which $\tilde{f} \in Lip 1$.

The following lemmas are required for the proof of the theorem.

LEMMA 2.1. *If*

$$\|f(x) - N_n(x)\| = o\left[\frac{p_n}{P_n}\right]$$

then f is a constant.

Proof. From (1.3) we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} N_n(x) \cos rx \, dx &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{k=0}^n \frac{P_{n-k}}{P_n} A_k(x) \cos rx \, dx \\ &= \frac{1}{\pi} \sum_{k=0}^n \frac{P_{n-k}}{P_n} \int_{-\pi}^{\pi} A_k(x) \cos rx \, dx \\ &= \frac{P_{n-r}}{P_n} a_r . \end{aligned}$$

Thus,

$$\begin{aligned} a_r - \frac{P_{n-r}}{P_n} a_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos rx \, dx - \frac{1}{\pi} \int_{-\pi}^{\pi} N_n(x) \cos rx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos rx [f(x) - N_n(x)] \, dx , \end{aligned}$$

hence

$$\left| a_r - \frac{P_{n-r}}{P_n} a_r \right| \leq \|f(x) - N_n(x)\| \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cdot dx = o\left[\frac{p_n}{P_n}\right] .$$

Consequently

$$(2.3) \quad a_r \left\{ \frac{p_n + \dots + p_{n-r+1}}{p_n} \right\} = o(1) ,$$

and since $p_r > 0$ for all r , we have $(p_n + \dots + p_{n-r+1})/p_n \geq 1$ for $r \geq 1$.

Thus from (2.3) it follows that $a_r = 0$, for each $r \geq 1$. Similarly we can show that $b_r = 0$ for each $r \geq 1$. Hence $f(x) = 1/2a_0$, a constant.

LEMMA 2.2. *If*

$$\|f(x) - N_n(x)\| = O\left[\frac{p_n}{P_n}\right]$$

and condition (2.1) is satisfied, then $\tilde{f}(x) \in Lip 1$.

Proof. It can be shown without much difficulty that if

$$\|f(x) - N_n(x)\| = O\left[\frac{p_n}{P_n}\right] ,$$

then

$$\left\| \sum_{k=1}^N \frac{p_n + \dots + p_{n-k+1}}{p_n} A_k(x) \left[1 - \frac{k}{N+1} \right] \right\| = O(1), \quad N \leq n .$$

Taking the limit as $n \rightarrow \infty$, and using condition (2.1), we obtain

$$(2.4) \quad \left\| \sum_{k=1}^N kA_k(x) \left[1 - \frac{k}{N+1} \right] \right\| = O(1).$$

The left hand side of the above equation represents the $(C, 1)$ mean of the series

$$\sum_{k=1}^{\infty} -kA_k(x).$$

Since $-kA_k(x) = B'_k(x)$, where $\sum_{k=1}^{\infty} B_k(x) \equiv \sum_{k=1}^{\infty} (b_k \cos kx - \alpha_k \sin kx)$ is the conjugate series of (1.2), then (2.4) is equivalent to

$$\|\tilde{\sigma}'_N(f)\| < M$$

which implies that $\tilde{f}(x) \in Lip 1$, [1].
 $(\tilde{\sigma}'_N(f))$ represents the $(C, 1)$ mean of the conjugate series.)

LEMMA 2.3. Assume $\tilde{f} \in Lip 1$. If the sequence $\{p_n\}$ satisfies condition (2.2), then

$$\|f(x) - N_n(x)\| = O\left[\frac{p_n}{P_n}\right].$$

Proof. Since, by definition

$$\tilde{S}_n(\tilde{f}, x) = \frac{1}{\pi} \int_0^\pi [\tilde{f}(x, t) - \tilde{f}(x - t)] \frac{\cos \frac{t}{2} - \cos \left[n + \frac{1}{2} \right] t}{2 \sin \frac{t}{2}} dt$$

where $\tilde{S}_n(\tilde{f}, x)$ denotes the partial sums of the conjugate series associated with $\tilde{f}(x)$, we have

$$\begin{aligned} N_n(\tilde{S}_n(\tilde{f}, x)) &= \frac{1}{P_n} \sum_{k=0}^n p_{n-k} \tilde{S}_k(\tilde{f}, x) \\ &= \frac{1}{P_n} \sum_{k=0}^n p_{n-k} \frac{1}{2\pi} \int_0^\pi [\tilde{f}(x+t) - \tilde{f}(x-t)] \cot \frac{1}{2} t dt \\ &\quad - \frac{1}{P_n} \sum_{k=0}^n p_{n-k} \frac{1}{2\pi} \int_0^\pi [\tilde{f}(x+t) - \tilde{f}(x-t)] \frac{\cos \left[k + \frac{1}{2} \right] t}{\sin \frac{1}{2} t} dt. \end{aligned}$$

Since the function $\tilde{f}(x) \in Lip 1$, $-f + (1/2)a_0$ is identical to \tilde{f} , therefore

$$(2.5) \quad f(x) - N_n(f, x) = \frac{1}{2\pi} \int_0^\pi [\tilde{f}(x+t) - \tilde{f}(x-t)] K_n(t) dt,$$

where

$$K_n(t) = \frac{1}{P_n \sin \frac{1}{2}t} \sum_{k=0}^n p_{n-k} \cos \left[k + \frac{1}{2} \right] t .$$

Now by partial summation

$$\begin{aligned} K_n(t) &= \frac{1}{2P_n \sin^2 \frac{1}{2}t} \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \sin (k + 1)t \\ &= \frac{1}{P_n} \left\{ \frac{2}{t^2} + O(1) \right\} \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \sin (k + 1)t \\ &= \frac{2}{P_n t^2} \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \sin (k + 1)t + O \left[\frac{p_n}{P_n} \right], \end{aligned}$$

by hypothesis. Since $\tilde{f}(x)$ is certainly bounded, the right hand side of (2.5) becomes

$$(2.6) \quad \frac{1}{\pi P_n} \int_0^\pi [\tilde{f}(x + t) - \tilde{f}(x - t)] \frac{1}{t^2} \left\{ \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \sin (k + 1)t \right\} dt + O \left[\frac{p_n}{P_n} \right].$$

Let us write

$$F_n(t) = \frac{1}{P_n} \int_t^\pi \frac{1}{u^2} \left\{ \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \sin (k + 1)u \right\} du .$$

Since $\tilde{f}(u) \in Lip 1$, it is an indefinite integral of a bounded function, say $\tilde{f}'(u)$. Further, since $\tilde{f}(x + t) - \tilde{f}(x - t) = O(t)$, as $t \rightarrow 0$, while for fixed n , $F_n(t) = O(\log (1/t))$, we can integrate (2.6) by parts to obtain

$$\frac{1}{\pi} \int_0^\pi [\tilde{f}'(x + t) + \tilde{f}'(x - t)] F_n(t) dt + O \left[\frac{p_n}{P_n} \right],$$

noting that the integrated term vanishes at both limits. The absolute value of this above expression is now,

$$(2.1) \quad O \left\{ \int_0^\pi |F_n(t)| dt \right\} + O \left[\frac{p_n}{P_n} \right] \text{ since } \tilde{f}' \text{ is bounded .}$$

Now

$$\begin{aligned} F_n(t) &= \frac{1}{P_n} \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) \int_t^\pi \frac{\sin (k + 1)u}{u^2} du \\ &= \frac{1}{P_n} \sum_{k=0}^n (p_{n-k} - p_{n-k-1})(k + 1) \int_{(k+1)t}^{(k+1)\pi} \frac{\sin \nu}{\nu^2} d\nu . \end{aligned}$$

However,

$$\int_{(k+1)t}^{(k+1)\pi} \frac{\sin v}{v^2} dv = \begin{cases} O(\log 1/(k+1)t) & \text{if } (k+1)t < 1 \\ O(1/(k+1)t^2) & \text{if } (k+1)t \geq 1. \end{cases}$$

Hence

$$\begin{aligned} \int_0^\pi |F_n(t)| dt &= O\left\{ \frac{1}{P_n} \int_0^\pi \left[\sum_{\substack{(k+1)t < 1/t \\ k \geq 0}} |p_{n-k} - p_{n-k-1}| (k+1) \log(1/(k+1)t) \right. \right. \\ &\quad \left. \left. + \sum_{\substack{(k+1)t \geq 1/t \\ k \leq n}} |p_{n-k} - p_{n-k-1}| 1/(k+1)t^2 \right] dt \right\} \\ &= O\left\{ \frac{1}{P_n} \sum_{k=0}^n |p_{n-k} - p_{n-k-1}| \left[\int_0^{1/(k+1)} (k+1) \log(1/(k+1)t) dt \right. \right. \\ &\quad \left. \left. + \int_{1/(k+1)}^\pi \frac{1}{(k+1)t^2} dt \right] \right\}. \end{aligned}$$

Further,

$$\int_0^{1/(k+1)} \log(1/(k+1)t) dt = \int_0^1 \log\left(\frac{1}{u}\right) du = \text{constant}$$

and

$$\int_{1/(k+1)}^\pi \frac{1}{(k+1)t^2} dt < M \text{ (constant)},$$

therefore

$$\int_0^\pi |F_n(t)| dt = O\left\{ \frac{1}{P_n} \sum_{k=0}^n |p_{n-k} - p_{n-k-1}| \right\} = O\left[\frac{p_n}{P_n} \right]$$

from (2.2).

Thus (2.7) and hence (2.6) is $O[p_n/P_n]$.

Consequently from (2.5), we have that

$$\|f(x) - N_n(f, x)\| = O\left[\frac{p_n}{P_n} \right]$$

which proves the lemma.

The proof of the theorem now follows from Lemmas 2.1, 2.2, and 2.3.

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