

Pacific Journal of Mathematics

CHARACTERISTIC IDEALS IN GROUP ALGEBRAS

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I. SINHA

If $\mathfrak{F}G$ is the group-algebra of a group G over a field \mathfrak{F} , and \mathfrak{A} is any subgroup of the automorphism group of the \mathfrak{F} -algebra $\mathfrak{F}G$, then an ideal I of $\mathfrak{F}G$, is called \mathfrak{A} -characteristic if $I^\alpha \subseteq I, \forall \alpha \in \mathfrak{A}$. If \mathfrak{A} is the whole automorphism group itself, then we merely say that I is characteristic. Then D.S. Passman has proved the following result:

“Let $H \trianglelefteq G$ such that G/H is \mathfrak{F} -complete. Then for each characteristic ideal I of $\mathfrak{F}G, I = (I \cap \mathfrak{F}H)\mathfrak{F}G$.”

The main concern in this paper is to consider the converse of this result.

2. Some preliminaries. For a given ideal $I \trianglelefteq \mathfrak{F}G$, let $\mathcal{R}(I)$ be the set of all $H \leq G$ such that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$. Let $C(I)$ be the set of all H in G such that if for some right $\mathfrak{F}H$ -module $\mathfrak{M}, I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$, then $I \subseteq \text{Ann } \mathfrak{M}^G$, the induced $\mathfrak{F}G$ -module. We first of all have:

THEOREM 1. (i) For any $I \trianglelefteq \mathfrak{F}G, C(I) \subseteq \mathcal{R}(I)$.
 (ii) If $H \trianglelefteq G$, then $H \in \mathcal{R}(I)$ if and only if $H \in C(I)$.

Proof. (i) Let $I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$ imply that $I \subseteq \text{Ann } \mathfrak{M}^G$. Let $\sum p_i x_i \in I$ with $p_i \in \mathfrak{F}H$, where $G = \cup Hx_i$ is a coset-decomposition. We have $(\sum \mathfrak{M} \otimes x_i)(\sum p_i x_i) = 0$ if $I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$. In particular $(m \otimes I)(\sum p_i x_i) = 0, \forall m \in \mathfrak{M}$, i.e., $\sum m p_i \otimes x_i = 0, \forall m \in \mathfrak{M}$. So $\mathfrak{M} \cdot p_i = 0$ for each i . Thus $p_i \in \text{Ann } \mathfrak{M}$. Since \mathfrak{M} is arbitrary with the property that $I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$, so we may take $\mathfrak{M} = \mathfrak{F}H/I \cap \mathfrak{F}H$, and conclude that each $p_i \in \text{Ann } \mathfrak{M} = I \cap \mathfrak{F}H$. Thus $\sum p_i x_i \in (I \cap \mathfrak{F}H)\mathfrak{F}G$.

(ii) Suppose $I = \mathfrak{F}G(I \cap \mathfrak{F}H)$ and $I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$, for some $\mathfrak{F}H$ -module \mathfrak{M} . Note that $H \trianglelefteq G$ implies that $\mathfrak{F}G(I \cap \mathfrak{F}H) = (I \cap \mathfrak{F}H)\mathfrak{F}G$. Let $a = \sum x_i p_i \in I$ where $p_i \in I \cap \mathfrak{F}H$. So $a \mathfrak{M}^G = (\sum x_i p_i)(\sum x_j \otimes \mathfrak{M}) = \sum x_i x_j \otimes p_i^j \mathfrak{M} = 0$ since $p_i^j \in I \cap \mathfrak{F}H \subseteq \text{Ann } \mathfrak{M}$. Thus $a \mathfrak{M}^G = 0$ and $I \subseteq \text{Ann } \mathfrak{M}^G$.

Theorem 17.4 of [1] then gives us:

COROLLARY 1. Let $H \trianglelefteq G$ such that G/H is \mathfrak{F} -complete. Then $H \in C(I)$ for every characteristic ideal I of $\mathfrak{F}G$.

Also Theorem 17.7 of [1] implies:

COROLLARY 2. If $H \trianglelefteq G \ni G/H$ is abelian and has no elements of order $p = \text{Char. } \mathfrak{F}$, then $H \in C(J(G))$, where J denotes the

Jacobson-radical of $\mathfrak{F}G$.

3. Main result. We will prove:

THEOREM 2. *For $I = [\mathfrak{F}G, \mathfrak{F}G]$, the commutator ideal and for $J = J(G)$, if $H \leq G$ such that $H \in \mathcal{R}(I)$ and $H \in \mathcal{R}(J)$ then $H \trianglelefteq G$, G/H is abelian with no elements of order p . In particular, $\mathfrak{F}(G/H)$ is semi-simple.*

Further, if \mathfrak{F} is algebraically closed then G/H is \mathfrak{F} -complete.

We observe that the last two statements in the theorem follows from 17.8 and 17.1 (i) respectively of [1]. The rest of the theorem will be proved by a series of results proved below.

LEMMA 1. *Let $H \leq G$, $I \trianglelefteq \mathfrak{F}G$ and $H \in \mathcal{R}(I)$. Then $H \supseteq \mathfrak{X}^{-1}(I) = \{g \in G \mid g - 1 \in I\}$.*

Proof. Let $G = \cup Hx_i$ be a coset-decomposition, and $g \in \mathfrak{X}^{-1}(I)$ such that $g \notin H$. Then $g = hx_i$ for some i , where $x_i \neq 1$, and $h \in H$; and $hx_i - 1 \in (I \cap \mathfrak{F}H)\mathfrak{F}G = \sum (I \cap \mathfrak{F}H)x_i$. Since $\{x_i\}$ are linearly independent over $\mathfrak{F}H$, $h \in I \cap \mathfrak{F}H$, and $x_i \neq 1$, so $g \in I$ which implies that $1 \in I$, a contradiction.

LEMMA 2. *If $I = [\mathfrak{F}G, \mathfrak{F}G]$, and $H \in \mathcal{R}(I)$ then $H \trianglelefteq G$ and G/H is abelian.*

Proof. Observe that I is a proper ideal in $\mathfrak{F}G$, since $\mathfrak{X}(I) = 0$. Also by Lemma 1, $H \supseteq \mathfrak{X}^{-1}(I)$. Since $(ghg^{-1}h^{-1} - 1)hg = gh - hg \in I$, for all $g, h \in G$, so $(ghg^{-1}h^{-1} - 1) \in I$. Hence $ghg^{-1}h^{-1} \in \mathfrak{X}^{-1}(I) \subseteq H$; i.e., G' , the commutator-subgroup is in H . Hence $H \trianglelefteq G$ and G/H is abelian.

Now let H satisfy the hypothesis of Lemma 2. Then we have:

LEMMA 3. *Let $I = J(G)$ and $H \in \mathcal{R}(I)$. Then $\mathfrak{F}(G/H)$ is semi-simple and G/H has no elements of order $p = \text{Char. } \mathfrak{F}$.*

Proof. $J(G) = (J(G) \cap \mathfrak{F}H)\mathfrak{F}G \cong J(H) \cdot \mathfrak{F}G$ by 16.9 of [1]. Now $\mathfrak{F}H[\mathfrak{X}_H(H) \cong \mathfrak{F}$ where $\mathfrak{X}_H(H)$ is the ideal of $\mathfrak{F}H$, generated by $\{h - 1 \mid h \in H\}$. So $\mathfrak{X}_H(H) \supseteq J(H)$. Hence $\mathfrak{X}_H(H)\mathfrak{F}G = \mathfrak{X}_G(H) \cong J(H) \cdot \mathfrak{F}G \supseteq J(G)$, where $\mathfrak{X}_G(H)$ is the ideal in $\mathfrak{F}G$, generated by $\{h - 1 \mid h \in H\}$. Now $\mathfrak{X}_G(H)$ is the kernel of the natural map of $\mathfrak{F}G$ onto $\mathfrak{F}(G/H)$; {see for example proof of Theorem 1 in [2]}. Thus $\mathfrak{F}(G/H) \cong \mathfrak{F}G/\mathfrak{X}_G(H)$ is semi-simple. Since G/H is abelian by Lemma

2, so it is clear that it has no elements of order p , as $\mathfrak{F}(G/H)$ is semi-simple.

This also completes the proof of Theorem 2.

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