Pacific Journal of Mathematics

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Vol. 55, No. 1 September 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

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If $C_1, \cdots, C_n$ are n convex surfaces or sets in d-dimensional Euclidean space E^d, then it is of some interest to study the invariance properties of $\bigcap_{i=1}^n (C_i + \boldsymbol{a}_i)$ for all choices of vectors $\boldsymbol{a}_i$ in E^d. Such considerations occur naturally in identifying an object irrespective of the direction in which it approaches the observer.

For example, Melzak [2] and Lewis [1] have investigated the conditions under which the intersection $\bigcap_{i=1}^d (C_i + \boldsymbol{a}_i)$ of certain convex surfaces always is a single point. These surfaces arise from the work of Ratcliff and Hartline [3] concerning varying light intensities upon different visual elements of the eye.

In this article we study such intersections and in Theorem 1, we show that the result of Melzak [1] has an associated Helly number in E^2 but not in E^3. In Theorem 2 we give a necessary and sufficient condition for $\bigcap_{i=1}^n C_i + \boldsymbol{a}_i$ to be nonempty, whenever $C_1, \cdots, C_n$ are convex sets, in terms of the outward normals. This condition is not easy to apply in that it involves the outward normals to intersections of d-membered subsets. So in Theorem 3 we give a sufficient condition in terms of inner and outer apertures which is widely applicable. Finally, in Theorem 4, we give a characterization of the sets which can arise as inner apertures. I am indebted to Z. A. Melzak for suggesting these problems to me.

To define the inner and outer aperture, let D be a convex subset of E^d. If $l \equiv l(\boldsymbol{u}, \boldsymbol{v})$,

$$l = \{\boldsymbol{u} + \lambda\boldsymbol{v}, \lambda \geqq 0\}$$

is a typical ray in E^d, $\boldsymbol{u}, \boldsymbol{v} \in E^d$, $\boldsymbol{v} \neq \boldsymbol{o}$, define

$$\theta(\lambda, D) = \text{dist.}\, \{\boldsymbol{u} + \lambda\boldsymbol{v}, E^d \backslash D\}$$

and

$$\theta(D) = \sup_{\lambda \geqq 0} \theta(\lambda)$$

where

$$\text{dist.}\, \{A, B\} = \inf_{\substack{\boldsymbol{a} \in A \\ \boldsymbol{b} \in B}} ||\boldsymbol{a} - \boldsymbol{b}||$$

when A, B are nonempty subsets of E^d. The inner aperture $\mathscr{I}(D)$ of D is the union of those rays $l(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{u}$ emanating from the origin

o such that $\theta(l(u, v), D) = +\infty$. So, if D contains o, $\mathscr{I}(D)$ is the union of those rays $l \equiv l(o, u)$ in D such that λu can be made an arbitrarily large distance from the boundary of D for λ sufficiently large. The outer cone $O(D)$ of D is what is usually known as the characteristic cone namely the set of all rays $l(u, v) - u$ emanating from o with $l(u, v)$ contained in D. Both $O(D)$ and $\mathscr{I}(D)$ are convex cones and $O(D)$ is closed whenever D is closed. In general, of course, $O(D)$ can be any convex cone in E^d but this is not the case for $\mathscr{I}(D)$. It will follow from Theorem 4 that $\mathscr{I}(D)$ is a G_δ-convex cone with the property that whenever a ray $l \in \text{cl.}\{\mathscr{I}(D)\}\backslash\mathscr{I}(D)$ then the smallest exposed face $F(l)$ of $\text{cl.}\{\mathscr{I}(D)\}$ containing l also is contained in $\{\text{cl.}\,\mathscr{I}(D)\}\backslash\mathscr{I}(D)$.

THEOREM 1. *Let $C_1^*, \cdots, C_n^*$ be n convex sets in E^d whose d-dimensional interiors are nonempty and do not contain a line. Let $C_1, \cdots, C_n$ be the convex surfaces bounding $C_1^*, \cdots, C_n^*$ respectively. Then $\bigcap_{j=1}^n (C_j + a_j)$ is at most a single point for all choices $a_1, \cdots, a_n$ of points in E^d if and only if there does not exist n parallel lines of support $l_1, \cdots, l_n$ to $C_1^*, \cdots, C_n^*$ respectively. In E^2 this is true if and only if some four membered subset $C_{j_1}^*, \cdots, C_{j_4}^*$ do not have parallel lines of support. However, in E^3 and for every $n \geqq 3$ there exist convex sets $C_1^*, \cdots, C_n^*$, whose relative interiors do not contain a line, such that every $n-1$ membered subset have parallel lines of support but this is not so for $C_1^*, \cdots, C_n^*$.*

LEMMA 1. *Let $A_1, \cdots, A_n$ be spherically convex subsets (possibly open, half-open or closed semicircles) of the unit circle S^1 such that*

$$\bigcap_{\nu=1}^{4} (A_{i_\nu} \cup -A_{i_\nu}) \neq \varnothing,\ 1 \leqq i_\nu \leqq n,\ \nu = 1, \cdots, 4\,.$$

Then

$$\bigcap_{i=1}^{n} (A_i \cup -A_i) \neq \varnothing\,.$$

Proof. We parametrise S^1 in terms of the angle θ made with some fixed line through the origin and consider the semicircular interval $[0, \pi]$. The intersection $A_i \cup -A_i$ with $[0, \pi]$ is either

(i) an interval $\langle c_i, d_i\rangle$ not containing either 0 or π,

or (ii) $[0, \pi]$,

or (iii) two intervals $[0, a_i >, < b_i, \pi]$, the first containing 0 and the second containing π.

The classification yields a corresponding subdivision I_1, I_2, I_3 of $\{1, \cdots, n\}$. Let

$$[0, a_{i_1}\rangle = \bigcap_{i \in I_3} [0, a_i\rangle$$
$$\langle b_{i_2}, \pi] = \bigcap_{i \in I_3} \langle b_i, \pi] .$$

If $\langle c_i, d_i\rangle$ and $\langle c_j, d_j\rangle$, $i, j \in I_1$ both meet $[0, a_{i_1}\rangle$ and

(1) $$\langle c_i, d_i\rangle \cap \langle c_j, d_j\rangle \cap [0, a_{i_1}\rangle = \varnothing$$

then at least one of these intervals is contained in $[0, a_{i_1}\rangle$. But then

$$(A_i \cup -A_i) \cap (A_j \cup -A_j) \cap (A_{i_1} \cup -A_{i_1}) \cap (A_{i_2} \cup -A_{i_2})$$

is contained in $[0, a_{i_1}\rangle \cup -[0, a_{i_1}\rangle$ and consequently, by (1), is empty, which is contradiction. So, if

$$I_1^1 = \{i \in I_1 : \langle c_i, d_i\rangle \cap [0, a_{i_1}\rangle \neq \varnothing\}$$

we have, from Helly's theorem, that

(2) $$[0, a_{i_1}\rangle \cap \bigcap_{i \in I_1^1} \langle c_i, d_i\rangle \neq \varnothing .$$

Similarly, if

(3) $$I_1^2 = \{i \in I_1 : \langle c_i, d_i\rangle \cap \langle b_{i_2}, \pi] \neq \varnothing\}$$
$$\langle b_{i_2}, \pi] \cap \bigcap_{i \in I_1^2} \langle c_i, d_i\rangle \neq \varnothing .$$

If there exists $i_3 \in I_1 \backslash I_1^1$ and $i_4 \in I_1 \backslash I_1^2$ then

$$\bigcap_{\nu=1}^{4} A_{i_\nu} \cup -A_{i_\nu} = \varnothing ,$$

so either $I_1^1 = I_1$ or $I_1^2 = I_1$ and, using (2) and (3),

$$\bigcap_{i=1}^{n} A_i \cup -A_i \neq \varnothing .$$

REMARK. This is the best possible result for if $A_1 = [0, \pi/2]$, $A_2 = [\pi/4, 3\pi/4]$, $A_3 = [\pi/2, \pi]$, $A_4 = [3\pi 4, 5\pi/4]$ then

$$\bigcap_{\nu=1}^{3} A_{i_\nu} \cup -A_{i_\nu} \neq \varnothing, 1 \leqq i_1 < i_2 < i_3 \leqq 4$$

but

$$\bigcap_{i=1}^{4} A_i \cup -A_i = \varnothing .$$

LEMMA 2. *There exist n closed spherically convex two dimensional subsets $D_1, \cdots, D_n$ on S^2, none of which contain antipodal points, such that for every $n-1$ membered subset $D_{i_1}, \cdots, D_{i_{n-1}}$ there exists* *a great circle of* S^2 *which meets each* D_{i_ν}, *but there does not exist a great circle meeting each of* $D_1, \cdots, D_n$.

Proof. In [4], Santalo constructs, for each $n \geqq 3$, a family of n compact convex two dimensional sets $F_1, \cdots, F_n$ in E^2 so that each $n-1$ members of the family admit a common transversal but the entire family does not have a common transversal. We mention that such an example is the family of n circular discs whose centers have polar coordinates $\rho = 1$ and $\theta = 2k\pi/n$, $k = 1, \cdots, n$ and whose radii are all equal to $\cos^2 \pi/n$ or $\cos^2 \pi/n + \cos^2 \pi/2n - 1$ according as whether n is even or odd.

Now, if we place the configuration $F_1, \cdots, F_n$ into a plane tangent to S^2, let $D_1, \cdots, D_n$ be the corresponding closed spherically convex subsets of S^2 obtained by the projection of $F_1, \cdots, F_n$ into S^2 from the origin. Clearly $D_1, \cdots, D_n$ satisfy the requirements of the lemma.

Proof of Theorem 1. The proof of the first part is essentially due to Melzak [1] but as he makes the restriction that $d = n$ we repeat the details.

If there exist n parallel lines of support $l_1, \cdots, l_n$ to $C_1^*, \cdots, C_n^*$ respectively then by translating the line l_j into the relative interior of C_j if necessary, $j = 1, \cdots, n$ we obtain n nondegenerate similarly orientated chords $[\boldsymbol{p}_j, \boldsymbol{q}_j]$ of C_j^* parallel to l_j such that

$$||\boldsymbol{p}_1 - \boldsymbol{q}_1|| = \cdots = ||\boldsymbol{p}_n = \boldsymbol{q}_n|| \,.$$

Hence, if $\boldsymbol{a}_j = \boldsymbol{p}_i - \boldsymbol{p}_j$, $j = 1, \cdots, n$

$$\bigcap_{j=1}^{n} C_j^* + \boldsymbol{a}_j \supset \{\boldsymbol{p}_1, \boldsymbol{q}_1\}$$

and so contains at least two points.

On the other hand, if there exist vectors $\boldsymbol{a}_j$, $j = 1, \cdots, n$ such that $\bigcap_{j=1}^{n} C_j^* + \boldsymbol{a}_j$ contains at least two points say $\boldsymbol{p}, \boldsymbol{q}$ then, by considering two dimensional sections of C_j, C_j has a line of support l_j parallel to $[\boldsymbol{p}, \boldsymbol{q}]$ and hence $l_1, \cdots, l_n$ are parallel lines of support to $C_1, \cdots, C_n$ respectively which completes the proof of the first part.

In E^2 we may select a set A_i of unit tangent vectors $\boldsymbol{u}$ to C_i^* by ensuring that the outward normal lies on the left hand side of $\boldsymbol{u}$ when viewed from the point of contact on C_i in a clockwise direction. Then A_i is a spherically convex subset of S^1 which is either S^1 or is contained in semicircle according to whether or not C_i is bounded. Now $C_1^*, \cdots, C_n^*$ do not have parallel lines of support if and only if

$$\bigcap_{i=1}^{n} (A_i \cup -A_i) = \varnothing \,.$$

This, by Lemma 1, is true if and only if there exists some four membered subset of $C_1^*, \cdots, C_n^*$ which do not possess parallel lines of support which completes the proof of the second part of the theorem.

In E^3 and for each $n \geqq 2$ consider the n closed spherically convex subsets $D_1, \cdots, D_n$ of S^2 afforded by Lemma 2. If $\langle , \rangle$ denotes scalar product consider the set of closed half-spaces $\mathscr{H}_i$ such that $H^- \in \mathscr{H}_i$ if

$$H^- = \{x: \langle x, u \rangle \leqq 1\} \quad \text{for some} \quad u \in D_i .$$

Let

$$C_i^* = \bigcap_{\mathscr{H}_i} H^- , \qquad i = 1, \cdots, n .$$

Then D_i is the set of outward normals to C_i^* and so as D_i is two dimensional, C_i^* does not contain a line, $i = 1, \cdots, n$. Also for every $n - 1$ membered subset $C_{i_1}^*, \cdots, C_{i_{n-1}}^*$ of $C_1, \cdots, C_n$ the corresponding set of outward normals $D_{i_1}, \cdots, D_{i_{n-1}}$ all meet some great sphere $S \equiv S(i_1, \cdots, i_{n-1})$. Consequently, if l is a line perpendicular to aff. S, $C_{i_1}, \cdots, C_{i_{n-1}}$ each possess lines of support parallel to l.

On the other hand, if $C_1, \cdots, C_n$ possess parallel lines of support then there would exist a great sphers S^1 of S^2 which meets each of $D_1, \cdots, D_n$ which, by Lemma 2, is not so. Hence $C_1, \cdots, C_n$ do not possess parallel lines of support, which completes the proof of Theorem 1.

We observe the following lemma which is easily established by separating two disjoint convex sets by a hyperplane.

LEMMA 3. *Two convex sets C_1, C_2 in E^d cannot be separated by translation if and only if $N(C_1) \cap (-N(C_2)) = \boldsymbol{o}$, where $N(C_i)$ is the convex cone of outward normals to C_i, $i = 1, 2$.*

Using Helly's theorem we readily verify the following lemma.

LEMMA 4. *If $C_1, \cdots, C_n$ are convex sets in E^d, then $\bigcap_{i=1}^n (C_i + \boldsymbol{a}_i) \neq \emptyset$ for all points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ in E^d if and only if $\bigcap_{\nu=1}^{d+1} (C_{i_\nu} + \boldsymbol{a}_{i_\nu}) \neq \emptyset$ for all points $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ in E^d and for every $d + 1$ membered subset $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{i=1}^n$.*

Using Lemmas 3 and 4 we obtain

THEOREM 2. *If $C_1, \cdots, C_n$ are convex sets in E^d then $\bigcap_{i=1}^n (C_i + \boldsymbol{a}_i) \neq \emptyset$ for all points $\boldsymbol{a}_1, \cdots \boldsymbol{a}_n$ in E^d if and only if*

$$\{-N(C_{i_1})\} \cap N\Big(\bigcup_{\nu=2}^{d+1} C_{i_\nu}\Big) = \emptyset$$

for all $d+1$ membered subcollections $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{i=1}^{n}$.

However, this condition is not completely satisfactory in that $N(\bigcup_{\nu=2}^{d+1} C_{i_\nu})$ is a function of $\bigcup_{\nu=2}^{d+1} C_{i_\nu}$ rather than a combination of functions of each C_{i_ν}. We shall resolve this problem to a certain extent in Theorem 3 by giving a widely applicable sufficient condition.

THEOREM 3. *Let $C_1, \cdots, C_n$ be n convex sets in E^d. Then*

$$\bigcap_{i=1}^{n} (C_i + \boldsymbol{a}_i) \neq \emptyset \tag{4}$$

for all choices of $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ if there exists j such that

$$O(\text{cl. } C_j) \cap \bigcap_{\nu=1}^{d+1} \mathscr{I}(C_{i_\nu}) \neq \emptyset$$

for all $d+1$ membered subcollections $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{\substack{i=1\\ i\neq j}}^{n}$. Further, if at least of cl. $C_1, \cdots,$ cl. C_n *does not contain a line, each is unbounded and $C_1, \cdots, C_n$ cannot be separated by translation,* i.e., (4) *holds for all $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ then*

$$\bigcap_{j=1}^{n} O(\text{cl. } C_j) \neq \emptyset \ .$$

Proof. Let l be a ray of $O(\text{cl. } C_j) \cap \bigcap_{i=1}^{n} \mathscr{I}(C_i)$ which, by Helly's theorem, is nonempty. We may suppose, without loss of generality, that $\boldsymbol{o} \in C_1 \cap \cdots \cap C_n$. Then, if $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ are points of E^d,

$$l + \boldsymbol{a}_i \subset C_i + \boldsymbol{a}_i \ , \qquad i = 1, \cdots, n \ .$$

If $l = \{\lambda \boldsymbol{u}, \lambda \geqq 0\}$, then, as $l \subset \mathscr{I}(C_i)$, $i \neq j$, there exists λ_i such that $\lambda \boldsymbol{u} + \boldsymbol{a}_j$ is in C_i, $\lambda \geqq \lambda_i$.

So, if $\lambda^* = \max_{1 \leqq i \leqq n} \lambda_i$,

$$\lambda^* u + a_j \in \bigcap_{i=1}^{n} C_i \quad \text{as required} \ .$$

To prove the second part, let C_i^* denote the closure of C_i, $i = 1, \cdots, n$. We may assume that C_1 and C_1^* do not contain a line and that for some n, $\bigcap_{i=1}^{n-1} C_i^*$ is unbounded, which is certainly true for $n = 2$. As $\bigcap_{i=1}^{n-1} C_i^*$ is convex closed and unbounded it follows that $O(\bigcap_{i=1}^{n-1} C_i^*)$ is nonempty. Further, as $\bigcap_{i=1}^{n-1} C_i^*$ is contained in C_1^*, $\bigcap_{i=1}^{n-1} C_i^*$ and $O(\bigcap_{i=1}^{n-1} C_i^*)$ do not contain a line. Let l be a ray of $O(\bigcap_{i=1}^{n-1} C_i^*)$, say $l = \{\lambda \boldsymbol{u}, \lambda \geqq 0\}$. If $O(\bigcap_{i=1}^{n} C_i^*)$ is empty then, in particular, $\bigcap_{i=1}^{n} C_i^*$ must be a compact convex set.

If $\lambda \geqq 0$,

$$\lambda \boldsymbol{u} + \bigcap_{i=1}^{m-1} C_i \subset \bigcap_{i=1}^{m-1} C_i \ ,$$

and consequently,

$$(5)\qquad \Big(\lambda u + \bigcap_{i=1}^{m-1} C_i\Big) \cap C_m = \Big(\lambda u + \bigcap_{i=1}^{m-1} C_i\Big) \cap \Big(\bigcap_{i=1}^{m} C_i\Big) .$$

If no matter how large λ is taken, $(\lambda u + \bigcap_{i=1}^{m-1} C_i) \cap C_m$ contains a point $z(\lambda)$ say then, by (5), $z(\lambda)$ is confined to a compact set $\bigcap_{i=1}^{m} C_i$ and $z(\lambda) - \lambda u \in \bigcap_{i=1}^{m-1} C_i$, $\lambda \geqq 0$. It follows that $-l$ is a ray of $O(\bigcap_{i=1}^{m-1} C_i^*)$ which is a contradiction to C_1^* not containing a line. So $\bigcap_{i=1}^{m} C_i^*$ is an unbounded closed convex set and hence $O(\bigcap_{i=1}^{m} C_i^*)$ is nonempty. So repeating this process for $m = 1, 2, \cdots, n$ we conclude that $O(\bigcap_{i=1}^{n} C_i^*)$ is nonempty as required.

DEFINITION. We say that a collection $\mathscr{H}$ of closed half-spaces in E^d is *closed* if whenever $\{H_i^-\}_{i=1}^{\infty}$ is a sequence of closed half-spaces in $\mathscr{H}$, where

$$H_i^- = \{x: \langle x, u_i\rangle \leqq \alpha_i\},\ u_i \text{ a unit vector},$$

and $u_i \rightarrow u$, $\alpha_i \rightarrow \alpha$ as $i \rightarrow \infty$ then the closed half-space

$$H^- = \{x: \langle x, u\rangle \leqq \alpha\}$$

is in $\mathscr{H}$. We say that a collection $\mathscr{H}$ of closed half-spaces is F_σ if it is the countable union of closed collections.

If $\mathscr{H}$ is a closed collection of closed half-spaces notice that the set $\bigcup_{H^-\in\mathscr{H}} H$, where H is the bounding hyperplane of H^-, is a closed set and consequently $\bigcap_{H^-\in\mathscr{H}} \operatorname{int} H^-$ is a relatively open subset of $\bigcap_{H^-\in\mathscr{H}} H^-$.

THEOREM 4. *A set C in E^d is the inner aperture of some convex subset of E^d if and only if*

$$C = o \cup \bigcap_{\mathscr{H}} \operatorname{int.} H^-$$

where $\mathscr{H}$ is an F_σ-collection of closed half-spaces and $o \in H$, the bounding hyperplane of H^-, for all $H^- \in \mathscr{H}$.

REMARK. So, in particular, C has to be a G_δ-convex cone with apex the origin such that if $x \in \{\text{cl.}\, C\}\backslash C$ then the smallest exposed face $F(x)$ of cl. C that contains x is also contained in $\{\text{cl.}\, C\}\backslash C$. In E^3 the converse is also true.

Proof. We shall assume that the theorem is true in $d-1$ dimensions, the theorem being trivial for $d = 1$.

(i) *Necessity.* Let C be the inner aperture of some convex set D in E^d where, since $\mathscr{I}(D) = \mathscr{I}(\text{cl.}\, D)$ we may suppose that D is closed. If $D = E^d$ then $C = E^d$ and, by convention,

$$C = \bigcap_{\mathscr{H}} \text{int. } H^- = E^d$$

where $\mathscr{H}$ is the empty set of closed half-spaces.

Otherwise $D \neq E^d$ and so possesses at least one hyperplane of support M say with D contained in the closed half-space M^-. We may suppose, without loss of generality, that $\boldsymbol{o} \in M$. If D contains a (maximal) linear subspace L of dimension at least one then $L \subset M$ and

$$D = F + L$$

where F is a closed convex subset of $L^\perp$. By the inductive assumption the inner aperture $\mathscr{I}(F)$ of F can be written

$$\mathscr{I}(F) = \boldsymbol{o} \cup \bigcap_{\mathscr{H}^*} \text{int. } H^{*-}$$

where $\mathscr{H}^*$ is a closed subset of the closed half-spaces in $L^\perp$. Then

$$C = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int. } H^-$$

where $\mathscr{H}$ is the closed collection of closed half-spaces in E^d formed by taking H^- in $\mathscr{H}$ if

$$H^- = L + H^{*-}$$

where $H^{*-} \in \mathscr{H}^*$.

If D does not contain a line then the set of rays in D is a closed convex cone K which has a hyperplane of support say $\{x_d = 0\}$ with

$$K \cap \{x_d = 0\} = \boldsymbol{o} .$$

Let π_ν denote the hyperplane $x_d = \nu$, $\nu \geqq 0$. Let l be a typical ray of K,

$$\alpha_\nu(l) = \text{dist. } \{(l\pi_\nu), \pi_\nu(E^d \backslash D)\} ,$$

and

$$\alpha(l) = \sup_{\nu \geqq 0} \alpha_\nu(l) .$$

By considering two dimensional sections through l it is easily verified that $\alpha_\nu(l)$ increases with ν. Also

$$l \subset C \quad \text{if and only if} \quad \alpha(l) = +\infty .$$

So, if

$$C_i = \{l : l \text{ is a ray in } K, \alpha(l) > i\} ,$$

then

$$C = \bigcap_{i=1}^{\infty} C_i \,. \tag{6}$$

Now $C_i K$, $i = 1, 2, \cdots$ and

$$K = o \cup \bigcap_{\mathscr{H}} \text{int.}\, H^- \tag{7}$$

where $\mathscr{H}$ is the collection of closed half-spaces, whose bounding hyperplanes contain o, such that $K \backslash o \subset \text{int.}\, H^-$. If $\hat{K} = K \cap S^{d-1}$, let $\mathscr{H}_j^*$ denote the closed set of the closed half-spaces H^-,

$$H^- = \{x\colon \langle x, u\rangle \leqq 0\}$$

where

$$\langle -u, k\rangle \leqq -2^{-j}\,, \qquad \text{for all} \quad k \in \hat{K}\,.$$

Then $\mathscr{H} = \bigcup_{j=1}^{\infty} \mathscr{H}_j^*$ and so, using (6), (7) it is enough to show that

$$C_i = K \cap \bigcap_{\mathscr{H}_i} \text{int.}\, H^-$$

where $\mathscr{H}_i$ is a closed collection of closed half-spaces of E^d whose bounding hyperplanes goes through o.

Suppose now that l is a ray of $K \backslash C_i$. Then

$$\alpha(l) \leqq i\,.$$

For $j = 1, 2, \cdots$, there exist points $a_1, a_2, \cdots$, with $a_j \in \pi_j \cap \text{bdy.}\, D$ such that

$$\|a_j - \{\pi_j \cap l\}\| \leqq i\,. \tag{8}$$

Let H_j denote a hyperplane of support to D at a_j, with $D \subset H_j^-$. As we may suppose that $K \neq o$, H_j is not parallel to the hyperplane π_1. So $H_j \cap \pi_1$ is a line in π_1. If we consider the two plane σ_j through l and a_j then H_j meets σ_j in a line l_j. As l_j supports $\sigma_j \cap D$, it follows, using (8), that

$$\|l_j \cap \pi_1 - l \cap \pi_1\| \leqq i\,. \tag{9}$$

Consequently the $(d-2)$ affine space $\pi_1 \cap H_j$ lies within a distance i of $l \cap \pi_1$. So we may suppose, by picking subsequences if necessary, that $\pi_1 \cap H_j \to \pi_1 \cap H_0$ as $j \to \infty$ and $l_j \cap \pi_1$ tends to a point which, with a view to later developments, we denote by $l_0 \cap \pi_1$. Let the line through the points a_j and $l_j \cap \pi_1$ be l_j^*, $j = 1, 2, \cdots$. As (8), (9) hold, l_j^* converges to a line l_0 through $l_0 \cap \pi_1$ and parallel to l. Consequently $H_j \to H_0$ as $j \to \infty$. So $D \subset H_0^-$ and

(10) $$\|\pi_\nu \cap l_0 - \pi_\nu \cap l\| = \beta \leqq i\,, \qquad \text{if} \quad \nu \geqq 0\,,$$

β a constant. We claim that

$$H_0^- + \{\pi_1 l - \pi_1 l_0\} = H_0'^- \text{ say}\,,$$

contains K and H_0' supports K and passes through $\boldsymbol{o}$. Certainly

(11) $$l \subset H_0'$$

and so H_0' passes through $\boldsymbol{o}$. If there exists a ray l^* in $K \backslash H_0'^-$, then l^* meets H_0 which contradicts $D \subset H_0^-$.

Now let $\mathscr{H}_i$ denote those closed half-spaces H^- such that the bounding hyperplane H supports K and there exists a closed half-space H^{*-} containing H^- such that H^* supports D; H^* is parallel to H and a distance, in the hyperplane π_1, at most i from H.

By (11),

(12) $$C_i \supset K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\,,$$

where $\mathscr{H}_i$ is a closed set of closed half-spaces.

Conversely, if l is a ray of

$$K \backslash \{K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\}$$

then there exists H^- in $\mathscr{H}_i$ such that $l \subset H$. Then there exists a closed half-space H^{*-} which contains D such that H^* is parallel to H and the distance between H and H^* is at most i. Consequently

$$\alpha_\nu(l) \leqq i,\ \nu \geqq 0$$

and so $l \not\subset C_i$. Hence

(13) $$C_i \subset K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\,.$$

Combining (12) and (3),

$$C_i = K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-$$

which completes the proof of the necessity of the conditions.

(ii) *Sufficiency*. Suppose now that

$$C = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int. } H^-$$

where $\mathscr{H}$ is an F_σ-collection of closed half-spaces and $\boldsymbol{o} \in H$ for all $H^- \in \mathscr{H}$. So we may write $\mathscr{H} = \bigcup_{i=1}^{\infty} \mathscr{H}_i$ where the $\mathscr{H}_i$ form an increasing sequence of closed collections.

Consider the closed convex cone

$$C_0 = \text{cl.}\, C = \bigcap_{\mathscr{H}} H^- .$$

If $C_0 = E^d$ then $C = E^d$ and C is its own inner aperture. Otherwise C_0 possesses one hyperplane of support M through $\boldsymbol{o}$ with C_0 contained in the closed half-space M^-. If $M \cap C_0$ contains a maximal linear subspace L of dimension at least 1 then we may write $C_0 = F + L$ where F is a proper closed convex cone in L. Notice that $L \subset H$ for each $H^- \in \mathscr{H}$ and consequently we may write

$$H^- = L + H^{*-} \quad \text{for each} \quad H^- \in \mathscr{H},$$

where H^{*-} is a closed half-space in L whose bounding hyperplane H^* passes through $\boldsymbol{o}$. Consequently

$$C = \boldsymbol{o} \cup \{\{\bigcap_{\mathscr{H}} \text{int.}\, H^{*-}\} + L\} .$$

By the inductive assumption, there exists a closed convex set D^* in L such that

$$\boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int.}\, H^{*-}$$

is the inner aperture of D^* in L. Let

$$D = D^* + L$$

and then C is the inner aperture of D.

Henceforth therefore we may suppose that C_0 is a proper closed convex cone in E^d i.e., C_0 does not contain a line and we can also suppose that the ray

$$X_d^+ = \{(0, \cdots, 0, x_d), x_d \geqq 0\}$$

is in C_0 and that the hyperplane $\pi_0 = \{x_d = 0\}$ supports C_0 with $\pi_0 \cap C_0 = \boldsymbol{o}$. Then, as for K in the proof of necessity,

$$C_0 = \boldsymbol{o} \cup \bigcap_{\mathscr{H}_0} \text{int.}\, H^-$$

where $\mathscr{H}_0$ is a closed set of closed half-spaces whose bounding hyperplanes pass through $\boldsymbol{o}$. We may suppose that

$$\mathscr{H}_0 \subset \mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots$$

and let

$$C_i = \boldsymbol{o} \cup \bigcap_{\mathscr{H}_i} \text{int.}\, H^- , \qquad i = 0, 1, 2, \cdots .$$

We shall produce inductively a nested sequence of closed convex sets $\{C_i^*\}_{i=0}^{*}$ such that C_i is the inner aperture of C_i^* and indeed

(14) $$C_{i+1}^* = C_i^* \cap \bigcap_{\mathscr{H}_i} H^{*-},\ i \geqq 0$$

where, if $H^- \in \mathscr{H}_i$ then H^{*-} is that closed half-space containing H^- such that H^* and H are parallel and at a distance i apart in the hyperplane π_1.

We begin the induction by taking

$$C_0^* = \{\boldsymbol{x} = (x_1, \cdots, x_d),\ x_d \geqq 0 \quad \text{and} \quad \text{dist.}\,(\boldsymbol{x}, C_0 \cap \pi_{x_d}) \leqq x_d^{1/2}\}\ .$$

Clearly C_0^* is closed and it is convex since, from above, $C_0^* \cap \pi_\nu$ is convex, $\nu \geqq 0$ and so C_0^* cannot possess a point of concavity. We shall show that

(15) $$\mathscr{I}(C_0^*) = C_0\ .$$

First notice that if $\boldsymbol{u} = (u_1, \cdots, u_d)$ is a unit vector in C_0 then $u_d > 0$. So, if $l = \{\lambda u: \lambda \geqq 0\}$ is the corresponding ray in C_0

$$\theta_\lambda = \alpha_{\lambda u_d}(l) \geqq \sqrt{\lambda u_d} > 0\ .$$

So, if m is a positive number

(16) $$\theta_\lambda \geqq m$$

provided $m^2/u_d \leqq \lambda$. It is an almost immediate consequence of (16) that $l \subset \mathscr{I}(C_0^*)$ and hence $C_0 \subset \mathscr{I}(C_0^*)$.

Suppose next that the ray

$$l' = \{\lambda \boldsymbol{v},\ \lambda \geqq 0\}$$

is not in C_0. If $v_d \leqq 0$ then $\lambda \boldsymbol{v} \notin C_0^*$ for all $\lambda > 0$ and then certainly $l' \not\subset \mathscr{I}(C_0^*)$. If $v_d > 0$ then $l' \cap \pi_\nu$ is a single point for each $\nu \geqq 0$ and there exists $\eta > 0$ such that

$$\text{dist.}\,(\boldsymbol{v}, C_0 \cap \pi_{v_d}) > \eta\ .$$

So

(17) $$\text{dist.}\,(\lambda \boldsymbol{v}, C_0 \pi_{\lambda v_d}) > \lambda \eta\ .$$

But, if $l' \subset \mathscr{I}(C_0^*)$ then, in particular, $\lambda \boldsymbol{v} \in C_0^*$ for each $\lambda \geqq 0$. So

(18) $$\text{dist.}\,(\lambda \boldsymbol{v}, C_0 \pi_{\lambda v_d}) \leqq (\lambda v_d)^{1/2},\ \lambda \geqq 0\ .$$

However, provided $\lambda > v_d/\eta^2$ it follows from (17) that (18) is false. Consequently $l' \not\subset \mathscr{I}(C_0^*)$ which establishes (15).

Suppose inductively that for some $m \geqq 1$ we have constructed m closed convex sets $C_0^*, \cdots, C_{m-1}^*$ in E^d with C_i being the inner aperture of C_i^*, $i = 0, \cdots, m-1$. Indeed,

(19) $$C^*_{i+1} = C^*_i \cap \bigcap_{\mathscr{H}_{i+1}} H^{*-}\,, \qquad i = 0, 1, \cdots, m-2\,,$$

where, if $H^- \in \mathscr{H}_{i+1}$ then H^{*-} is that closed half-space containing H^- such that H^* and H are parallel and at a distance $i+1$ apart in the plane π_1.

For each $H^- \in \mathscr{H}_m$, let H^{*-} be that closed half-space containing H^- such that H^* and H are parallel and at a distance m apart in the plane π_1. Define

(20) $$C^*_m = C^*_{m-1} \cap \bigcap_{\mathscr{H}_m} H^{*-}\,.$$

We claim that the inner aperture of C^*_m is C_m i.e.,

(21) $$\mathscr{I}(C^*_m) = C_m\,.$$

If l is a ray of C_0 not in C_m then l is in some hyperplane H where $H^- \in \mathscr{H}_m$. Consequently, by considering the corresponding closed half-space H^{*-}, we deduce that $\alpha(l) \leqq m$, and so $l \not\subset \mathscr{I}(C^*_m)$. Hence $\mathscr{I}(C^*_m) \subset C_m$.

On the other hand, suppose that $l \in C_m$. That the set

$$\bigcup_{\mathscr{H}_m} H^* = H_m \text{ say}$$

is a closed set and does not meet the ray $l\backslash \boldsymbol{o}$. As each hyperplane H, with $H^- \in \mathscr{H}_m$, passes through $\boldsymbol{o}$, it follows that

(22) $$\text{dist.}\,(l \cap \pi_\nu, H_m) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Also $l \in \mathscr{I}(C^*_{m-1})$ and so

(23) $$\text{dist.}\,(l \cap \pi_\nu, E^d\backslash C^*_{m-1}) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Consequently using (20), (22), (23),

$$\text{dist.}\,(l \cap \pi_\nu, E^d\backslash C^*_m) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Therefore, $l \subset \mathscr{I}(C^*_m)$ and so $C_m \subset \mathscr{I}(C^*_m)$ which completes the verification of (21).

The results (20), (21) verify (19) for m and we can now suppose that the C^*_m have been defined so that (20), (21) hold for $m = 0, 1, 2, \cdots$. Define

$$C^* = \bigcap_{m=0}^{\infty} C^*_m$$

and we shall show that $\mathscr{I}(C^*) = C$.

Suppose that l is a ray of C_0 not in $\mathscr{I}(C^*)$. Then there exists m such that $\alpha_\nu(l) \leqq m$, $\nu \geqq 0$. So l is not in $\mathscr{I}(C^*_{m+1}) = C_{m+1}$. Consequently l is not in C. So $C \subset \mathscr{I}(C^*)$.

On the other hand, suppose that l is a ray of C_0 which is not in C. Then l is not in C_m for some $m \geqq 0$. So

$$l \not\subset \mathscr{I}(C_m^*) \supset \mathscr{I}(C^*) .$$

Hence $\mathscr{I}(C^*) \subset C$ and this finally establishes that

$$\mathscr{I}(C^*) = C$$

which completes the proof of Theorem 4.

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Received April 30, 1973 and in revised form August 21, 1973. Research supported by the National Research Council of Canada, while visiting the University of British Columbia.

UNIVERSITY COLLEGE LONDON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 55, No. 1 September, 1974