

Pacific Journal of Mathematics

STIELTJES DIFFERENTIAL-BOUNDARY OPERATORS. II

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The differential boundary system

$$Ly = (y + H[Cy(0) + Dy(1)] + H_1\Psi)' + Py,$$

$$Ay(0) + By(1) + \int_0^1 dK(t)y(t) = 0,$$

$$\int_0^1 dK_1(t)y(t) = 0,$$

and its adjoint system are written as Stieltjes integral equation systems with end point boundary conditions. Fundamental matrices are exhibited and, from these, a spectral analysis and a Green's matrix are produced. These are used to achieve spectral resolutions in both self-adjoint and nonself-adjoint situations.

1. Introduction. This article is a continuation of [2] and [6] which showed the density of the domain of L in $\mathcal{L}_n^p[0, 1]$, $1 \leq p < \infty$, when the boundary functionals satisfied certain conditions, and which derived the dual operator in $\mathcal{L}_n^q[0, 1]$, $1/p + 1/q = 1$, in those circumstances. Rather than repeat those results, we prefer to refer the reader to the articles mentioned. For our purposes here it is sufficient to state that y is an n dimensional vector in $\mathcal{L}_n^p[0, 1]$; A and B are $m \times n$ matrices, $m \leq 2n$, such that $\text{rank } (A: B) = m$; C and D are $(2n - m) \times n$ matrices such that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular; K is an $m \times n$ matrix valued function of bounded variation such that the measure it generates satisfies $dK(0) = A$, $dK(1) = B$; K_1 is an $r \times n$ matrix valued function of bounded variation which is not absolutely continuous, satisfying $dK_1(0) = 0$, $dK_1(1) = 0$; H and H_1 are, respectively, $n \times (2n - m)$ and $n \times s$ matrix valued functions of bounded variation, H_1 not absolutely continuous; P is a continuous $n \times n$ matrix; and, finally, Ψ is an s dimensional constant vector.

Because we wish to exhibit the contributions of K , K_1 , H , H_1 at 0 and 1 separately, integrals involving their resulting measures will not include contributions at 0 or 1. At all other points, however, we do assume that these functions are regular as defined by Hildebrandt [4]. This results in considerable simplification throughout. Of course, all integrals are Lebesgue or Lebesgue-Stieltjes integrals.

It is convenient to note that the adjoint system has the form

$$L^*z = -(z + K^*[\tilde{A}z(0) + \tilde{B}z(1)] + K_1^*\phi)' + P^*z,$$

$$\tilde{C}z(0) + \tilde{D}z(1) + \int_0^1 dH^*(t)z(t) = 0 ,$$

$$\int_0^1 dH_1^*(t)z(t) = 0 ,$$

where ϕ is an r dimensional constant vector, and \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} satisfy

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} = \begin{pmatrix} -\tilde{A}^* & -\tilde{C}^* \\ \tilde{B}^* & \tilde{D}^* \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{2n} .$$

2. Integral equation representation. Let us make the following definitions. Let

$$\begin{aligned} \xi_1 &= y , \\ \xi_2 &= Ay(0) + \int_0^t dK(x)y(x) , \\ \xi_3 &= Cy(0) + Dy(1) , \\ \xi_4 &= \int_0^t dK_1(x)y(x) , \\ \xi_5 &= \Psi . \end{aligned}$$

Then the equation $Ly = 0$, together with the boundary conditions is equivalent to the system

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} (t) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} (0) + \int_0^t d \begin{pmatrix} -Q & 0 & -H & 0 & -H_1 \\ K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ K_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} (x) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} (x) ,$$

where $Q(t) = \int_0^t P(x)dx$,

$$\begin{pmatrix} A & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C & 0 & -\frac{1}{2}I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} (0) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ B & I & 0 & 0 & 0 \\ D & 0 & -\frac{1}{2}I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix} (1) = 0 .$$

If $M(t)$ represents the Stieltjes measure in the integral equation, then Hildebrandt's $\Delta M^\pm(t)$ has zero entries along the diagonal. Hence $I \pm \Delta M^\pm$ is always nonsingular.

The adjoint system $L^*z = 0$, together with the boundary conditions is

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}(t) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}(0) - \int_0^t d \begin{pmatrix} -Q^* & K^* & 0 & K_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -H^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -H^* & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}(x),$$

$$\begin{pmatrix} I & A^* & C^* & 0 & 0 \\ 0 & 0 & -D^* & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}(0) + \begin{pmatrix} 0 & 0 & -C^* & 0 & 0 \\ I & -B^* & D^* & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}(1) = 0.$$

These representations should be compared to those found in [5] which they generalize under certain conditions.

In addition we note that the problem $Ly = \lambda y$ has a similar representation. The only change necessary is to replace $Q(t) = \int_0^t P(x)dx$ by $Q(t) - \lambda t$. The nonhomogeneous problem $Ly = f$ has a representation as a nonhomogeneous integral equation with an additional term

$$F(t) = \int_0^t \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}(x)dx$$

on the right side.

3. Fundamental matrices. We can express the homogeneous integral problem generated by $(L - \lambda I)y = 0$ together with the boundary conditions in a more compact way by the expressions

$$\xi(t) = \xi(0) + \int_0^t dM_\lambda(x)\xi(x),$$

$$R\xi(0) + S\xi(1) = 0;$$

likewise the adjoint system by

$$\eta(t) = \eta(0) - \int_0^t dM_\lambda^*(x)\eta(x),$$

$$\tilde{R}\eta(0) + \tilde{S}\eta(1) = 0.$$

We shall assume in addition that $M_\lambda(t)$ is regular:

$$M_\lambda(t) = 1/2[M_\lambda(t+) + M_\lambda(t-)] ,$$

$$M(0) = M(0+) , \quad M(1) = M(1-) .$$

Hildebrandt [4] and Vejvoda and Tvrdy [8] have shown that under these conditions the first integral equation has a solution given by $\xi(t) = U_\lambda(0, t)\xi(0)$, where $U_\lambda(s, t)$ is the uniform limit of Picard-like approximations beginning with I (hence U_λ is analytic in λ) satisfying

$$U_\lambda(s, t) = I + \int_s^t dM_\lambda(x) U_\lambda(s, x) .$$

U_λ has the additional properties $U_\lambda(t, t) = I$, and $U_\lambda(r, t)U_\lambda(s, r) = U_\lambda(s, t)$. U_λ is therefore a fundamental matrix when M_λ is absolutely continuous.

Similarly the adjoint equation has a solution given by $\eta(t) = V_{\lambda^*}(0, t)\eta(0)$, where $V_{\lambda^*}(s, t)$ satisfies

$$V_{\lambda^*}(s, t) = I - \int_s^t dM_\lambda^*(x) V_{\lambda^*}(s, x) ,$$

$$V_{\lambda^*}(t, t) = I, \quad V_{\lambda^*}(r, t)V_{\lambda^*}(s, r) = V_{\lambda^*}(s, t).$$

Since M_λ is regular, it is possible to show that U_λ and V_{λ^*} are related through the formula

$$U_\lambda(s, t) = V_\lambda(t, s) .$$

Hence $U_\lambda(s, t)^{-1} = V_\lambda(s, t)$. Regularity, however, is not inherited from M_λ unless $(\Delta^+ M_\lambda)^2 \equiv 0$. This occurs only when $\Delta^+ K \Delta^+ H \equiv 0$, $\Delta^+ K_1 \Delta^+ H \equiv 0$, $\Delta^+ K \Delta^+ H_1 \equiv 0$, and will not be necessary.

The fundamental matrices U_λ and V_λ may be easily calculated in the same way as was done in [5]. If $Y(t)$ is a fundamental matrix for $Y' + PY = 0$ satisfying $Y(0) = I$, and

$$\mathcal{H}(t) = \int_0^t e^{-\lambda x} Y(t) Y(x)^{-1} dH(x) ,$$

$$\mathcal{H}_1(t) = \int_0^t e^{-\lambda x} Y(t) Y(x)^{-1} dH_1(x) ,$$

$$\mathcal{K}(t) = \int_0^t dK(x) e^{\lambda x} Y(x) ,$$

$$\mathcal{K}_1(t) = \int_0^t dK_1(x) e^{\lambda x} Y(x) ,$$

$$\mathcal{L}(t) = \int_0^t dK(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH(x) ,$$

$$\mathcal{L}_{01}(t) = \int_0^t dK(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH_1(x) ,$$

$$\mathcal{L}_{10}(t) = \int_0^t dK_1(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH(x) ,$$

$$\mathcal{L}_{11}(t) = \int_0^t dK_1(z) \int_0^z e^{\lambda(z-x)} Y(z) Y(x)^{-1} dH_1(x) ,$$

and $\mathcal{M}(t)$, $\mathcal{M}_{01}(t)$, $\mathcal{M}_{10}(t)$, $\mathcal{M}_{11}(t)$ are defined by the same formulae as $\mathcal{L}(t)$, $\mathcal{L}_{01}(t)$, $\mathcal{L}_{10}(t)$, $\mathcal{L}_{11}(t)$ with only the limits of integration with respect to x changed to from z to t , then

$$U_\lambda(0, t) = \begin{pmatrix} e^{\lambda t} Y(t) & 0 & -e^{\lambda t} \mathcal{H}(t) & 0 & -e^{\lambda t} \mathcal{H}_1(t) \\ \mathcal{K}(t) & I & -\mathcal{L}(t) & 0 & -\mathcal{L}_{01}(t) \\ 0 & 0 & I & 0 & 0 \\ \mathcal{K}_1(t) & 0 & -\mathcal{L}_{10}(t) & I & -\mathcal{L}_{11}(t) \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

and

$$V_\lambda(0, t) = \begin{pmatrix} e^{-\lambda t} Y(t)^{-1} & 0 & Y(t)^{-1} \mathcal{H}(t) & 0 & Y(t) \mathcal{H}_1(t) \\ -\mathcal{K}(t) e^{-\lambda t} Y(t)^{-1} & I & -\mathcal{M}(t) & 0 & -\mathcal{M}_{01}(t) \\ 0 & 0 & I & 0 & 0 \\ -\mathcal{K}_1(t) e^{-\lambda t} Y(t)^{-1} & 0 & -\mathcal{M}_{10}(t) & I & -\mathcal{M}_{11}(t) \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

By applying the boundary condition of U_λ the following theorem immediately follows.

THEOREM 3.1. *If $Y(t)$ is a fundamental matrix for $Y' + PY = 0$ satisfying $Y(0) = I$, then the system*

$$Ly = \lambda y,$$

$$Ay(0) + By(1) + \int_0^1 dK(t)y(t) = 0,$$

$$\int_0^1 dK_1(t)y(t) = 0$$

is compatible if and only if the rank of

$$\begin{pmatrix} A & -I & 0 & 0 & 0 \\ Be^\lambda Y(1) + \mathcal{K}(1) & I - Be^\lambda \mathcal{H}(1) - \mathcal{L}(1) & 0 & -Be^\lambda \mathcal{H}_1(1) - \mathcal{L}_{01}(1) \\ De^\lambda Y(1) + C & 0 & -De^\lambda \mathcal{H}(1) - I & 0 & -De^\lambda \mathcal{H}_1(1) \\ 0 & 0 & 0 & I & 0 \\ \mathcal{K}_1(1) & 0 & -\mathcal{L}_{10}(1) & I & -\mathcal{L}_{11}(1) \end{pmatrix}$$

is less than $3n + r + s$. If $m = n$, the system is compatible if and only if the determinant of the matrix above is zero.

We shall assume throughout the remainder of this article that $m = n$ in order to derive eigenfunction expansions under various conditions.

4. **The Green's matrix.** Whenever the homogeneous problem is not comparable, the nonhomogeneous problem possesses a unique solution generated by a Green's matrix, just as is the case for the regular Sturm-Liouville problem. Hildebrandt [4] shows that the solution to

$$\begin{aligned}\xi(t) &= \int_0^t dM_\lambda(s)\xi(s) + \mathcal{F}(t), \\ \xi(0) &= \mathcal{F}(0)\end{aligned}$$

is given by

$$\xi(t) = U_\lambda(0, t)\mathcal{F}(0) + \int_0^t U_\lambda(s, t)d\mathcal{F}(s)$$

whenever $\Delta^\pm \mathcal{F} \equiv 0$. Since in our situation $\mathcal{F}(t) = F(t) + \xi(0)$, where $F(t)$ is absolutely continuous, $F'(t) = f_0(t) = (f(t), 0 \dots 0)^T$, we find that the solution can be expressed by

$$\xi(t) = U_\lambda(t, 0)y(0) + \int_0^t U_\lambda(s, t)f_0(s)ds.$$

If $\xi(1)$ is calculated and $R\xi(0) + S\xi(1)$ is set equal to 0, $\xi(0)$ is determined, and the solution takes the form

$$\xi(t) = \int_0^1 \mathcal{G}_\lambda(s, t)f_0(s)ds,$$

where the Green's function \mathcal{G} is given by

$$\begin{aligned}\mathcal{G}_\lambda(s, t) &= U(0, t)[R + SU_\lambda(0, 1)]^{-1}RU_\lambda(0, s)^{-1}, s < t, \\ &= -U(0, t)[R + SU_\lambda(0, 1)]^{-1}SU_\lambda(0, 1)U_\lambda(0, s)^{-1}, s > t.\end{aligned}$$

This is the same formula as that encountered in the regular Sturm-Liouville problem. The Green's function \mathcal{G} possesses the properties, including the adjoint properties, usually attributed to Green's functions.

We note in particular that λ is in the spectrum of the operator L if and only if

$$\det [R + SU_\lambda(0, 1)] = 0.$$

Since $[R + SU_\lambda(0, 1)]$ is analytic in λ , this implies that either the entire complex plane is in the point spectrum of L , or else the spectrum of L consists only of isolated eigenvalues, accumulating only at ∞ .

5. **Self-adjoint Stieltjes differential-boundary expansions.** It was shown earlier in [6] that the operator $T = iL$ is self-adjoint in

$\mathcal{L}_n^2[0, 1]$ if and only if

1. $P^* = -P$
2. $m = n, r = s.$
3. $K = [BD^* - AC^*]H^*$ a.e.
4. $AA^* = BB^*$
5. $H[CC^* - DD^*] = 0$ a.e.
6. $K_1 = MH_1^*$, where M is a nonsingular $r \times r$ matrix.

This being the case, then the spectrum of T is contained in the real axis. Every point with nonzero imaginary part lies in the resolvent. This implies that $\det[R + U_\lambda(0, 1)S] = 0$ only at isolated real points with ∞ their only limit. An application of the spectral resolution theorem for self-adjoint operators on a Hilbert space results in the following.

THEOREM 5.1. *If T is self-adjoint, then*

1. *The spectrum of T consists of a denumerable set of real eigenvalues, accumulating only at ∞ .*
2. *Each eigenvalue corresponds to at most n eigenfunctions. Eigenfunctions corresponding to different eigenvalues are orthogonal.*
3. *For each complex number λ , not an eigenvalue, $(T - \lambda I)^{-1}$ exists and can be represented by a unique linear integral operator*

$$(T - \lambda I)^{-1}f(t) = \int_0^1 G_\lambda(s, t)f(s)ds .$$

4. *The Green's function $G_\lambda(s, t)$ satisfies*
- a. *As a function of $t, s \neq t$,*

$$(T - \lambda I)G_\lambda(s, t) = 0 .$$

- b. $AG_\lambda(s, 0) + BG_\lambda(s, 1) + \int_0^1 dK(t)G_\lambda(s, t) = 0$

a.e. in s .

- c. $\int_0^1 dK_1(t)G_\lambda(s, t) = 0$ a.e. in s .

- d. $G_\lambda(t, s) = G_\lambda^*(s, t)$ a.e. in s and t .

- e. *The eigenfunctions of T are complete in $\mathcal{L}_n^2[0, 1]$.*

If those corresponding to the same eigenvalue have been made orthonormal (denote them by $\{y_i\}_1^\infty$), then for all f in $\mathcal{L}_n^2[0, 1]$

$$f = \sum_1^\infty (f, y_i)y_i .$$

Operators self-adjoint under a transformation are substantially more complex and will be discussed in a subsequent paper. At this point the existence of such a transformation except in trivial cases is doubtful.

6. Nonself-adjoint Stieltjes differential-boundary expansions. Expansions for nonself-adjoint systems have been derived in certain earlier circumstances. First, for the case where $H = 0$, $H_1 = 0$, $K_1 = 0$ or when $H = 0$, $H_1 = 0$, $K = 0$ (the adjoint of the former), an expansion was derived in [2] using familiar techniques. Second, when $H_1 = 0$, $K_1 = 0$ (so $r = 0$, $s = 0$) and H and K are absolutely continuous, an expansion was derived in [5].

In the present situation troubles arise. The bottom terms in the matrix of Theorem 3.1 do not all asymptotically have nice limits as $\text{Re}(\lambda) \rightarrow \infty$, a necessary sort of condition previously. For example, when

$$\begin{aligned} K_{j/6}(t) &= 0, \quad 0 \leq t < \frac{j}{6}, \\ &= 1, \quad \frac{j}{6} < t \leq 1, \end{aligned}$$

the system

$$Ly = (y + K_{1/6}[y(0) - y(1)] + K_{2/6}\Psi)'$$

$$y(0) + y(1) + \int_0^1 dK_{3/6}y = 0,$$

$$\int_0^1 d[K_{4/6} + K_{5/6}]y = 0,$$

has eigenvalues which are zeros of the determinant of

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ e^\lambda + e^{3\lambda/6} & 1 & -e^{5\lambda/6} - e^{2\lambda/6} & 0 & -e^{4\lambda/6} - e^{\lambda/6} \\ e^\lambda + 1 & 0 & -e^{5\lambda/6} - e^{2\lambda/6} & 0 & -e^{4\lambda/6} \\ 0 & 0 & 0 & 1 & 0 \\ e^{4\lambda/6} + e^{5\lambda/6} & 0 & -e^{3\lambda/6} - e^{4\lambda/6} & 1 & -e^{2\lambda/6} - e^{3\lambda/6} \end{bmatrix}.$$

These are $\lambda = (2k + 1)6\pi i$; $k = 0, \pm 1, \dots$. As $\text{Re } \lambda \rightarrow -\infty$, however, the matrix has a singular limit.

However, the system

$$Ly = (y + K_{3/6}\Psi)'$$

$$y(0) + y(1) = 0,$$

$$\int_0^1 dK_{3/6}y = 0,$$

has as its eigenvalue determining matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -e^\lambda & 1 & 0 & 0 & -e^{\lambda/2} \\ 1 + e^\lambda & 0 & -1 & 0 & -e^{\lambda/2} \\ 0 & 0 & 0 & 1 & 0 \\ -2e^{\lambda/2} & 0 & 0 & 1 & -1 \end{bmatrix}.$$

The eigenvalues are easily seen to be $\lambda = 2k\pi i$, $k = 0, \pm 1, \dots$. The limit of the matrix above as $\operatorname{Re} \lambda \rightarrow -\infty$ is nonsingular. Frankly, the author does not entirely understand what is going on.

It is possible to extend the results of [5] under some rather severe restrictions. Let us assume that $H_1 = 0$ and $K_1 = 0$ so that a 3 dimensional vector representation (with $\xi_4 = 0$ and $\xi_5 = 0$) is possible. In addition assume that H is continuous (or by considering the adjoint problem that K is continuous). One system has the form

$$Ly = (y + H[Cy(0) + Dy(1)])' + Py$$

$$Ay(0) + By(1) + \int_0^1 dKy = 0.$$

If y is replaced by \tilde{y} under the invertible transformation $y = Y\tilde{y}$ ($Y' + PY = 0$), then we find the equations $Ly = f$, $Ly = \lambda y$ are equivalent to

$$\left(\tilde{y} + \left[Y^{-1}H - \int_0^1 Y^{-1}Pdx \right] [CY(0)\tilde{y}(0) + DY(1)\tilde{y}(1)] \right)' = Y^{-1}f \text{ or } = \lambda \tilde{y}.$$

The new equations are of the same form as the old, with the same assumptions, with the absence in the second set of the term Py . This results in an equivalent system in which the terms Y and Y^{-1} are missing, a considerable simplification in calculation. We shall henceforth assume that $P = 0$.

The following lemma is the analog of Lemmas 6.4-6.8 of [5].

LEMMA 6.1. (a) $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} \mathcal{H}(t) = 0$ a.e.

In particular $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} \mathcal{H}(1) = 0$.

(b) $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} e^{\lambda t} [\mathcal{H}(1) - \mathcal{H}(t)] = 0$ a.e.

(c) $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} e^{-\lambda t} \mathcal{H}(t) = 0$ a.e.

In particular $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} e^{-\lambda} \mathcal{H}(1) = 0$.

(d) $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} [\mathcal{H}(t) \cdot \mathcal{H}(1) - \mathcal{L}(t)] = 0$ a.e.

(e) $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} \mathcal{M}(t) = 0$ a.e.

In particular $\lim_{\operatorname{Re}(\lambda) \rightarrow \infty} \mathcal{M}(1) = 0$.

Proof. Let V_α^β stand for the total variation from α to β .

(a) If $0 < a < t$, then for an appropriate norm

$$\begin{aligned}
\|\mathcal{H}(t)\| &= \left\| \int_0^t e^{-\lambda x} d\mathcal{H}(x) \right\| \\
&\leq \left\| \int_0^a e^{-\lambda x} d\mathcal{H}(x) \right\| + \left\| \int_a^t e^{-\lambda x} d\mathcal{H}(x) \right\| \\
&\leq V_0^a \|\mathcal{H}\| + e^{-a\lambda} V_a^t \|\mathcal{H}\|.
\end{aligned}$$

The first can be made less than half of any preassigned ε if a is sufficiently close to 0. The second is less than $\varepsilon/2$ if $\operatorname{Re}(\lambda)$ is sufficiently large.

$$\begin{aligned}
(b) \quad \|e^{\lambda t}[\mathcal{H}(1) - \mathcal{H}(t)]\| &= \left\| e^{\lambda t} \int_t^1 e^{-\lambda x} d\mathcal{H}(x) \right\| \\
&\leq \left\| e^{\lambda t} \int_{t+\delta}^1 e^{\lambda x} d\mathcal{H}(x) \right\| + \left\| e^{\lambda t} \int_t^{t+\delta} e^{\lambda x} d\mathcal{H}(x) \right\|
\end{aligned}$$

when $t \leq t + \delta \leq 1$. The second term is less than $V_{t+\delta}^t \|\mathcal{H}\|$. This can be made less than any $\varepsilon/2$ by choosing δ small. The first is bounded by $e^{-\lambda\delta} V_0^1 \|\mathcal{H}\|$ which becomes small as $\operatorname{Re}(\lambda) \rightarrow \infty$.

(c) This is shown by the same technique as was used in (a).

$$\begin{aligned}
(d) \quad \|\mathcal{H}(t)\mathcal{H}(1) - \mathcal{L}(t)\| &= \left\| \int_0^t d\mathcal{H}(z) \int_z^1 e^{\lambda(z-x)} d\mathcal{H}(x) \right\| \\
&\leq \left\| \int_0^t d\mathcal{H}(z) \int_{z+\delta}^1 e^{\lambda(z-x)} d\mathcal{H}(x) \right\| \\
&\quad + \left\| \int_0^t d\mathcal{H}(z) \int_z^{z+\delta} e^{\lambda(z-x)} d\mathcal{H}(x) \right\|.
\end{aligned}$$

The second term is bounded by $V_0^1 \|\mathcal{H}\| \cdot \sup_z V_z^{z+\delta} \|\mathcal{H}\|$. Since \mathcal{H} is continuous on $[0, 1]$ this can be made uniformly small if δ is sufficiently close to 0. The first term is then bounded by $e^{-\lambda\delta} V_0^1 \|\mathcal{H}\|$ which has zero limit as $\operatorname{Re}(\lambda) \rightarrow \infty$.

(e) This is shown by the same technique as was used in (d).

It is now possible to determine the location of the eigenvalues of L .

THEOREM 6.2. *The eigenvalues of L are the zeros of the determinant of*

$$\Delta_1 = \begin{pmatrix} A & -I & 0 \\ Be^\lambda + \mathcal{H}(1) & I & -Be^\lambda \mathcal{H}(1) - \mathcal{L}(1) \\ De^\lambda + C & 0 & -De^\lambda \mathcal{H}(1) - I \end{pmatrix}.$$

If A is nonsingular, they are bounded on the left in the complex plane. If B is nonsingular, they are bounded on the right in the complex plane. Hence when both A and B are nonsingular, the eigenvalues of L lie in a vertical strip.

Since $\det \Delta_1$ is almost periodic in $\operatorname{Im}(\lambda)$, when A and B are nonsingular, the number of zeros lying in a vertical strip $|\operatorname{Re}(\lambda)| < h$ also satisfying $\varepsilon < \operatorname{Im}(\lambda) < \varepsilon + 1$ is bounded by some number

independent of ϵ . For any $\delta > 0$ there is a corresponding $m(\delta) \gg 0$ such that

$$|\det \Delta_1| > m(\delta)$$

for λ lying in the strip $|\operatorname{Re}(\lambda)| < h$ and outside circles of radius δ with centers at the zeros of $\det \Delta_1$.

Proof. An elementary calculation shows, when A is nonsingular, that as $\operatorname{Re}(\lambda) \rightarrow -\infty$, $\det \Delta_1 = (\det A + o(1))$, which ultimately cannot be zero. Similarly, using Lemma 6.1, when B is nonsingular, as $\operatorname{Re}(\lambda) \rightarrow \infty$, $\det \Delta_1 = -e^2(\det B + o(1))$, which is also ultimately non-zero. The final statements follow from [7, pp. 264–269].

We are now in a position to quote directly the results in §6 of [5]. Please note that the phrases “uniformly in \dots ” appearing there should be replaced by “for all x, ξ in $(0, 1)$ ”. Actually a.e. will do fine. Such is our present situation. Assuming A and B are nonsingular, we quote:

THEOREM 6.3. *Let λ_0 be in the resolvent set for L . Let $\{\lambda_i\}_1^\infty$ be the eigenvalues of L (which for convenience we assume to be simple). Let $\{Y_i\}_1^\infty$ and $\{Z_i\}_1^\infty$ be the associated eigenfunctions and adjoint eigenfunctions, assuming that $\int_0^1 Z_i^* Y_i dx = 1$. Then the Green's function for L , $G_{\lambda_0}(s, t) = \mathcal{G}_{11}(s, t)$ satisfies*

$$G_{\lambda_0}(s, t) = \sum_{i=1}^{\infty} \frac{Y_i(t) Z_i^*(s)}{\lambda_i - \lambda_0} \quad \text{a.e.}$$

The proof is by contour integration using the asymptotic estimates established in this section as well as that in [5, §6], suitably avoiding the zeros of $\det \Delta_1$ as we know is possible.

By integrating $G_{\lambda_0}(s, t) \cdot f(s)$ with respect to s before the contour approaches ∞ and appealing to the Lebesgue dominated convergence theorem, we find:

THEOREM 6.4. *Let f in $\mathcal{L}_n^p[0, 1]$ be in the domain of L , then*

$$f(t) = \sum_{i=1}^{\infty} Y_i(t) \int_0^1 Z_i^*(s) f(s) ds.$$

COROLLARY 6.5. *If f in $\mathcal{L}_n^p[0, 1]$ is in the domain of L and g in $\mathcal{L}_n^q[0, 1]$ is in the domain of L^* , then (Parseval's Equality)*

$$\int_0^1 g^*(t) f(t) dt = \sum_{i=1}^{\infty} \int_0^1 g^*(t) Y_i(t) dt \int_0^1 Z_i^*(s) f(s) ds.$$

The problem of expansions in the general case remains open.

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Received March 7, 1974.

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Pacific Journal of Mathematics

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