CHARACTERISTIC IDEALS IN GROUP ALGEBRAS

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If $\mathbb{G}$ is the group-algebra of a group $G$ over a field $\mathbb{F}$, and $A$ is any subgroup of the automorphism group of the $\mathbb{F}$-algebra $\mathbb{G}$, then an ideal $I$ of $\mathbb{G}$, is called $A$-characteristic if $I^a \subseteq I$, $\forall a \in A$. If $A$ is the whole automorphism group itself, then we merely say that $I$ is characteristic. Then D.S. Passman has proved the following result:

"Let $H \triangleleft G$ such that $G/H$ is $\mathbb{F}$-complete. Then for each characteristic ideal $I$ of $\mathbb{G}$, $I = (I \cap \mathbb{G})\mathbb{G}$." The main concern in this paper is to consider the converse of this result.

2. Some preliminaries. For a given ideal $I \triangleleft \mathbb{G}$, let $\mathcal{R}(I)$ be the set of all $H \triangleleft G$ such that $I = (I \cap \mathbb{G})\mathbb{G}$. Let $C(I)$ be the set of all $H$ in $G$ such that if for some right $\mathbb{F}H$-module $\mathcal{M}$, $I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$, then $I \subseteq \text{Ann} \mathcal{M}^\alpha$, the induced $\mathbb{G}$-module. We first of all have:

**Theorem 1.** (i) For any $I \triangleleft \mathbb{G}$, $C(I) \subseteq \mathcal{R}(I)$.
(ii) If $H \triangleleft G$, then $H \in \mathcal{R}(I)$ if and only if $H \in C(I)$.

**Proof.** (i) Let $I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$ imply that $I \subseteq \text{Ann} \mathcal{M}^\alpha$. Let $\sum p_i x_i \in I$ with $p_i \in \mathbb{G}H$, where $G = \bigcup Hx_i$ is a coset-decomposition. We have $(\sum \mathcal{M} \otimes x_i)(\sum p_i x_i) = 0$ if $I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$. In particular $(m \otimes I)(\sum p_i x_i) = 0$, $\forall m \in \mathcal{M}$, i.e., $\sum m p_i \otimes x_i = 0$, $\forall m \in \mathcal{M}$. So $\mathcal{M} \cdot p_i = 0$ for each $i$. Thus $p_i \in \text{Ann} \mathcal{M}$. Since $\mathcal{M}$ is arbitrary with the property that $I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$, so we may take $\mathcal{M} = \mathbb{G}H/I \cap \mathbb{G}H$, and conclude that each $p_i \in \text{Ann} \mathcal{M} = I \cap \mathbb{G}H$. Thus $\sum p_i x_i \in (I \cap \mathbb{G})\mathbb{G}$.

(ii) Suppose $I = \mathbb{G}(I \cap \mathbb{G})$ and $I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$, for some $\mathbb{F}H$-module $\mathcal{M}$. Note that $H \triangleleft G$ implies that $\mathbb{G}(I \cap \mathbb{G}) = (I \cap \mathbb{G})\mathbb{G}$. Let $a = \sum x_i p_i \in I$ where $p_i \in I \cap \mathbb{G}H$. So $a \mathcal{M}^\alpha = (\sum x_i p_i)(\sum x_j \otimes \mathcal{M}) = \sum x_i x_j \otimes p_i \mathcal{M} = 0$ since $p_i \mathcal{M} \subseteq I \cap \mathbb{G} \subseteq \text{Ann} \mathcal{M}$. Thus $a \mathcal{M}^\alpha = 0$ and $I \subseteq \text{Ann} \mathcal{M}^\alpha$.

Theorem 17.4 of [1] then gives us:

**Corollary 1.** Let $H \triangleleft G$ such that $G/H$ is $\mathbb{F}$-complete. Then $H \in C(I)$ for every characteristic ideal $I$ of $\mathbb{G}$.

Also Theorem 17.7 of [1] implies:

**Corollary 2.** If $H \triangleleft G \triangleright G/H$ is abelian and has no elements of order $p = \text{Char. } \mathbb{F}$, then $H \in C(J(G))$, where $J$ denotes the
Jacobson-radical of \( \mathfrak{g} G \).

3. Main result. We will prove:

**Theorem 2.** For \( I = [\mathfrak{g} G, \mathfrak{g} G] \), the commutator ideal and for \( J = J(G) \), if \( H \leq G \) such that \( H \in \mathcal{R}(I) \) and \( H \in \mathcal{R}(J) \) then \( H \leq G \), \( G/H \) is abelian with no elements of order \( p \). In particular, \( \mathfrak{g}(G/H) \) is semi-simple.

Further, if \( \mathfrak{g} \) is algebraically closed then \( G/H \) is \( \mathfrak{g} \)-complete.

We observe that the last two statements in the theorem follows from 17.8 and 17.1 (i) respectively of [1]. The rest of the theorem will be proved by a series of results proved below.

**Lemma 1.** Let \( H \leq G \), \( I \leq \mathfrak{g} G \) and \( H \in \mathcal{R}(I) \). Then \( H \supseteq \mathfrak{a}^{-1}(I) = \{ g \in G \mid g - 1 \in I \} \).

**Proof.** Let \( G = \bigcup Hx_i \) be a coset-decomposition, and \( g \in \mathfrak{a}^{-1}(I) \) such that \( g \notin H \). Then \( g = hx_i \) for some \( i \), where \( x_i \neq 1 \), and \( h \in H \); and \( hx_i - 1 \in (I \cap \mathfrak{g} H)\mathfrak{g} G = \sum (I \cap \mathfrak{g} H)x_i \). Since \( \{ x_i \} \) are linearly independent over \( \mathfrak{g} H \), \( h \in I \cap \mathfrak{g} H \), and \( x_i \neq 1 \), so \( g \in I \) which implies that \( 1 \in I \), a contradiction.

**Lemma 2.** If \( I = [\mathfrak{g} G, \mathfrak{g} G] \), and \( H \in \mathcal{R}(I) \) then \( H \leq G \) and \( G/H \) is abelian.

**Proof.** Observe that \( I \) is a proper ideal in \( \mathfrak{g} G \), since \( \mathfrak{a}(I) = 0 \). Also by Lemma 1, \( H \supseteq \mathfrak{a}^{-1}(I) \). Since \( (ghg^{-1}h^{-1} - 1)hg = gh - hg \in I \), for all \( g, h \in G \), so \( (ghg^{-1}h^{-1} - 1) \in I \). Hence \( ghg^{-1}h^{-1} \in \mathfrak{a}^{-1}(I) \subseteq H \); i.e., \( G' \), the commutator-subgroup is in \( H \). Hence \( H \leq G \) and \( G/H \) is abelian.

Now let \( H \) satisfy the hypothesis of Lemma 2. Then we have:

**Lemma 3.** Let \( I = J(G) \) and \( H \in \mathcal{R}(I) \). Then \( \mathfrak{g}(G/H) \) is semi-simple and \( G/H \) has no elements of order \( p = \text{Char.} \mathfrak{g} \).

**Proof.** \( J(G) = (J(G) \cap \mathfrak{g} H)\mathfrak{g} G \subseteq J(H) \cdot \mathfrak{g} G \) by 16.9 of [1]. Now \( \mathfrak{g} H[\mathfrak{a}_\mathfrak{a}(H) \mathfrak{a}(H) \mathfrak{g} G = \mathfrak{a}_\mathfrak{a}(H) \subseteq J(H) \cdot \mathfrak{g} G \subseteq J(G) \), where \( \mathfrak{a}_\mathfrak{a}(H) \) is the ideal in \( \mathfrak{g} G \), generated by \( \{ h - 1 \mid h \in H \} \). Now \( \mathfrak{a}_\mathfrak{a}(H) \) is the kernel of the natural map of \( \mathfrak{g} G \) onto \( \mathfrak{g}(G/H) \); (see for example proof of Theorem 1 in [2]). Thus \( \mathfrak{g}(G/H) \cong \mathfrak{g} G/\mathfrak{a}_\mathfrak{a}(H) \) is semi-simple. Since \( G/H \) is abelian by Lemma
2, so it is clear that it has no elements of order $p$, as $\mathfrak{g}(G/H)$ is semi-simple.

This also completes the proof of Theorem 2.

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