ON \((J, M, m)\)-EXTENSIONS OF BOOLEAN ALGEBRAS

Dwight Webster Read
ON \((J, M, m)\)-EXTENSIONS OF BOOLEAN ALGEBRAS

DWIGHT W. READ

The class \(\mathcal{K}\) of all \((J, M, m)\)-extensions of a Boolean algebra \(\mathcal{A}\) can be partially ordered and always contains a maximum and a minimal element, with respect to this partial ordering. However, it need not contain a smallest element. Should \(\mathcal{K}\) contain a smallest element, then \(\mathcal{K}\) has the structure of a complete lattice. Necessary and sufficient conditions under which \(\mathcal{K}\) does contain a smallest element are derived. A Boolean algebra \(\mathcal{A}\) is constructed for each cardinal \(m\) such that the class of all \(m\)-extensions of \(\mathcal{A}\) does not contain a smallest element. One implication of this construction is that if a Boolean algebra \(\mathcal{A}\) is the Boolean product of a least countably many Boolean algebras, each of which has more than one \(m\)-extension, then the class of all \(m\)-extensions of \(\mathcal{A}\) does not contain a smallest element. The construction also has as implication that neither the class of all \((m, 0)\)-products nor the class of all \((m, n)\)-products of an indexed set \(\{\mathcal{A}_i\}_{i \in \tau}\) of Boolean algebras need contain a smallest element.

1. Sikorski [2] has investigated the question of imbedding a given Boolean algebra \(\mathcal{A}\) into a complete or \(m\)-complete Boolean algebra \(\mathcal{B}\) and has shown that in the case where the imbedding map is not a complete isomorphism, the imbedding need not be unique up to isomorphism. He further has shown that if \(\mathcal{K}\) is the class of all \((J, M, m)\)-extensions of a Boolean algebra \(\mathcal{A}\), then \(\mathcal{K}\) has a naturally defined partial ordering on it and always contains a maximum and a minimal element. He has left as an open question whether it always contains a smallest element. La Grange [1] has given an example which implies that \(\mathcal{K}\) need not always contain a smallest element. However, the question of when does \(\mathcal{K}\) in fact contain a smallest element is of interest as it turns out that should \(\mathcal{K}\) contain a smallest element, it has the structure of a complete lattice.

In §2, necessary and sufficient conditions are given for \(\mathcal{K}\) to contain a smallest element. In addition, the principle behind La Grange’s example is generalized in Proposition 2.10 to show that if \(\mathcal{A}\) is not \(m\)-representable then the class \(\mathcal{K}\) of all \((J, M, m')\)-extensions of \(\mathcal{A}\), where \(J, \overline{M} < \sigma\) and \(m' > M\), will not contain a smallest element.

Since the proof of this result requires that \(J\) and \(M\) have cardinality \(\leq \sigma\), it is of interest to ask if the class of all \(m\)-extensions
contain a smallest element in general, and the answer is no.

In § 3, a Boolean algebra $\mathcal{A}$ is constructed for each cardinal $m$ such that the class $\mathcal{K}$ of all $m$-extensions of $\mathcal{A}$ does not contain a smallest element. The construction has as implication (Theorems 3.1 and 3.2; Corollary 3.1) that for each algebra in a rather broad group of Boolean algebras, the class of all $m$-extensions will not contain a smallest element. In particular, this group includes all Boolean algebras which are the Boolean product of at least countably many Boolean algebras each of which has more than one $m$-extension.

Finally, in the last section, Sikorski’s result that there is an equivalence between the class $\mathcal{P}$ of all $(m, 0)$-products of an indexed set $\{\mathcal{A}_t\}_{t \in \mathcal{T}}$ of Boolean algebras and the class of all $(J, M, m)$-extensions of the Boolean product $\mathcal{A}_0$ of $\{\mathcal{A}_t\}_{t \in \mathcal{T}}$, for suitably defined $J$ and $M$, is generalized to show there is an equivalence between the class $\mathcal{P}_n$ of all $(m, n)$-products of $\{\mathcal{A}_t\}_{t \in \mathcal{T}}$ and all $(J, M, m)$-extensions of $\mathcal{F}$, where $\mathcal{F}$ is the field of sets generated by a certain set $\mathcal{I}$, for suitably defined $J$ and $M$. Then the above results imply that neither $\mathcal{P}$ nor $\mathcal{P}_n$ need contain a smallest element.

The notation throughout follows that of Sikorski [2].

2. Let $n$ be the cardinality of a set of generators for the Boolean algebra $\mathcal{A}$, let $\mathcal{A}_{m,n}$ be a free Boolean $m$-algebra with a set of $n$ free $m$-generators, let $\mathcal{A}_n$ be the free Boolean algebra generated by this set of $n$ free $m$-generators and let $g$ be a homomorphism from $\mathcal{A}_n$ to $\mathcal{A}$. Let $\Delta_0$ be the kernel of this homomorphism and let $I$ be the set of all $m$-ideals $\mathcal{A}$ in $\mathcal{A}_{m,n}$ such that:

a. $\Delta_0 \cap \mathcal{A}_n = \Delta_0$;

b. $\mathcal{A}$ contains all the elements

$$A_0 - \bigcup_{A \in S_1} A, \quad \bigcup_{A \in S_1} A - A_0,$$

$$A_0 - \bigcap_{A \in S_2} A, \quad \bigcap_{A \in S_2} A - A_0,$$

where $A_0 \in \mathcal{A}_n$ and $S_1, S_2$ are any subsets of $\mathcal{A}_n$ of cardinality $\leq m$ such that:

$$g(S_1) \in J, \quad g(A_0) = \bigcup_{A \in S_1} g(A)$$

$$g(S_2) \in M, \quad g(A_0) = \bigcap_{A \in S_2} g(A).$$

For each $\mathcal{A} \in I$ let

$$\mathcal{A}_\mathcal{A} = \mathcal{A}_{m,n} / \mathcal{A}$$

and

$$g_\mathcal{A}(\mathcal{A}/\mathcal{A}) = g(\mathcal{A}), \quad \text{for all } \mathcal{A} \in \mathcal{A}_{m,n}.$$
**Proposition 2.1.** The ordered pair \( \{i_\Delta, A_\Delta\} \) is a \((J, M, m)\)-extension of the Boolean algebra \( A \) and if \( \{i, B\} \) is a \((J, M, m)\)-extension of \( A \) there is a \( \Delta \in I \) such that \( \{i_\Delta, A_\Delta\} \) is isomorphic to \( \{i, B\} \). Further, if \( \Delta, \Delta' \in I \) then
\[
\{i_\Delta, A_\Delta\} \leq \{i_{\Delta'}, A_{\Delta'}\} \text{ if, and only if, } \Delta \supseteq \Delta'.
\]

**Lemma 2.1.** If \( S \) is a set of elements in \( \mathcal{X} \) then the least upper bound (lub) of \( S \) exists in \( \mathcal{X} \).

Now let \( \mathcal{X}(J, M, m) \) denote the class of all \((J, M, m)\)-extensions of \( A \).

**Theorem 2.1.** Let \( \mathcal{X} \) be the class of all \((J, M, m)\)-extensions of a Boolean algebra \( A \). The following are equivalent:
1. \( \mathcal{X} \) contains a smallest element;
2. \( \mathcal{X} \) is a lattice;
3. \( \mathcal{X} \) is a complete lattice.

**Proof.**
1. \( \Rightarrow \) 3. It suffices to show that if \( S \) is a set of \((J, M, m)\)-extensions of \( A \) then the greatest lower bound (glb) of \( S \) exists in \( \mathcal{X} \), which follows from noting that if \( L \) is the set of all lower bounds for the set \( S \) then \( L \neq 0 \) and by Lemma 2.1 the lub of \( L \) exists in \( \mathcal{X} \), hence is in \( L \).

3. \( \Rightarrow \) 2. By definition.

2. \( \Rightarrow \) 1. If \( \{i, B\} \) is an \( m \)-completion of \( A \), \( \{j, C\} \in \mathcal{X} \), and \( \mathcal{X} \) a lattice, then there is an element \( \{j', C'\} \in \mathcal{X} \) such that
\[
\{j', C'\} \leq \{j, C\}.
\]
Thus
\[
\{j', C'\} \leq \{i, B\},
\]
so
\[
\{j', C'\} = \{i, B\},
\]
implying
\[
\{i, B\} \leq \{j, C\}.
\]
Hence \( \{i, B\} \) is a smallest element in \( \mathcal{X} \).

**Corollary 2.1.** If \( J' \supseteq J \) and \( M' \supseteq M \) then the following are equivalent:
1. \( \mathcal{X}(J, M, m) \) contains a smallest element;
2. $\mathcal{H}(J', M', m)$ is a sublattice of $\mathcal{H}(J, M, m)$;
3. $\mathcal{H}(J', M', m)$ is a complete sublattice of $\mathcal{H}(J, M, m)$.

Proof.
1. $\Rightarrow$ 3. Since $\mathcal{H}(J', M', m)$ contains a smallest element, so does $\mathcal{H}(J, M, m)$ hence $\mathcal{H}(J', M', m)$ and $\mathcal{H}(J, M, m)$ are complete lattices. If $\{\{i, \mathcal{B}_i\}\} \in S$ is a set of elements in $\mathcal{H}(J', M', m)$, $\{i, \mathcal{C}\}$ is the lub of $S$ in $\mathcal{H}(J, M, m)$ and $\{i', \mathcal{C}'\}$ is the lub of $S$ in $\mathcal{H}(J', M', m)$, then there is an $m$-homomorphism $h$ mapping $\mathcal{C}'$ onto $\mathcal{C}$ such that $hi' = i$. Hence $i$ is a $(J', M', m)$-isomorphism. Thus $\{i, \mathcal{C}\} \in \mathcal{H}(J', M', m)$, implying

$$\{i, \mathcal{C}\} = \{i', \mathcal{C}'\}.$$

If $\{i, \mathcal{C}\}$ is the glb of $S$ in $\mathcal{H}(J, M, m)$ and $\{i', \mathcal{C}'\} \in S$, then by a similar argument, $i$ is a $(J', M', m)$-isomorphism, which implies $\{i, \mathcal{C}\}$ is the glb of $S$ in $\mathcal{H}(J', M', m)$.

3. $\Rightarrow$ 2. By definition.

2. $\Rightarrow$ 1. The proof is the same as that for showing 2. $\Rightarrow$ 1, in Theorem 2.1.

Thus it is of particular interest to know whether $\mathcal{H}(J, M, m)$ contains a smallest element, in general. Although, as it turns out, $\mathcal{H}(J, M, m)$ need not contain a smallest element in general, a minimal $(J, M, m)$-extension is always an $m$-completion, hence there is always a unique minimal $(J, M, m)$-extension in $\mathcal{H}(J, M, m)$.

**Proposition 2.2.** An $m$-completion $\{i, \mathcal{B}\}$ of the Boolean algebra $\mathcal{A}$ is a unique minimal element in $\mathcal{H}$.

**Proof.** That a minimal element in $\mathcal{H}$ is an $m$-completion is clear.

If $\{i', \mathcal{B}'\}$ is another minimal element in $\mathcal{H}$, there are $\Delta, \Delta' \in I$ such that

$$\{i, \mathcal{B}\} = \{i_s, \mathcal{A}_s\}$$

and

$$\{i', \mathcal{B}'\} = \{i_{s'}, \mathcal{A}_{s'}\}.$$

Now $\{i, \mathcal{B}\}$ and $\{i', \mathcal{B}'\}$ minimal in $\mathcal{H}$ imply $\Delta$ and $\Delta'$ are maximal $m$-ideals in $I$, but if $\hat{\Delta}$ is a maximal $m$-ideal in $I$ then $g_{\hat{\Delta}}(\mathcal{A}_s, \mathcal{A}_s)$ is dense in $\mathcal{A}_{\hat{\Delta}}$. The ideal $\hat{\Delta}' = \langle \hat{\Delta}, A \rangle$ in $\mathcal{A}_s, \mathcal{A}_s$ is an $m$-ideal and $\hat{\Delta}' \in I$, contradicting the maximality of $\hat{\Delta}$. So $\{i', \mathcal{B}'\}$ is an $m$-completion of $\mathcal{A}$, hence isomorphic to $\{i, \mathcal{B}\}$, implying
\{i', \mathcal{B}'\} = \{i, \mathcal{B}\}.

**Proposition 2.3.** If \( \mathcal{A} \) is a Boolean \( m \)-algebra that satisfies the \( m \)-chain condition and

\[
\bigcup_{t \in T} A_t
\]

is the join of an indexed set \( \{A_t\}_{t \in T} \) in \( \mathcal{A} \), then there is an indexed set \( \{A'_t\}_{t \in T} \) of disjoint elements of \( \mathcal{A} \) such that

1. \( \bigcup_{t \in T} A'_t = \bigcup_{t \in T} A_t ; \)
2. \( A'_t \subseteq A_t \) for all \( t \in T \).

**Proof.** Let \( \mathcal{I} \) be the collection of all sets \( S \) of disjoint elements in \( \mathcal{A} \) such that for each \( s \in S \) there is a \( t \in T \) with \( s \subseteq A_t \). If

\[
S_1 \subseteq S_2 \subseteq \cdots \subseteq S_i \subseteq \cdots
\]

is a chain of sets in \( \mathcal{I} \) indexed by \( I \) and ordered by set theoretical inclusion, then

\[
\bigcup_{t \in I} S_t = S \in \mathcal{I}.
\]

By Zorn's lemma there is a maximal set in \( \mathcal{I} \), say \( S' = \{A_r\}_{r \in R} \), and it immediately follows that

\[
\bigcup_{r \in R} A_r \neq A.
\]

Now let

\[
\varphi: S' \longrightarrow T
\]

be a mapping such that if \( A_r \in S' \) then

\[
A_r \subseteq A_{\varphi(A_r)}.
\]

For each \( t \in T \) define

\[
A'_t = \bigcup \{A_r \in S' : \varphi(A_r) = t\}
\]

if there is an \( A_r \in S' \) such that \( \varphi(A_r) = t \), otherwise define

\[
A'_t = \Lambda.
\]

Then

\[
\{A'_t\}_{t \in T}
\]

is the desired set.

**Proposition 2.4.** Let \( \mathcal{A} \) be a Boolean algebra. The following are equivalent:
1. \( \mathcal{A} \) satisfies the m-chain condition:
2. for all sets \( S \) in \( \mathcal{A} \) such that \( \bigcup_{s \in S} s \) exists,

\[
\bigcup_{s \in S} s = \bigcup_{s \in S'} s
\]

for some set \( S' \subseteq S \) with \( S' \leq m \); and dually for meets.

**Proof.**

1. \( \implies \) 2. Suppose \( \mathcal{A} \) satisfies the m-chain condition. It suffices to show that if

\[
S = \{A_t\}_{t \in T} \quad \text{and} \quad V = \bigcup_{t \in T} A_t, \quad \bar{T} = m' > m,
\]

then there is a set \( T' \subseteq T, \bar{T}' \leq m, \) such that

\[
\bigcup_{t \in T'} A_t = V.
\]

Let \( \{i, \mathcal{B}\} \) be an \( m' \)-completion of \( \mathcal{A} \). Then \( \mathcal{B} \) satisfies the m-chain condition and

\[
V_{\mathcal{B}} = i(V_{\mathcal{A}})
\]

\[
= \bigcup_{t \in T'} i(A_t).
\]

By Proposition 2.3, there is a set \( \{B_t\}_{t \in T} \) of disjoint elements in \( \mathcal{B} \) such that

\[
B_t \subseteq i(A_t) \quad \text{and} \quad \bigcup_{t \in T} B_t = \bigcup_{t \in T} i(A_t).
\]

Since this set contains at most \( m \)-distinct elements,

\[
\bigcup_{t \in T'} B_t = \bigcup_{t \in T'} i(A_t),
\]

\( T' \subseteq T \) and \( \bar{T}' \leq m \). Thus

\[
V_{\mathcal{B}} = \bigcup_{t \in T'} i(A_t)
\]

or

\[
V_{\mathcal{A}} = \bigcup_{t \in T'} A_t.
\]

2. \( \implies \) 1. Suppose \( \{A_t\}_{t \in T} \) is an \( m' \)-indexed set of disjoint elements of \( \mathcal{A} \), \( m' > m \). It may be assumed that \( \{A_t\}_{t \in T} \) is a maximal set of disjoint elements of \( \mathcal{A} \). Then for some \( T' \subseteq T, \bar{T}' \leq m, \)

\[
V_{\mathcal{A}} = \bigcup_{t \in T'} A_t.
\]

Since \( \bar{T}' \neq \bar{T} \), there is a \( t_0 \in T - T' \) such that
Thus
\[ \bigcup_{i \in T} A_i \neq \bigvee \mathcal{A}, \]
a contradiction. Hence \( T \leq m. \)

This gives, as an immediate corollary, the following result due to Sikorski [2].

**Corollary 2.2.** If \( \mathcal{A} \) is a Boolean \( m \)-algebra and satisfies the \( m \)-chain condition, it is a complete Boolean algebra.

**Proposition 2.5.** The class \( \mathcal{X}(J, M, m') \) contains a smallest element if \( \mathcal{X}(J, M, m) \) contains a smallest element, \( m' < m. \)

**Proof.** Let \( \{i, B\} \) be the smallest element in \( \mathcal{X}(J, M, m). \) If \( \{j', C'\} \in \mathcal{X}(J, M, m'), \) let \( \{k, C\} \) be an \( m \)-completion of \( C'. \) Then \( \{kj, C\} \in \mathcal{X}(J, M, m). \)

By the fact that \( \{i, B\} \) is the smallest element in \( \mathcal{X}(J, M, m), \) there is an \( m \)-homomorphism \( h \) such that
\[ h: C \rightarrow B \text{ and } hkj = i. \]

Also \( \{i, B\} \) an \( m \)-completion of \( \mathcal{A} \) implies that there is an \( m' \)-completion \( \{i, B'\} \) of \( \mathcal{A} \) such that \( B' \subseteq B. \) Thus \( hk(C') \) is an \( m \)-subalgebra of \( B, \) hence \( B' \subseteq hk(C') \) and is an \( m \)-subalgebra of \( C. \)

Now \( kj(\mathcal{A}) \) \( m \)-generates \( k(C') \) in \( C \) and \( kj(\mathcal{A}) \subseteq h^{-1}(B'), \) hence
\[ h^{-1}(B') \supseteq k(C'), \]
or
\[ h(h^{-1}(B')) \supseteq hk(C'). \]
But
\[ h(h^{-1}(B')) = B', \]
thus
\[ B' \supseteq hk(C'), \]
so
\[ B' = hk(C'). \]

Since \( hkj = i, \)
But a complete isomorphism implies that

\[ \{kj, k(C')\} \leq \{kj, k(C')\} , \]

and since isomorphic elements in \( \mathcal{A}(J, M, m) \) have been identified,

\[ \{i, \mathcal{B}'\} = \{i, \mathcal{C}'\} . \]

**Lemma 2.2.** If \( \bar{J} \leq \sigma \) and \( \bar{M} \leq \sigma \) then there is a \((J, M, m)\)-isomorphism \( i \) of a Boolean algebra \( \mathcal{A} \) into the field \( \mathcal{F} \) of all subsets of a space.

**Proposition 2.6.** If the Boolean algebra \( \mathcal{A} \) is \( m \)-representable but not \( m^+ \)-representable, \( m^+ \) the smallest cardinal greater than \( m \), then \( \mathcal{A}(J, M, m^+) \) does not contain a smallest element if

\[ \mathcal{A}^+(J, M, m^+) \neq \emptyset . \]

If \( J \leq \sigma \) and \( M \leq \sigma \) then \( \mathcal{A}^+(J, M, m^+) \neq \emptyset . \)

**Proof.** Suppose \( \{j, C\} \in \mathcal{A}^+(J, M, m^+) \). Then \( C \) is \( m \)-representable and if an \( m^+ \)-completion \( \{i, \mathcal{B}\} \) of \( \mathcal{A} \) is a smallest element in \( \mathcal{A}(J, M, m^+) \), there is a surjective \( m^+ \)-homomorphism

\[ h: C \rightarrow \mathcal{B} , \]

which implies \( \mathcal{B} \) is \( m^+ \)-representable, hence \( \mathcal{A} \) is \( m^+ \)-representable, a contradiction. Thus \( \mathcal{A}(J, M, m^+) \) does not contain a smallest element if \( \mathcal{A}^+(J, M, m^+) \neq \emptyset . \)

If \( J \leq \sigma \) and \( M \leq \sigma \) then \( \mathcal{A} \) is \((J, M, m^+)\)-representable by Lemma 2.2, hence \( \mathcal{A}^+(J, M, m^+) \neq \emptyset . \)

The next proposition is an easy generalization of Sikorski's [2] Proposition 25.2 and will be needed for the last theorem in this section.

**Proposition 2.7.** A Boolean algebra \( \mathcal{A} \) is completely distributive, if, and only if, it is atomic.

**Corollary 2.3.** A Boolean algebra \( \mathcal{A} \) is completely distributive, if, and only if, \( \mathcal{A} \) is \( m \)-distributive, \( m = \mathcal{A} . \)

The following proposition is due to Sikorski [2] and will be given without proof.

**Proposition 2.8.** If the Boolean algebra \( \mathcal{A} \) is \( m \)-distributive, then \( \mathcal{A}(J, M, m) \) contains a smallest element for arbitrary \( J \) and \( M \).
LEMMA 2.3. If \( \{i, B\} \) is an \( m \)-extension of the Boolean algebra \( A \) and \( B \) is \( m \)-representable, then \( A \) is \( m \)-representable.

Proof. This follows immediately from the fact that \( A \) is \( m \)-regular in \( B \).

Now to prove the main theorem of this section.

THEOREM 2.2. Let \( \mathcal{A} \) be a Boolean algebra. Then the following are equivalent:
1. \( \mathcal{A} \) contains a smallest element for arbitrary \( J, M, \) and \( m \);
2. \( \mathcal{A} \) is \( m \)-representable for all \( m \);
3. \( \mathcal{A} \) is completely distributive;
4. \( \mathcal{A} \) is atomic;
5. an \( m \)-completion of \( \mathcal{A} \) is atomic for all \( m \);
6. an \( m \)-completion of \( \mathcal{A} \) is in \( \mathcal{A}, \mathcal{(J, M, m)} \) for arbitrary \( J, M, \) and \( m \);
7. \( \mathcal{A}(J, M, 2^m) \) contains a smallest element, where \( J = M = \emptyset \) and \( \mathcal{A} = m^* \).

Proof.
1. \( \Rightarrow \) 2. If \( \mathcal{A} \) is \( m \)-representable but not \( m^* \)-representable, then Proposition 2.6 implies \( \mathcal{A}(J, M, m^*) \) does not contain a smallest element if \( \mathcal{J}, \mathcal{M} < \sigma \).

2. \( \Rightarrow \) 3. This follows from the fact that if a Boolean algebra \( \mathcal{A} \) is \( 2^m \)-representable, it is \( m \)-distributive.

3. \( \Rightarrow \) 4. This follows from Proposition 2.7.

3. \( \Rightarrow \) 1. This follows from Proposition 2.8.

4. \( \Rightarrow \) 5. If \( \{i, B\} \) is an \( m \)-completion of \( \mathcal{A} \) then \( i(\mathcal{A}) \) is dense in \( B \), so \( B \) is atomic, and conversely.

2. \( \Rightarrow \) 6. This follows from noting that 2. \( \Rightarrow \) 3. and \( \mathcal{A} \) completely distributive implies an \( m \)-completion of \( \mathcal{A} \) is completely distributive, hence \( m \)-representable for all cardinals \( m \).

6. \( \Rightarrow \) 2. This follows from Lemma 2.3.

3. \( \Rightarrow \) 7. If \( J = M = \emptyset \) and \( \mathcal{A}(J, M, 2^m) \) contains a smallest element, then by Proposition 2.6, \( \mathcal{A} \) is \( 2^m \)-representable, hence \( m^* \)-distributive. Since \( m^* = \mathcal{A} \), \( \mathcal{A} \) is completely distributive, by Corollary 2.3. The converse is clear.
3. The example in § 2 of a Boolean algebra $\mathcal{J}$ such that the class of all $(J, M, m)$-extensions of $\mathcal{J}$ does not contain a smallest element depends on the assumption that $\overline{J}, \overline{M} \leq \sigma$. Thus it is of interest to know whether an example can be found showing that the class of all $m$-extensions of $\mathcal{J}$ does not contain a smallest element, since this corresponds to the case where $J$ and $M$ are as large as possible. As it turns out, there are Boolean algebras $\mathcal{J}$ such that the class of all $m$-extensions $\mathcal{K}$ does not contain a smallest element. In this section such an example will be constructed for each infinite cardinal $m$ and several general types of Boolean algebras such that $\mathcal{K}$ does not contain a smallest element will be given.

Throughout this section $\mathcal{K}$ will denote the class of all $m$-extensions of a Boolean algebra $\mathcal{J}$ and $\mathcal{K}(J, M, m)$ the class of all $(J, M, m)$-extensions.

If $\mathcal{J}$ is a Boolean algebra and $\{i, \mathcal{C}\} \in \mathcal{K}(J, M, m)$, let

$$K(\mathcal{C}) = \{C \in \mathcal{C}: \text{if } i(A) \subseteq C, A \in \mathcal{J}, \text{ then } A = \bigwedge_{\mathcal{J}}\} ,$$

and

$$K_p(\mathcal{C}) = \{C \in \mathcal{C}: \text{if } P = \{A \in \mathcal{J}: i(A) \supseteq C\} \text{ then } \bigcap_{A \in P} A = \bigwedge_{\mathcal{J}}\} .$$

Note that $K_p(\mathcal{C}) \subseteq K(\mathcal{C})$.

**Lemma 3.1.** The set $K_p(\mathcal{C})$ is an ideal and $K(\mathcal{C}) = K_p(\mathcal{C})$, if, and only if, $K(\mathcal{C})$ is an ideal.

**Proof.** It follows easily that $K_p(\mathcal{C})$ is an ideal.

If $K(\mathcal{C})$ is an ideal and $\mathcal{C} \in K(\mathcal{C})$ let

$$P = \{A \in \mathcal{J}: i(A) \supseteq C\} .$$

If $A' \in \mathcal{J}$ and $A' \subseteq A$ for all $A \in P$, then

$$i(A') - C \in K(\mathcal{C}) .$$

Now $i(A') \cap C \in K(\mathcal{C})$, hence

$$i(A') = (i(A') - C) \cup (i(A') \cap C) \in K(\mathcal{C}) ,$$

which implies $i(A') = \bigwedge_{\mathcal{J}}$ or $A' = \bigwedge_{\mathcal{J}}$. Thus

$$\bigcap_{A \in P} A = \bigwedge_{\mathcal{J}} ,$$

so $C \in K_p(\mathcal{C})$, and

$$K_p(\mathcal{C}) = K(\mathcal{C}) .$$

Since $K_p(\mathcal{C})$ is an ideal, the converse is true.
**Proposition 3.1.** If $\mathcal{A}$ is a Boolean algebra the following are equivalent:

1. $\mathcal{K}(J, M, m)$ contains a smallest element;
2. $K(\mathcal{C}) = K_P(\mathcal{C})$ for all $\{i, \mathcal{C}\} \in \mathcal{K}(J, M, m)$;
3. $K(\mathcal{C}) = K_P(\mathcal{C})$ if $\{i, \mathcal{C}\}$ is the maximum element in $\mathcal{K}(J, M, m)$.

**Proof.**

1. $\Rightarrow$ 2. Suppose $\mathcal{K}(J, M, m)$ contains a smallest element $\{i, \mathcal{C}\}$, and there is an element $\{j, \mathcal{E}\} \in \mathcal{K}(J, M, m)$ with the property that

$$K(\mathcal{C}) \neq K_P(\mathcal{C}) .$$

Let $h$ be the unique $m$-homomorphism mapping $\mathcal{C}$ onto $\mathcal{D}$ such that $h_j = i$. Let $\ker h$ be the kernel of this mapping. Then

$$K_P(\mathcal{C}) \subseteq \ker h \subseteq K(\mathcal{C}) ,$$

and

$$\ker h \neq K(\mathcal{C}) .$$

Pick $x \in K(\mathcal{C}) - \ker h$ and let

$$\Delta = \langle x \rangle ,$$

so $\Delta$ is a complete ideal. Thus

$$\{i_j, \mathcal{C}/\Delta\} \in \mathcal{K}(J, M, m) ,$$

where

$$i_\Delta: \mathcal{A} \to \mathcal{C}/\Delta$$

is defined by

$$i_\Delta(A) = [i(A)]_\Delta .$$

Consequently, there are unique homomorphisms $h_\Delta$ and $h'$ mapping $\mathcal{C}$ onto $\mathcal{C}/\Delta$, $\mathcal{C}/\Delta$ onto $\mathcal{D}$, and satisfying $h_\Delta j = i_j$, $h'i_\Delta = i$, respectively. Hence

$$h'h_\Delta j = h'i_\Delta = i$$

and by the uniqueness of $h$,

$$h = h'h_\Delta .$$

This implies

$$h(x) = h'h_\Delta(x) = \bigwedge_{\mathcal{C}} ,$$
a contradiction. Thus

\[ K(\mathcal{C}) = K_p(\mathcal{C}) . \]

2. \(\implies\) 3. Obvious.

3. \(\implies\) 1. To show that \(\mathcal{A}(J, M, m)\) contains a smallest element, let \(\{j, \mathcal{C}\}\) be the largest element in \(\mathcal{A}(J, M, m)\) and suppose \(\{j', \mathcal{C}'\} \in \mathcal{A}(J, M, m)\). Let \(\{i, \mathcal{B}\}\) be an \(m\)-completion of \(\mathcal{A}\). Then there is an \(m\)-homomorphism \(h'\) mapping \(\mathcal{C}\) onto \(\mathcal{C}'\) such that \(h'j = j'\) and an \(m\)-homomorphism \(h\) mapping \(\mathcal{C}\) onto \(\mathcal{B}\) such that \(hj = i\). Thus

\[ K_p(\mathcal{C}) \subseteq \ker h \subseteq K(\mathcal{C}) , \]

which implies, by assumption, that

\[ K_p(\mathcal{C}) = \ker h = K(\mathcal{C}) , \]

so \(K_p(\mathcal{C})\) and \(K(\mathcal{C})\) are \(m\)-ideals in \(\mathcal{C}\). Further,

\[ h'(K_p(\mathcal{C})) \subseteq K_p(\mathcal{C}') \subseteq K(\mathcal{C}') \subseteq h'(K(\mathcal{C})) . \]

This implies that

\[ h'(K_p(\mathcal{C})) = K_p(\mathcal{C}') = K(\mathcal{C}') = h'(K(\mathcal{C})) , \]

hence \(K(\mathcal{C}')\) is an \(m\)-ideal. Let

\[ \Delta = K(\mathcal{C}') . \]

Then \(\mathcal{C}'/\Delta\) is an \(m\)-algebra and

\[ j'_*(\mathcal{A}) = \{[j' (A)]_\Delta : A \in \mathcal{A}\} \]

\(m\)-generates \(\mathcal{C}'/\Delta\). Finally, \(j'_*(\mathcal{A})\) is dense in \(\mathcal{C}'/\Delta\). Thus \(\{j', \mathcal{C}'/\Delta\}\) is an \(m\)-completion of \(\mathcal{A}\), hence is equal to \(\{i, \mathcal{B}\}\), as isomorphic elements of \(\mathcal{A}(J, M, m)\) have been identified. The \(m\)-homomorphism

\[ h_\Delta : \mathcal{C}' \longrightarrow \mathcal{C}'/\Delta \]

defined by

\[ h_\Delta(C') = [C']_\Delta \]

has the property that

\[ h_\Delta j = j' \quad \text{for all} \quad A \in \mathcal{A} , \]

implying that

\[ \{i_\Delta, \mathcal{C}'/\Delta\} \subseteq \{j', \mathcal{C}'\} . \]
Hence \( \mathcal{K}(J, M, m) \) contains a smallest element.

This, then, gives a way to construct a Boolean algebra \( \mathcal{A} \) such that \( \mathcal{K} \) does not contain a smallest element. Namely, by finding a Boolean algebra \( \mathcal{A} \) with an \( m \)-extension \( \{i, \emptyset\} \) such that \( K_p(\emptyset) \neq K(\emptyset) \). The next task is to construct such a Boolean algebra.

If \( \bar{T} = m \) and \( \mathcal{A} = \mathcal{A}_t \) for all \( t \in T \), the Boolean product of \( \{\mathcal{A}_t\}_{t \in T} \) will be called the \( m \)-fold product of \( \mathcal{A} \). Note that if \( \mathcal{A} \) is a subalgebra of the Boolean algebra \( \mathcal{A}' \), \( \mathcal{F} \) is the \( m \)-fold product of \( \mathcal{A} \) and \( \mathcal{F}' \) is the \( m \)-fold product of \( \mathcal{A}' \), then \( \mathcal{F} \subseteq \mathcal{F}' \).

**Lemma 3.2.** If \( \mathcal{A} \) is an \( m \)-regular subalgebra of the Boolean algebra \( \mathcal{A}' \) then the Boolean \( m \)-fold product \( \mathcal{F} \) of \( \mathcal{A} \) is isomorphic to an \( m \)-regular subalgebra of the Boolean \( m \)-fold product \( \mathcal{F}' \) of \( \mathcal{A}' \).

**Proof.** Since \( \mathcal{A} \) is a subalgebra of \( \mathcal{A}' \), \( \mathcal{F} \subseteq \mathcal{F}' \). Let \( \mathcal{I}(\mathcal{F}') \) be the set of all \( \mathcal{P}_t(A) \), \( A \in \mathcal{A} \) and \( t \in T(A \in \mathcal{A}' \) and \( t \in T \). Then \( F \in \mathcal{I}(\mathcal{F}(\mathcal{F}')) \) implies \( -F \in \mathcal{I}(\mathcal{F}(-F \in \mathcal{F}')) \) and \( \mathcal{I}(\mathcal{F}') \) are sets of generators for \( \mathcal{F}(\mathcal{F}') \). For elements \( F \in \mathcal{F}' \) of the form

\[
F = \bigcap_{t=1}^{N} F_t, \quad F_t \in \mathcal{I},
\]

define

\[
\lambda_t(F) = \left\{ \pi_t(x) : x \in \bigcap_{t=1}^{N} F_t \right\}.
\]

Note that if \( F \in \mathcal{F}' \) and \( t \in T \) is such that \( \lambda_t(F) \neq \bigvee_{\mathcal{F}} \), then \( \mathcal{P}_t(\lambda_t(F)) = F \).

In order to show \( \mathcal{F} \) is \( m \)-regular in \( \mathcal{F}' \), it suffices to prove that if \( \{F_t\}_{t \in T} \) is an \( m \)-indexed set of elements of \( \mathcal{F} \) such that

\[
\bigcap_{t \in T} F_t = \Lambda_{\mathcal{F}}
\]

then

\[
\bigcap_{t \in T} F_t = \Lambda_{\mathcal{F}'}.
\]

Now \( F_t \in \mathcal{F} \) so \( F_t \) may be rewritten as

\[
F_t = \bigcap_{p=1}^{P_t} \bigcup_{q=1}^{Q_t} F_{p,q,t},
\]

where \( P_t, Q_t \) are finite numbers and \( F_{p,q,t} \in \mathcal{I} \), for all \( p \in P_t, q \in Q_t, \) and \( t \in T \). Thus
$\Lambda_{\mathcal{S}} = \bigcap_{t \in T} \bigcap_{p=1}^{P_t} \bigcup_{q=1}^{Q_t} F_{p,q,t}$

$$= \bigcap_{s \in S} \bigcup_{q=1}^{Q_s} F_{s,q}$$

after a suitable re-indexing, where $\bar{S} \leq m$ and $F_{s,q} = F_{p,q,t}$ for suitable $p \in P_t, t \in T$. Without loss of generality, assume that for each $s \in S, \lambda_t(F_{s,q}) \neq \Lambda_{\mathcal{S}'}$ implies $\lambda_t(F_{s,q'}) = \bigvee_{\mathcal{S}'}$ for all $t \in T$ and $q' \neq q$, and that $F_{s,q} \neq \bigvee_{\mathcal{S}'}$ for all $q, 1 \leq q \leq Q_s$, and all $s \in S$. Suppose $F'' \in \mathcal{S}'$ and $F'' \subseteq F_t$ for all $t \in T$. Then

$$F'' = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N} F'_{m,n}, \quad F'_{m,n} \in \mathcal{S}'$$

so

$$\bigcap_{n=1}^{N} F'_{m,n} \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for $1 < m \leq M$, and all $s \in S$. Thus to show $F' = \Lambda_{\mathcal{S}'}$, it suffices to prove that if

$$\bigcap_{n=1}^{N} F'_{n} \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}$$

for all $s \in S$, where $F'_{n} \in \mathcal{S}'$, then

$$\bigcap_{n=1}^{N} F'_{n} = \Lambda_{\mathcal{S}'}.$$ 

It may be assumed that for each $n, 1 \leq n \leq N, \lambda_t(F'_{n}) \neq \Lambda_{\mathcal{S}'}$ implies $\lambda_t(F'_{n'}) = \bigvee_{\mathcal{S}'}$ for all $t \in T$ and $n' \neq n$, and that $F'_{n} \neq \bigvee_{\mathcal{S}'}$ for all $n, 1 \leq n \leq N$.

Now

$$\bigcap_{n=1}^{N} F'_{n} \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}$$

implies

$$\bigcap_{n=1}^{N} F'_{n} \cap \bigcup_{q=1}^{Q_s} F_{s,q} = \Lambda_{\mathcal{S}'}$$

and as each $F'_{n}$ and $-F_{s,q}$ is of the form $\varphi_t(A)$ for some $A \in \mathcal{A}'$ and $t \in T$, the independence of the indexed set $\{\varphi_t(A')\}_{t \in T}$ of subalgebras of $\mathcal{S}'$ implies that for some $n_s, 1 \leq n_s \leq N$, and some $q_s, 1 \leq q_s \leq Q_s$, $F'_{n_s} \cap -F_{s,q_s} = \Lambda_{\mathcal{S}'}$,

$$F'_{n_s} \subseteq F_{s,q_s}.$$ 

This argument may be repeated for each $s \in S$. 

The set \( \{n_s: s \in S\} \) is finite so let \( \{n_s: s \in S\} = \{n_i: 1 \leq i \leq N'\} \). Let \( S_i = \{s \in S: F'_{n_i} \subseteq F_{s,q}\} \). If \( t_s \in T \) is such that 
\[
\lambda_{t_s}(F_{s,q}) \neq \bigvee \mathcal{S}, \quad \text{for all } s \in S
\]
then \( \lambda_{t_s}(F_{s,q}) \in \mathcal{S} \) and 
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{S}.
\]
Thus 
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{S},
\]
or 
\[
\bigcap_{s \in S_i} \lambda_{t_s}(F_{s,q}) \neq \bigwedge \mathcal{S},
\]
hence there is an \( A_i \in \mathcal{S}, A_i \neq \bigwedge \mathcal{S}, \) with 
\[
A_i \subseteq \lambda_{t_s}(F_{s,q}) \text{ for all } s \in S_i.
\]
Let \( A_{t,i} \) be the set of all \( x \in X \) such that \( \pi_{t_s}(x) \in A_i \). Thus \( A_{t,i} \in \mathcal{S} \) and this argument may be repeated for each \( i, 1 \leq i \leq N' \). Now 
\[
\bigwedge \mathcal{S} \neq \bigcap_{i=1}^{N'} A_{t,i}
\]
and 
\[
\bigcap_{i=1}^{N'} A_{t,i} \subseteq \bigcup_{q=1}^{Q_s} F_{q,s}
\]
for all \( s \in S \). But then 
\[
\bigcap_{i=1}^{N'} A_{t,i} \subseteq \bigcap_{s \in S} \bigcup_{q=1}^{Q_s} F_{q,s} = \bigwedge \mathcal{S},
\]
a contradiction. Thus \( \mathcal{S} \) is \( m \)-regular in \( \mathcal{S}' \).

The next lemma assumes there is a Boolean algebra \( \mathcal{A} \) such that an \( m \)-extension is not an \( m \)-completion. Sikorski [2] cites an example due to Katetov of such a Boolean algebra for the case \( m = \sigma \). As Lemmas 3.5 and 3.6 imply, there is such an \( \mathcal{A} \) for all infinite cardinal numbers \( m \).

Assume for the moment that \( \mathcal{A} \) is a Boolean algebra such that \( \mathcal{A} \) contains more than one element and \( \{i, B\} \in \mathcal{A} \) is an \( m \)-extension that is not an \( m \)-completion. Thus there is a \( B \in \mathcal{B} \) such that \( i(A) \subseteq B, A \in \mathcal{A} \), implies \( A = \bigwedge \mathcal{A} \). Let \( \mathcal{F}' \) be the Boolean \( m \)-fold product of \( \mathcal{B}, h_0 \) an isomorphism of \( \mathcal{B} \) onto the Stone space \( \mathcal{F} \) of
\( R, X \) the Cartesian product of \( R \) with itself \( m \) times and indexed by \( T \), and
\[
B_t = \mathcal{P}_t h_0(B) \quad \text{for all} \quad t \in T.
\]
Let
\[
B_0 = \bigcup_{t \in T'} B_t,
\]
where \( T' \) is a fixed, but arbitrary subset of \( T \) such that \( \bar{T}' \geq \sigma \), and define
\[
\mathcal{F}_0 = \langle \mathcal{F}', B_0 \rangle.
\]
Since \( \bar{T}' \geq \sigma \), \( \mathcal{F}_0 \neq \mathcal{F}' \).

**Lemma 3.3.** If \( \mathcal{F} \) is the Boolean \( m \)-fold product of \( \mathcal{N} \) then \( \mathcal{F} \) is isomorphic to an \( m \)-regular subalgebra of \( \mathcal{F}_0 \).

**Proof.** It may be assumed, without loss of generality, that \( \mathcal{A} \subseteq \mathcal{B} \). Thus \( \mathcal{F} \subseteq \mathcal{F}_0 \). Let \( \mathcal{I}(\mathcal{F}') \) be a generating set for \( \mathcal{F}(\mathcal{F}') \).

Let
\[
\mathcal{I}_0 = \mathcal{I}' \cup \{B_0\},
\]
so \( \mathcal{I}_0 \) is a generating set for \( \mathcal{F}_0 \). As in the previous lemma, to prove \( \mathcal{F} \) is \( m \)-regular in \( \mathcal{F}_0 \) it suffices to show that if
\[
\bigcap_{n=1}^N F_n' \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}
\]
for all \( s \in S, \bar{S} \leq m \); and
\[
\bigcap_{s \in S} \bigcup_{q=1}^{Q_s} F_{s,q} = \bigwedge \mathcal{S};
\]

\( F_{s,q} \in \mathcal{I} \) for all \( s \in S \) and \( 1 \leq q \leq Q_s, F_n' \in \mathcal{I}_0, 1 \leq n \leq N \); then
\[
\bigcap_{n=1}^N F_n' = \bigwedge \mathcal{S}'.
\]
Since \( F_n' \in \mathcal{I}_0 \), there is an \( n, 1 \leq n \leq N \), such that \( F_n' = B_0 \) or \( F_n' = -B_0 \), otherwise there is nothing to prove. This may be reduced to two cases:

**Case 1.**
\[
\bigcap_{n=1}^N F_n' \cap B_0 \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}
\]
for all \( s \in S \), where \( F_n' \in \mathcal{I}' \) and \( F_{s,q} \in \mathcal{I} \).
Case 2.

\[
(-B_0) \cap \bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}
\]

for all \(s \in S\), where \(F'_n \in \mathcal{F}'\) and \(F_{s,q} \in \mathcal{F}\).

Proof of Case 1. If for each \(s \in S\) there is an \(n_s, 1 \leq n_s \leq N\), such that there is a \(q_s, 1 \leq q_s \leq Q_s\), with \(F'_{n_s} \subseteq F_{s,q_s}\), then

\[
\bigcap_{n=1}^{N} F'_n \subseteq \bigcup_{q=1}^{Q_s} F_{s,q}
\]

for all \(s \in S\), and

\[
\bigcap_{n=1}^{N} F'_n \in \mathcal{F}'
\]

implies

\[
\bigcap_{n=1}^{N} F'_n = \mathbb{1}_{\mathcal{F}'}.
\]

Thus it may be assumed there is an \(s_0\) such that

\[
\bigcap_{n=1}^{N} F'_n \not\subseteq \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q}.
\]

Hence for all \(n\), \(F'_n \not\subseteq F_{s_0,q}\) for some \(q\), is false. If

\[
\bigcap_{n=1}^{N} F'_n \cap B_0 \neq \mathbb{1}_{\mathcal{F}'};
\]

let \(x \in X\) be defined as follows. Let \(t_1, \ldots, t_n \in T\) be such that \(\lambda_{t_i}(F'_i) \neq \mathbb{V}_\mathcal{F}, 1 \leq i \leq N\). Choose an \(x \in X\) such that it satisfies the following conditions:

(a) \( \pi_t(x) \in \{ \lambda_{t_i}(F'_i) \text{ if } \lambda_{t_i}(F_{s_0,q}) = \mathbb{V}_\mathcal{F} \text{ for all } q, 1 \leq q \leq Q_s \}
\)

for \(1 \leq i \leq N\);

(b) \( \pi_t(x) \in -\lambda_{t_i}(F_{s_0,q}) \) for each \(t_q \in T\) such that \(\lambda_{t_q}(F_{s_0,q}) \neq \mathbb{V}_\mathcal{F}, 1 \leq q \leq Q_{s_0}\) and \(t_q \neq t_i, 1 \leq i \leq n\);

(c) \( \pi_t(x) \in h_0(B) \) for all \(t \neq t_q; 1 \leq i \leq N, 1 \leq q \leq Q_{s_0}\).

Now \(x\) is well defined,

\[
x \in B_0 \quad \text{and} \quad x \in \bigcap_{n=1}^{N} F'_n,
\]

by its definition. But \(x \in F_{s_0,q}\) for all \(q, 1 \leq q \leq Q_s\), hence
Proof of Case 2. If

\[-B_0 \cap \bigcap_{n=1}^{N} F'_n \neq \Lambda_{S'},\]

and \(\lambda_{t_n}(F'_n) \neq V_{S'}, t_n \in T\), let \(A_n = \varphi_{t_n}(-B_0), 1 \leq n \leq N\). Then

\[\bigcap_{n=1}^{N} F'_n \cap (-B_0) = \bigcap_{n=1}^{N} (F'_n \cap A_n) \cap (-B_0)\]

and

\[\bigcap_{n=1}^{N} (F'_n \cap A_n) \in \mathcal{F}'.'\]

As before, an \(s_0 \in S\) may be found such that

\[\bigcap_{n=1}^{N} (F'_n \cap A_n) \not\subseteq \bigcup_{q=1}^{Q_{s_0}} F_{s_0,q}.\]

Define \(t_1, \cdots, t_N\) as before so that \(\lambda_{t_i}(F'_i \cap A_i) \neq V_{S'}, 1 \leq i \leq N\). Choose \(x \in X\) satisfying the following conditions:

(a) \(\pi_{t_i}(x) \in \lambda_{t_i}(F'_i \cap A_i)\) if \(\lambda_{t_i}(F_{s_0,q}) = V_{S'}, 1 \leq q \leq Q_{s_0}\); \(\lambda_{t_i}(F'_i \cap A_i) - \lambda_{t_i}(F_{s_0,q})\) if \(\lambda_{t_i}(F_{s_0,q}) \neq V_{S'}\)

for \(1 \leq i \leq N\).

(b) \(\pi_{t_q}(x) \in -\lambda_{t_q}(F_{s_0,q})\) for each \(t_q \in T\) such that \(\lambda_{t_q}(F_{s_0,q}) \neq V_{S'}; 1 \leq q \leq Q_{s_0}\), and \(t_q \neq t_i, 1 \leq i \leq N\).

(c) \(\pi_t(x) \in \lambda_t(-B_0)\) if \(t \neq t_i, t_q; 1 \leq i \leq n, 1 \leq q \leq Q_{s_0}\).

Now \(x\) is well defined and

\[x \in (-B_0) \cap \bigcap_{n=1}^{N} (F'_n \cap A_n) = -B_0 \cap \bigcap_{n=1}^{N} F'_n,\]

so

\[x \not\subseteq \bigcup_{q=1}^{Q_{s_0}} F_{s,q},\]

a contradiction.

Consequently, in either case

\[\bigcap_{n=1}^{N} F'_n = \Lambda_{S'}.\]
**Lemma 3.4.** If $j$ is the identity isomorphism of $\mathcal{F}$ into $\mathcal{F}_0$ and $\{i, \mathcal{C}\}$ is an $m$-completion of $\mathcal{F}_0$, then $\{ij, \mathcal{C}\}$ is an $m$-extension of $\mathcal{F}$.

**Proof.** All that needs to be shown is that $ij(\mathcal{F})$ $m$-generates $\mathcal{C}$. But this follows immediately from the fact that $\mathcal{X} m$-generates $\mathcal{B}$ and the definition of $\mathcal{F}$ and $\mathcal{F}_0$.

**Theorem 3.1.** If $\mathcal{X} m$-generates $\mathcal{B}$ then $\mathcal{X}(\mathcal{F})$ does not contain a smallest element.

**Proof.** $F \in \mathcal{F}$ and $F \supseteq B_0$ then $F = \bigvee F_0$, by definition of $B_0$. Thus if $j$ and $\{i, \mathcal{C}\}$ are defined as in Lemma 3.4, $\{ij, \mathcal{C}\}$ is an $m$-extension of $\mathcal{F}$ and $ij(B_0) \in K(\mathcal{C})$. By Proposition 3.1, $\mathcal{X}(\mathcal{F})$ does not contain a smallest element.

The results of this theorem may be generalized as follows. Let $\{\mathcal{A}_t\}_{t \in T}$ be an infinite indexed set of Boolean algebras and $\{\{i_t\}_{t \in T}, \mathcal{B}\}$ be the Boolean product of $\{\mathcal{A}_t\}_{t \in T}$. Let $T'$ be the set of all $t \in T$ such that $\mathcal{K}(\mathcal{A}_t)$ contains more than one element.

**Theorem 3.2.** The class of $m$-extensions $\mathcal{K}(\mathcal{B})$ does not contain a smallest element if $T' \geq \sigma$.

**Proof.** Define $\mathcal{F}'$ to be the Boolean product of $\{\{j_t, \mathcal{B}_t\}_{t \in T}$, where $\{j_t, \mathcal{B}_t\} \in \mathcal{K}(\mathcal{A}_t)$ for all $t \in T$ and $\{j_t, \mathcal{B}_t\}$ is not an $m$-completion of $\mathcal{A}_t$ for all $t \in T'$. For each $\mathcal{B}_t$, $t \in T'$, there is a $B_t \in \mathcal{B}_t$ such that $j_t(A) \subseteq B_t, A \in \mathcal{A}_t$, implies $A = \bigwedge A_t$. Let $\varphi_t$ map $\mathcal{B}_t$ into $\mathcal{B}$ and set

$$B_0 = \bigcup_{t \in T'} \varphi_t(B_t)$$

and

$$\mathcal{F}_0 = \langle \mathcal{F}', B_0 \rangle .$$

Then by an argument similar to the proofs of Lemmas 3.2, 3.3, and 3.4, and Theorem 3.1, $\mathcal{K}(\mathcal{B})$ does not contain a smallest element.

**Corollary 3.1.** If $\mathcal{A}_t = \mathcal{A}_t'$ for all $t, t' \in T$ then $\mathcal{K}(\mathcal{B})$ contains a smallest element if, and only if, an $m$-extension of $\mathcal{B}$ is an $m$-completion.

**Proof.** If $\mathcal{K}(\mathcal{B})$ contains an $m$-extension which is not an $m$-completion, let $\mathcal{B}$ play the role of $\mathcal{A}$ in Lemmas 3.2, 3.3, and 3.4. By Theorem 3.1, $\mathcal{K}(\mathcal{F})$ does not contain a smallest element. As
the $m$-fold product $\mathcal{F}$ of $\mathcal{B}$ is isomorphic to $\mathcal{B}$, $\mathcal{H}(\mathcal{B})$ does not contain a smallest element. The converse is clear.

Now to prove the assumption on which these results are based.

**Lemma 3.5.** For each infinite cardinal number $m$ there is a Boolean algebra $\mathcal{A}$ such that an $m$-completion $\{i, \mathcal{B}\}$ of $\mathcal{A}$ contains an element $B$ with

$$B \neq \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v},$$

for all $m$-indexed sets $\{A_{u,v}\}_{u \in U, v \in V}$ in $\mathcal{A}$.

**Proof.** The proof will be by constructing such an $\mathcal{A}$ for each $m$. Let $S$ be an indexing set of cardinality $m$. Let $\mathcal{D}_m$ be the Cartesian product of $S$ with itself $m$ times and indexed by $T$. Define

$$D_{t,s} = \{d \in \mathcal{D}_m : \pi_t(d) = s\}.$$

Fix $s', s'' \in S$, $s' \neq s''$, and set $S' = S - \{s', s''\}$. Let $D = \bigcup_{t \in T}(D_{t,s'_1} \cup D_{t,s''_2})$. Thus $D = 2^m$ and $d \in \mathcal{D}_m - D$ implies $\pi_t(d) \neq s_k, k = 1, 2$, for all $t \in T$.

Let

$$\mathcal{I} = \{\{d\} : d \in \mathcal{D}_m\} \cup \{D_{t,s} : t \in T, s \in S'\}.$$

Let $\mathcal{A}$ be generated by $\mathcal{I}$ in $\mathcal{D}_m$ and let $\mathcal{B}$ be the $m$-field of sets $m$-generated by $\mathcal{I}$ in $\mathcal{D}_m$. Then $\mathcal{A}$ is dense in $\mathcal{B}$ and $m$-generates $\mathcal{B}$, so if $i$ is the identity map of $\mathcal{A}$ into $\mathcal{B}$, $\{i, \mathcal{B}\}$ is an $m$-completion of $\mathcal{A}$.

Let

$$B = \mathcal{D}_m - D.$$

Suppose

$$B = \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v},$$

$\{A_{u,v}\}_{u \in U, v \in V}$ an $m$-indexed set in $\mathcal{A}$. This can be written in the form

$$\bigcup_{u \in U} \bigcup_{v \in V} A_{u,v,m} ;$$

$A_{u,v,m}$ or $-A_{u,v,m} \in \mathcal{I}$, $\overline{M_{u,v}} < \sigma$.

Let $B' = \{d \in \mathcal{D}_m : \{d\} = A_{u,v,m}$ for some $u \in U, v \in V, m \in M_{u,v}\}$. Then $\overline{B'} \leq m$, so if

$$M_{u,v}' = \{m \in M_{u,v} : A_{u,v,m}$ is not of the form $\{d\}, d \in \mathcal{D}_m\},$$

it follows that
It will now be shown that in fact

\[ B - \bigcup_{u \in U} \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} \leq m. \]

a contradiction. Hence it may be assumed that \( A_{u,v,m} \) is not of the form \( \{d\} \), \( d \in \mathcal{D}_m \), for all \( u \in U, v \in V \), and \( m \in M_{u,v} \).

If \( A_{u,v,m} = -\{d\}, d \in \mathcal{D}_m \), for some \( m \in M_{u,v} \), then either

(1) \[ \bigcup_{m \in M_{u,v}} A_{u,v,m} = -\{d\} \]

or

(2) \[ \bigcup_{m \in M_{u,v}} A_{u,v,m} = V. \]

If (1) occurs, it may be assumed that \( M_{u,v} = \{1\} \) and \( A_{u,v,1} = -\{d\} \).

If (2) occurs, the term \( \bigcup_{m \in M_{u,v}} A_{u,v,m} \) may be dropped. Thus for all \( u \in U, V \) may be written as \( V_u \cup V'_v \), where (1) \( V_u \cap V'_v = \emptyset \); (2) \( A_{u,v,m} = -\{d_{u,v}\}, d_{u,v} \in \mathcal{D}_m \), for all \( v \in V_u \); and (3) \( A_{u,v,m} \) is either of the form \( -D_{t,s} \) or \( D_{t,s} \) for all \( v \in V'_v \). Consequently, for all \( u \in U, V \)

\[ \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m} = \bigcap_{v \in V_u} -\{d_{u,v}\} \cap \bigcap_{v \in V'_v} \bigcup_{m \in M_{u,v}} A_{u,v,m}. \]

Let

\[ C_u = \bigcap_{v \in V} \bigcup_{m \in M_{u,v}} A_{u,v,m}. \]

Suppose \( U \) is the set of all ordinals \( u < \alpha \), where \( \alpha = \bar{U} \). Let \( D_1 = \{d \in \mathcal{D}_u : \pi_1(d) = s', s'_2\} \). Now \( \bar{D}_1 = 2^m \) implies there is a \( d_1 \in D_1 \) such that

\[ d_1 \in \bigcap_{v \in V_1} -\{d_{1,v}\}. \]

Since \( d_1 \notin B \), this implies

\[ d_1 \in \bigcap_{v \in V'_1} \bigcup_{m \in M_{1,v}} A_{1,v,m}. \]

hence for some \( v_1 \in V'_1 \),

\[ d_1 \in \bigcup_{m \in M_{1,v_1}} A_{1,v_1,m}. \]

Also, \( D_1 \subseteq -D_{t,s} \) for all \( t \in T \) and \( s \in S' \), hence

\[ A_{1,v_1,m} = D_{t_{1,m}, s_{1,m}} \]

for some \( t_{1,m} \in T \) and \( s_{t_{1,m}} \in S' \), for all \( m \in M_{1,v_1} \). Let \( T_1 = \{t_{1,m} : m \in M_{1,v_1}\} \).
and pick $s_i \in S'$ such that $s_i \neq s_{i,m}$ for all $m \in M_{i\!,\!v_i}$. Define

$$\varphi(t) = s_1$$

for all $t \in T_i$. Let $B_1 = \emptyset$ and define $B_2 = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in T_i\}$.

Note that $B_2 \cap C_i = \emptyset$.

Suppose $i > 1$ and a finite set $T_i$ has been defined for each $i' < i$ so that $T_i \cap T_{i'} = \emptyset$ if $i', i'' < i, i' \neq i''$; $s_i \in S'$ has been chosen; $\varphi$ has been defined on each $T_{i'}$, $i' < i$, so that $\varphi(t) = s_{i'}$ for all $t \in T_{i'}$; and if

$$B_i = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in \bigcup_{i' < i} T_{i'}\}$$

then

$$B_i \cap \bigcup_{i' < i} C_{i'} = \emptyset.$$

Let

$$\hat{T}_i = \bigcup_{i' < i} T_{i'}$$

and note that $\hat{T}_i < m$. Let

$$D_i = \{d \in \mathcal{D}_m : \pi_i(d) = \varphi(t) \text{ for all } t \in \hat{T}_i \text{ and } \pi_i(d) = s_{i', k = 1, 2, \text{ if } t \in T - \hat{T}_i}\}.$$ 

Then $D_i \subseteq D$ and $D_i = 2^m$, hence there is a $d_i \in D_i$ such that

$$d_i \in \bigcap_{v \in V_i} \left\{d \in D_i \mid d \notin \{d_{i,v}\}\right\}.$$ 

Since $d_i \notin B$, this implies

$$d_i \notin \bigcap_{v \in V_i} \bigcup_{m \in M_{i\!,\!v_i}} A_{i,v,m},$$

hence for some $v_i \in V_i$,

$$d_i \notin \bigcup_{m \in M_{i\!,\!v_i}} A_{i,v_i,m}.$$

If $B_i \cap C_i = \emptyset$ set $T_i = \emptyset$. If not, there is a $d'_i \in B_i$ such that $d'_i \in C_i$, so

$$d'_i \in \bigcup_{m \in M_{i\!,\!v_i}} A_{i,v_i,m}.$$

Note that $\pi_i(d'_i) = \pi_i(d_i)$ for all $t \in \hat{T}_i$.

It immediately follows that if
\[ d_i \in \bigcup_{m \in M_{i,v_i}} A_{i,v_i,m} \]

then

\[ A_{i,v_i,m} = D_{i,v_i,m} s_{i,m}, \]

where \( t_{i,m} \in \hat{T}_i \) and

\[ \pi_{t_{i,m}}(d_i) = s_{i,m}, \]

for some \( m \in M_{i,v_i} \).

Let

\[ T_i = \{ t_{i,m} \in T - \hat{T}_i : A_{i,v_i,m} = D_{i,v_i,m} s_{i,m} \text{ for some } m \in M_{i,v_i} \} \]

and pick \( s_i \in S' \) such that if \( t_{i,m} \in T_i \) then

\[ s_i \neq S_{i,m}, \]

for all \( m \in M_{i,v_i} \). Now define

\[ \varphi(t) = s_i \text{ for all } t \in T_i. \]

Thus \( T_i \cap \hat{T}_i = \emptyset \) which implies \( T_i \cap T_{i'} = \emptyset \) for all \( i' < i \). If

\[ B_{i+1} = \{ d \in \mathcal{D}_m : \pi_t(d) = \varphi(t) \text{ for all } t \in T_i \cup \hat{T}_i \} \]

then it is clear that

\[ B_{i+1} \cap \bigcup_{i' < i} C_i = \emptyset. \]

Now let \( \hat{T} = \bigcup_{i < \alpha} T_i \) and set

\[ \hat{B} = \{ d \in \mathcal{D}_m : \pi_t(d) = \varphi(t) \text{ for all } t \in \hat{T} \}
\]

and \( \pi_t(d) \neq s', s'' \text{ if } t \in T - \hat{T} \} \).

Then \( \hat{B} \neq \emptyset \) and \( \hat{B} \subseteq B \). But \( \hat{B} \cap \bigcup_{u \in U} C_u = \emptyset \) which implies

\[ B - \bigcup_{u \in U} C_u \neq \emptyset. \]

If \( B' = B - \bigcup_{u \in U} C_u \) then for each \( b \in B' \),

\[ b = \bigcap_{t \in T} D_{t,s_{t,b}}, \]

for some \( m \)-indexed set \( \{ s_{t,b} \}_{t \in T} \) in \( S' \). Thus

\[ B = \bigcup_{u \in U} \bigcap_{v \in Y} \bigcup_{m \in M_{u,v}} A_{u,v,m} \cup \bigcup_{b \in B'} \bigcap_{t \in T} D_{t,s_{t,b}}, \]

but the above construction shows that
\[ B - \left( \bigcup_{u \in U} \bigcap_{v \in V} A_{u,v,m} \bigcup_{b \in B'} \bigcap_{t \in T} D_{i(t,b)} \right) \neq \emptyset \]

if \( B' \leq m \). Hence

\[ B - \bigcup_{u \in U} C_u > m . \]

**Lemma 3.6.** If \( \{i, B\} \) is an \( m \)-completion of the Boolean algebra \( \mathcal{A} \) and there is a \( B \in \mathcal{B} \) such that

\[ B \neq \bigcup_{t \in T} \bigcap_{s \in S} i(A_{t,s}) \]

for all \( m \)-indexed sets \( \{A_{t,s}\}_{t \in T, s \in S} \) in \( \mathcal{A} \), then there is an \( m \)-ideal \( \Delta \) in \( \mathcal{B} \) such that \( \{j, \mathcal{B}_\Delta\} \) is an \( m \)-extension of \( i_\Delta(\mathcal{A}) \) but not an \( m \)-completion, where \( i_\Delta(A) = [i(A)]_\Delta \) for all \( A \in \mathcal{A} \), \( \mathcal{B}_\Delta = \mathcal{B} \setminus \Delta \) and \( j \) is the identity map of \( i_\Delta(\mathcal{A}) \) into \( \mathcal{B}_\Delta \).

**Proof.** Let

\[ \Delta' = \{B' \in \mathcal{B} : B' \subseteq B \text{ and } B' = \bigcap_{t \in T} i(A_t) \} , \]

for some \( m \)-indexed set \( \{A_t\}_{t \in T} \) in \( \mathcal{A} \) and let \( \Delta = \langle \Delta' \rangle_m \). Then if \( \delta \in \Delta, \delta \subseteq B \), so \( B \in \Delta \). If \( A \in \mathcal{A} \) and \([i(A)]_\Delta \subseteq [B]_\Delta \) then \( i(A) - B \in \Delta \) so \( i(A) - B \subseteq B \) which implies \( i(A) \subseteq B \), hence \( i(A) \in \Delta \) and \([i(A)]_\Delta = \bigwedge_{\mathcal{A}} j_\Delta \), implying \( i_\Delta(\mathcal{A}) \) is not dense in \( \mathcal{B}_\Delta \).

It only remains to show that \( i_\Delta(\mathcal{A}) \) is \( m \)-regular in \( \mathcal{B}_\Delta \). If

\[ \bigcap_{t \in T} [i(A_t)]_\Delta = \bigwedge_{\mathcal{A}} j_\Delta \]

then \( i(A) \subseteq i(A_t) \) for all \( t \in T \) implies \( i(A) \in \Delta \), so \( i(A) \subseteq B \). If

\[ \bigcap_{t \in T} i(A_t) \not\subseteq B , \]

then there is an \( A \neq \bigwedge_{\mathcal{A}} j_\Delta \) in \( \mathcal{A} \) such that

\[ i(A) \subseteq \bigcap_{t \in T} i(A_t) - B , \]

contradicting the above statement. Thus

\[ \bigcap_{t \in T} i(A_t) \subseteq B \]

so

\[ \bigcap_{t \in T} i(A_t) \in \Delta \]
and

$$\Lambda_i = \left[ \bigcap_{i \in T} i(A_i) \right] = \bigcap_{i \in T} [i(A_i)]_i.$$

Thus if \( \mathcal{A} \) is the Boolean algebra constructed in Lemma 3.5, \( i_\mathcal{A}(\mathcal{A}) \) is a Boolean algebra such that \( \mathcal{A}(i_\mathcal{A}(\mathcal{A})) \) contains more than one element. Hence it is justified to assume that for each infinite cardinal \( m \) there is a Boolean algebra \( \mathcal{A} \) such that \( \mathcal{A} \) has an \( m \)-extension which is not an \( m \)-completion.

4. Let \( \{\mathcal{A}_t\}_{t \in T} \) be a (fixed) indexed set of Boolean algebras. Let \( h_t \) be an isomorphism of \( \mathcal{A}_t \) onto the field \( \mathcal{F}_t \) of all open-closed subsets of the Stone space \( X_t \) of \( \mathcal{A}_t \). Let \( X \) denote the Cartesian product of all the spaces \( X_t \). Let \( \pi_t \) be the projection of \( X \) onto \( \mathcal{F}_t \) and define

$$\varphi_t : \mathcal{F}_t \rightarrow X$$

by:

$$\text{if } F \in \mathcal{F}_t \text{ then } \varphi_t(F) = \{ x \in X : \pi_t(x) \in F \}.$$

Let \( \mathcal{F} \) be the Boolean product of \( \{\mathcal{A}_t\}_{t \in T} \). Define \( h^*_t = \varphi_t h_t \) and let \( \mathcal{S} \) be the set of all sets \( \bigcap_{t \in T} h^*_t(A_t) ; A_t \in \mathcal{A}_t , T' \subseteq T , \mathcal{T}' \leq n \). Define \( \mathcal{H} \) to be the field of sets generated by \( \mathcal{S} \). Let \( J \) be the set of all sets \( S \subseteq \mathcal{H} \) such that

1. \( \bar{S} \leq m ; \)
2. there is a \( t \in T \) such that \( S \subseteq h^*_t(\mathcal{A}_t) ; \)
3. the join \( \bigcup_{\mathcal{A}_t} A \) exists.

Let \( M' \) be the set of all sets \( S \subseteq \mathcal{T} \) such that

1. \( \bar{S} \leq m ; \)
2. there is a \( t \in T \) such that \( S \subseteq h^*_t(\mathcal{A}_t) ; \)
3. the meet \( \bigcap_{\mathcal{A}_t} A \) exists.

Let \( M'' \) be the set of all sets \( S \subseteq \mathcal{T} \) such that

1. \( \bar{S} \leq n ; \)
2. if \( A \in S \) then \( A \in h^*_t(\mathcal{A}_t) \) for some \( t \in T \);
3. if \( A, B \in S, A \neq B \), then \( A \in h^*_t(\mathcal{A}_t) \) implies \( B \in h^*_t(\mathcal{A}_t) \). Let \( M = M' \cup M'' \).

The following lemma is due to La Grange [1] and will be given without proof.

**Lemma 4.1.** If \( \{i_t\}_{t \in T} \in P \), then there is one and only one \((J, M, m)\)-isomorphism \( h \) mapping \( \mathcal{F} \) into \( \mathcal{B} \) such that

$$hh^*_t = i_t \text{ for all } t \in T .$$
THEOREM 4.1. If \{\{i_t\}_{t \in T}, B\} \in \mathcal{P}_n then there is a mapping \( h \) of \( \hat{\mathcal{F}} \) into \( B \) such that \( \{h, B\} \) is a \((J, M, m)\)-extension of \( \hat{\mathcal{F}} \). If \( \{h, B\} \) is a \((J, M, m)\)-extension of \( \hat{\mathcal{F}} \) then the ordered pair \( \{\{h h_t^*\}_{t \in T}, B\} \in \mathcal{P}_n \).

Proof. Let \( h \) be the \((J, M, m)\)-isomorphism from \( \hat{\mathcal{F}} \) into \( B \) such that \( h h_t^* = i_t \) for all \( t \in T \). Then \( \{h, B\} \) is a \((J, M, m)\)-extension of \( \hat{\mathcal{F}} \).

Conversely, if \( \{h, B\} \) is a \((J, M, m)\)-extension of \( \hat{\mathcal{F}} \), it follows immediately that \( \{\{h h_t^*\}_{t \in T}, B\} \) is an \((m, n)\)-product of \( \{\mathcal{A}_t\}_{t \in T} \).

THEOREM 4.2. If \( \{\{i_t\}_{t \in T}, B\}, \{\{i'_t\}_{t \in T}, B'\} \) are two \((m, n)\)-products of \( \{\mathcal{A}_t\}_{t \in T} \) then

\[ \{\{i_t\}_{t \in T}, B\} \leq \{\{i'_t\}_{t \in T}, B'\} \]

if, and only if,

\[ \{i, B\} \leq \{i', B'\} \]

where \( \{i, B\} \) and \( \{i', B'\} \) are the \((J, M, m)\)-extensions of \( \hat{\mathcal{F}} \) induced by the \((J, M, m)\)-isomorphisms \( i' \) and \( i \) of \( \hat{\mathcal{F}} \) into \( B' \) and \( B \), respectively, given by Lemma 4.1.

Proof. Now

\[ \{\{i_t\}_{t \in T}, B\} \leq \{\{i'_t\}_{t \in T}, B'\} \]

if, and only if, there is an \( m \)-homomorphism \( h \) such that

\[ h: B' \rightarrow B \]

and \( h i'_t = i_t \) for all \( t \in T \). Similarly,

\[ \{i, B\} \leq \{i', B'\} \]

if, and only if, there is an \( m \)-homomorphism

\[ h: B' \rightarrow B \]

such that \( h' i' = i \). Thus it suffices to show that \( h i' = i \), if, and only if, \( h i'_t = i_t \). Let \( h_t^* \) be defined as above. Then \( i h_t^* = i_t \) and \( i' h_t^* = i'_t \), so if \( h i' = i \),

\[ h i'_t = h i' h_t^* = i h_t^* = i_t , \]

and if \( h i'_t = i_t \), then

\[ h i' = h i'_t h_t^{* -1} = i_t h_t^{* -1} = i . \]
La Grange [1] has given an example of an \((m, 0)\)-product for which \(\mathcal{I}\) does not contain a smallest element and an example of an \((m, n)\)-product for which \(\mathcal{I}_n\) does not contain a smallest element. Theorem 4.2 extends this result by showing that the question whether \(\mathcal{I}\) or \(\mathcal{I}_n\) contains a smallest element reduces to asking whether the class of all \((J, M, m)\)-extensions of \(\mathcal{A}_0\) or \(\mathcal{F}\) contains a smallest element for \(J\) and \(M\) defined appropriately in each case, where \(\mathcal{A}_0\) and \(\mathcal{F}\) are defined as above. Now the class of all \((J, M, m)\)-extensions of \(\mathcal{A}_0\) contains a smallest element only if the class of all \(m\)-extensions of \(\mathcal{A}\) contains a smallest element and Theorem 3.2 shows that the class of all \(m\)-extensions of \(\mathcal{A}\) need not contain a smallest element, which implies the same is true for \(\mathcal{I}\). Since Theorem 3.2 may be extended to Boolean algebras of the form \(\mathcal{F}\), it follows that \(\mathcal{I}_n\) need not contain a smallest element.

**References**


Received November 27, 1972 and in revised form March 28, 1973.

University of California, Los Angeles
Robert Lee Anderson, *Continuous spectra of a singular symmetric differential operator on a Hilbert space of vector-valued functions* .......................................................... 1
Michael James Cambern, *The isometries of $L^p (X, K)$* ................................. 9
R. H. Cameron and David Arne Storvick, *Two related integrals over spaces of continuous functions* ................................................................. 19
Gary Theodore Chartrand and Albert David Polimeni, *Ramsey theory and chromatic numbers* ................................................................. 39
John Deryck De Pree and Harry Scott Klein, *Characterization of collectively compact sets of linear operators* ........................................... 45
John Deryck De Pree and Harry Scott Klein, *Semi-groups and collectively compact sets of linear operators* ........................................... 55
George Epstein and Alfred Horn, *Chain based lattices* ........................................ 65
Paul Erdős and Ernst Gabor Straus, *On the irrationality of certain series* .......... 85
Zdeněk Frolík, *Measurable uniform spaces* ...................................................... 93
Stephen Michael Gagola, Jr., *Characters fully ramified over a normal subgroup* ........................................................................................................... 107
Frank Larkin Gilfeather, *Operator valued roots of abelian analytic functions* ........................................................................................................... 127
D. S. Goel, A. S. B. Holland, Cyril Nasim and B. N. Sahney, *Best approximation by a saturation class of polynomial operators* .................... 149
James Secord Howland, *Puiseux series for resonances at an embedded eigenvalue* ......................................................................................... 157
David Jacobson, *Linear GCD equations* ............................................................. 177
P. H. Karvellas, *A note on compact semirings which are multiplicative semilattices* ............................................................................................. 195
Allan Morton Krall, *Stieltjes differential-boundary operators. II* ...................... 207
D. G. Larman, *On the inner aperture and intersections of convex sets* ............. 219
S. N. Mukhopadhyay, *On the regularity of the $P^n$-integral and its application to summable trigonometric series* ...................................... 233
Dwight Webster Read, *On $(J, M, m)$-extensions of Boolean algebras* .......... 249
David Francis Rearick, *Multiplicativity-preserving arithmetic power series* ........................................................................................................... 277
Indranand Sinha, *Characteristic ideals in group algebras* ................................. 285
Charles Thomas Tucker, II, *Homomorphisms of Riesz spaces* ......................... 289
Kunio Yamagata, *The exchange property and direct sums of indecomposable injective modules* ............................................................................ 301