Pacific Journal of Mathematics

HOMOMORPHISMS OF RIESZ SPACES

CHARLES THOMAS TUCKER, II

Vol. 55, No. 1 September 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1. 1974

HOMOMORPHISMS OF RIESZ SPACES

C. T. TUCKER

If L is a Riesz space (lattice ordered vector space), a Riesz homomorphism of L is an order preserving linear map which preserves the finite operations "$\vee$" and "$\wedge$". It is shown here that if L is one of a large class of spaces and φ is a Riesz homomorphism from L onto an Archimedean Riesz space, then φ preserves the order limits of sequences.

The symbol θ will be used to denote the zero element of any vector space. Suppose L is a Riesz space (lattice ordered vector space). If $f \in L$ then $|f| = f \vee \theta - (f \wedge \theta)$. If M is a linear subspace of L then M is said to be an *ideal* of L if, whenever $|g| \leqq |f|$ and $f \in M$, then $g \in M$. If each of L_1 and L_2 is a Riesz space, a Riesz *homomorphism* φ from L_1 to L_2 is a linear map from L_1 to L_2 which preserves order and the finite operations "$\bigvee$" and "$\bigwedge$". A sequence $f_1, f_2, f_3, \cdots$ of points is said to *order converge* to the point f if there exists a sequence $u_1 \geqq u_2 \geqq u_3 \geqq \cdots$ and a sequence $v_1 \leqq v_2 \leqq v_3 \leqq \cdots$ of points such that $\bigvee v_p = f$, $\bigwedge u_p = f$, and $v_p \leqq f_p \leqq u_p$. Order convergence for nets is defined analogously. A sequence $f_1, f_2, f_3, \cdots$ of elements of the Riesz space L is said to converge *relatively uniformly* to the element f of L if there exists an element g of L (called the regulator) such that if $\varepsilon > 0$, there exists a number N_ε such that if n is a positive integer greater than N_ε, then $|f - f_n| \leqq \varepsilon g$. A Riesz space L is said to be *Archimedean* if, whenever f and g are two points of L such that $\theta \leqq nf \leqq g$ for all positive integers n, then $f = \theta$. Also L is said to be *σ-complete* if each countable set of positive elements has a greatest lower bound and *complete* if each set of positive elements has a greatest lower bound. If φ is a Riesz homomorphism which preserves the order limits of sequences then φ is said to be a Riesz *σ-homomorphism*. If φ preserves the order limits of nets it is said to be a *normal Riesz homomorphism*. A one-to-one onto map which is a Riesz homomorphism is a *Riesz isomorphism*. If H is a subset of L, H^+ will denote the set of all points f of H such that $f \geqq \theta$. If $f \in L$ then f^+ denotes $f \vee \theta$.

Suppose L is a Riesz space, M is an ideal of L, and the algebraic quotient L/M is partially ordered as follows: If each of H and K belongs to L/M and there is an element h of H and k of K such that $h \geqq k$, then $H \geqq K$. It follows that L/M is a Riesz space and the normal map $\pi: L \to L/M$ is a Riesz homomorphism (Luxemburg and Zaanen [3], p. 102). The coset of L/M containing f will be denoted $[f]$. Further, if M is the kernel of a Riesz homomorphism φ defined on a Riesz space L then the image of φ is Riesz isomorphic to L/M. (Luxemburg and Zaanen [3], p. 102).

If M is a subset of a Riesz space L with the property that whenever $m_1, m_2, m_3, \cdots$ is a sequence of points of M which converges relatively uniformly to a point b of L, b is in M, then M is said to be *uniformly closed*.

In many instances properties of Riesz homomorphisms can be related to properties of their kernels. The following four theorems which are examples of this are listed for future reference.

THEOREM A. *If L is a Riesz space and φ is a Riesz homomorphism defined on L then $\varphi(L)$ is Archimedean if and only if the kernel of φ is uniformly closed.* (See Veksler [8] or Luxemburg and Zaanen [3], Theorem 60.2.)

An ideal M of L is called a *σ-ideal* if, whenever $\{m_i\}$ is a countable subset of M and $b = \bigvee m_i$, then $b \in M$.

THEOREM B. *Suppose L is a Riesz space and φ is a Riesz homomorphism from L onto the Riesz space K. Then φ is a Riesz σ-homomorphism if and only if the kernel of φ is a σ-ideal.* (See Luxemburg and Zaanen [3], Theorem 18.11.)

THEOREM C. *Suppose L is a σ-complete Riesz space and φ is a Riesz σ-homomorphism defined on L. Then $\varphi(L)$ is σ-complete.* (See Veksler [7] or Luxemburg and Zaanen [3], Theorem 59.3.)

An ideal M of L is called a *band* if, whenever $\{m_\alpha\}$, $\alpha \in \lambda$, is a subset of M and $b = \bigvee m_\alpha$, then $b \in M$.

THEOREM D. *Suppose L is a Riesz space and φ is a Riesz homomorphism from L onto the Riesz space K. Then φ is a normal Riesz homomorphism if and only if the kernel of φ is a band.* (See Luxemburg and Zaanen [3], Theorem 18.13.)

A question of interest is when can properties of L imply properties of a class of Riesz homomorphisms defined on L. By combining some known results it can be noted that to place requirements on all the Riesz homomorphisms on L is quite strong.

The sequence $f_1, f_2, f_3, \cdots$ is called a *uniform Cauchy sequence* (with regulator g) if, for each $\varepsilon > 0$, there is a number N such that if n and m are positive integers and $n, m > N$, then $|f_n - f_m| \leqq \varepsilon g$. The Riesz space is *uniformly complete* whenever every uniform Cauchy sequence (with regulator g) converges uniformly (with regulator

g) to a point of L.

PROPOSITION 1. *Suppose L is a uniformly complete Archimedean Riesz space. Each two of the following four statements are equivalent:*

(1) *For each Riesz homomorphism φ defined on L, $\varphi(L)$ is Archimedean,*

(2) *For each Riesz homomorphism φ from L onto a Riesz space K, φ is a Riesz σ-homomorphism,*

(3) *For each Riesz homomorphism φ from L onto a Riesz space K, φ is a normal Riesz homomorphism, and*

(4) *There is a nonempty set X such that L is Riesz isomorphic to the space of all real functions which are zero except on some finite subset of X.*

Proof. By a theorem of Luxemburg and Moore [2], (1) $\rightarrow$ (4). By Theorems A, B, and D, (4) $\rightarrow$ (3) $\rightarrow$ (2) $\rightarrow$ (1).

On the other hand, if requirements are placed on only a subcollection of the collection of all Riesz homomorphisms on L, results of wider applicability can be obtained. In particular, in the following theorems, it is shown that for a large class of Riesz spaces every Riesz homomorphism onto an Archimedean Riesz space is a Riesz σ-homomorphism.

If ω is a subset of L, ω^d denotes the set of all elements g such that $|g| \wedge |f| = \theta$ for each point f of ω. If M is a band in L it is said to be a *projection band* if $L = M \oplus M^d$.

A *principal band* is a band generated by a single element. The Riesz space L is said to have the *principal projection property* if every principal band is a projection band. The Riesz space L has the principal projection property if and only if for each pair of points f and g of L^+, $\bigvee_{n=1}^{\infty} (nf \wedge g)$ exists. (See Luxemburg and Zaanan [3], Theorem 24.7.)

Order convergence in L is said to be *stable* if whenever $f_1, f_2, f_3, \cdots$ is a sequence order converging to θ there is an unbounded, nondecreasing sequence of positive numbers $c_1, c_2, c_3, \cdots$ such that $c_1 f_1, c_2 f_2, c_3 f_3, \cdots$ order converges to θ. Order convergence in the spaces L_p, $1 \leqq p < \infty$; l_p, $1 \leqq p < \infty$; and C_0 is stable.

If order convergence in L is stable then every uniformly closed ideal in L is a σ-ideal. Thus if φ is a Riesz homomorphism from L onto an Archimedean Riesz space K, then φ is a Riesz σ-homomorphism.

For certain sets X order convergence in R^X is not stable. This can be seen as follows: Let X be the set to which x belongs only if x is an unbounded, nondecreasing sequence of positive numbers. Let

f_n be the function defined on X such that if $c_1, c_2, c_3, \cdots$ is a point of X then $f_n(c_1, c_2, c_3, \cdots)$ is $1/c_n$. Then $f_1, f_2, f_3, \cdots$ order converges to θ, but if $c_1, c_2, c_3, \cdots$ is an unbounded, nondecreasing sequence of positive numbers then $c_1 f_1, c_2 f_2, c_3 f_3, \cdots$ does not order converge to θ since $c_n f_n(c_1, c_2, c_3, \cdots) = 1$ for each positive integer n. If X is made of larger cardinality then clearly order convergence in R^X still fails to be stable.

The author, in a paper concerned with the order properties of convergence of Baire functions [6], defined a positive element x of a Riesz space L to have *property* c if for each sequence $h_1 \leqq h_2 \leqq h_3 \leqq \cdots$ of elements of L such that $x = \bigvee h_i$, there exists an element b of L such that for each positive integer n, $b \leqq \sum_{i=1}^n h_i$.

EXAMPLE 2. The constant function 1 in R^x has property c.

The constant function 1 in $B[0, 1]$ (the space of all Baire functions on the interval $[0, 1]$) has property c.

Let ω be the set of all functions defined on the interval $[0, 1]$ whose ranges are a subset of the rational numbers and let Q be the vector space generated by ω. Then Q is a Riesz space with the principal projection property but is not uniformly complete. This can be seen as follows: If f is in ω, H is a subset of the interval $[0, 1]$, and $\tilde{f}$ is the function obtained by setting f to zero on H and leaving it unchanged off H, then $\tilde{f}$ is in ω. For Q to be a Riesz space it is sufficient that $f \vee \theta$ exists for each point f of Q. Thus, if f is in Q it is of the form $\sum_{i=1}^n c_i f_i$ where the f_i's are in ω. Let H be the set of numbers x for which $f(x) < 0$. Then $f \vee \theta = \sum_{i=1}^n c_i \tilde{f}_i$ and $f \vee \theta$ is in Q. Clearly Q has the principal projection property. Each point of Q has as range a countable number set, but a function which fails to have this property, say $g(x) = x$ on the interval $[0, 1]$, is the uniform limit of a sequence of points of Q. Further the constant function 1 in Q has property c.

Let L be a Riesz space and x a positive element of L which has property c and M be a sub Riesz space of L containing x with the property that if f belongs to L then there is a point g to M such that $g \geqq f$. Then x has property c in M.

THEOREM 3. *Suppose L is an Archimedean Riesz space containing a point x which has property c. Then each Riesz homomorphism φ of L into an Archimedean Riesz space K is a Riesz σ-homomorphism.*

Proof. If it can be shown that $f_1 \leqq f_2 \leqq f_3 \leqq \cdots \leqq \theta$ and $\bigvee f_p = \theta$ implies $\bigvee \varphi(f_p) = \theta$, then the theorem is proved.

Now

$$f_p \vee (-x) + f_p \wedge (-x) = f_p - x$$
$$\varphi(f_p \vee (-x)) + \varphi(f_p \wedge (-x)) = \varphi(f_p) - \varphi(x)$$
$$\varphi(f_p \wedge (-x)) + \varphi(x) = \varphi(f_p) - \varphi(f_p \vee (-x))$$
$$= \varphi(f_p \wedge (-x) + x) = \varphi((f_p + x) \wedge \theta)$$
$$\sum_{p=1}^{n} \varphi((f_p + x) \wedge \theta) = \sum_{p=1}^{n} \varphi(f_p) - \varphi(f_p \vee (-x))$$
$$\varphi\left(\sum_{p=1}^{n} (f_p + x) \wedge \theta\right) = \sum_{p=1}^{n} \varphi(f_p) - \varphi(f_p \vee (-x)) .$$

As x has property c there exists an element b such that $b \leqq \sum_{p=1}^{n} (f_p + x) \wedge \theta$ for each positive integer n. Thus,

$$\varphi(b) \leqq \varphi\left(\sum_{p=1}^{n} (f_p + x) \wedge \theta\right) = \sum_{p=1}^{n} \varphi(f_p) - \varphi(f_p \vee (-x)) .$$

Suppose that $u \leqq \theta$ is an upper bound for $\{\varphi(f_p)\}$. Then

$$\varphi(b) \leqq \sum_{p=1}^{n} (u - \varphi(f_p \vee (-x))) \leqq \sum_{p=1}^{n} (u - \varphi(-x)) = n(u - \varphi(-x)) .$$

Thus, $u - \varphi(-x) \geqq \theta$ as K is Archimedean and $u \geqq \varphi(-x)$.

But if x has property c, $(1/n)x$ has property c for each positive integer n. Therefore, $u \geqq (1/n)\varphi(-x)$ and $u = \theta$ as K is Archimedean. So $\bigvee \varphi(f_p) = \theta$ and φ is a Riesz σ-homomorphism.

Frequently inclusion maps do not preserve the order limits of sequences. For instance the inclusion map of the space of continuous functions on the interval [0, 1] into the space of all functions on the interval [0, 1] fails to preserve the order limits of sequences. For this reason most theorems which guarantee that a Riesz homomorphism is a Riesz σ-homomorphism require that the mappings be onto. Theorem B would not be true if φ was not specified to be an onto map because of the example just noted. However in view of Theorem 3, no such problem can arise in a space that contains an element with property c. Any embedding of such a space into an Archimedean space must preserve the order limits of sequences.

If in Theorem 3, x is assumed to be a strong unit (a point with the property that if $f \in L$ there is a number r such that $rx \geqq |f|$) rather than have property c, then the statement is no longer true. For instance, let L consist of the set of all bounded sequences and M be the set of all sequences $s_1, s_2, s_3, \cdots$ with the property that if $\varepsilon > 0$ there is only a finite number of positive integers n such that $|s_n| > \varepsilon$. Then M is a uniformly closed ideal but not a σ-ideal.

The Riesz space L is σ-complete if and only if it is uniformly complete and has the principal projection property (Luxemburg and Zaanan [3], Theorem 42.5). If L is uniformly complete and φ is a

Riesz homomorphism defined on L then $\varphi(L)$ is uniformly complete (Luxemburg and Moore [2]).

Thus the question of when the operation of taking a quotient preserves the property of σ-completeness can be included in the question of when this operation preserves the principal projection property.

The Riesz space L has the *quasi principal projection property* if for each point f of L, $L = \{f\}^d \oplus \{f\}^{dd}$. Then L has the principal projection property if and only if it has the quasi principal projection property and is Archimedean. If L has the quasi principal projection property then for each point f of L and g of L there is a unique element g_1 of $\{f\}^d$ and a unique element g_2 of $\{f\}^{dd}$ such that $g = g_1 + g_2$. Denote g_2 by $P_f(g)$.

THEOREM 4. *Suppose L is a Riesz space with the quasi principal projection property, M is an ideal of L, and π is the natural map of L onto L/M. Then the following two conditions are equivalent:*
(1) *If m is a point of M, $P_m L$ is a subset of M and*
(2) (a) *L/M has the quasi principal projection property and*
(b) *$\pi P_f = P_{\pi f}\pi$ for each point f of L.*

Proof. Suppose Condition 1 is true and each of H and K belongs to $(L/M)^+$. We wish to show that there exist points H_1 and H_2 belonging to K^d and K^{dd} respectively such that $H = H_1 + H_2$. There exist points h and k in L^+ such that $H = [h]$ and $K = [k]$. As L has the quasi principal projection property there exist points h_1 and h_2 of $\{k\}^d$ and $\{k\}^{dd}$ respectively such that $h = h_1 + h_2$. Now $H = [h_1] + [h_2]$ and $[h_1] \wedge [h_2] = \theta$. Since h_1 is in $\{k\}^d$, $h_1 \wedge k = \theta$, so $[h_1] \wedge [k] = [h_1 \wedge k] = \theta$ and $[h_1]$ belongs to $\{K\}^d$. Suppose $J \geqq \theta$ is in $\{K\}^d$, i.e., $J \wedge K = \theta$. There is a point j of L^+ such that $[j] = J$. There is a point m of M such that $j \wedge k = m$. By hypothesis there exists a point m_1 of M such that $P_m(j) = m_1$. Thus there is a point $j_1 \geqq \theta$ and a point $m_1 \geqq \theta$ such that $j_1 + m_1 = j$, j_1 is in $\{j \wedge k\}^d$, and m_1 is in $\{j \wedge k\}^{dd}$. Since $j_1 + m_1 = j$ and $m_1 \geqq \theta$, $j_1 \leqq j$ and $j_1 \wedge j = j_1$. Therefore, $\theta = j_1 \wedge (j \wedge k) = (j_1 \wedge j) \wedge k = j_1 \wedge k$ or $(j - m_1) \wedge k = \theta$. So $j - m_1$ is in $\{k\}^d$ and hence $(j - m_1) \wedge h_2 = \theta$. It follows that $[j] \wedge [h_2] = \theta$ and $[h_2]$ is in $\{K\}^{dd}$.

Also $\pi P_k(h) = \pi(h_2) = [h_2] = P_K(H) = P_{\pi k}\pi(h)$.

Suppose Condition 2 is true. If m is a point of M and h is a point of L

$$\theta = P_\theta \pi(h) = P_{\pi m}\pi(h) = \pi P_m(h) .$$

Thus $P_m(h)$ belongs to M.

COROLLARY 5. *Suppose L is a Riesz space with the quasi* *principal projection property, M is an ideal of L, and π is the natural map of L onto L/M. Then the following two conditions are equivalent:*

(1) (a) *If m is a point of M, $P_m L$ is a subset of M and*
(b) *M is relatively uniformly closed, and*
(2) (a) *L/M has the principal projection property and*
(b) *$\pi P_f = P_{\pi f}\pi$ for each point f of L.*

Proof. For L/M to have the principal projection property it is equivalent that L/M have the quasi principal projection property and be Archimedean. By Theorem A it is necessary and sufficient for L/M to be Archimedean that M be uniformly closed.

THEOREM 6. *Suppose L is a Riesz space with the quasi principal projection property and M is an ideal of L. Consider the following two properties:*

(1) (a) *If m is a point of M, $P_m L$ is a subset of M and*
(b) *M is relatively uniformly closed, and*
(2) *M is a σ-ideal.*

Then properties 1 *and* 2 *are independent. If L is assumed to have the principal projection property then property* 2 *implies property* 1 *but property* 1 *does not necessarily imply property* 2. *If L is assumed to be uniformly complete then property* 1 *implies property* 2, *but property* 2 *does not necessarily imply property* 1.

Proof. Suppose L is assumed to have the principal projection property and property 2. For each positive integer n and point m of M, $nm \wedge h$ belongs to M as M is an ideal. Now $P_m h = \bigvee (nm \wedge h)$, $P_m h$ belongs to M since M is a σ-ideal, and property 1(a) holds. Property 1(b) is clearly true.

An example of a space with the principal projection property in which property 1 does not imply property 2 is the following: Let L be the subspace of the space of all sequences generated by the collection of all constant sequences and all sequences which are zero except for a finite number of terms. Let M be the ideal consisting of the collection of all sequences which are zero except for a finite number of terms. Then M satisfies property 1 but not property 2.

Assume L is uniformly complete and property 1 is true. Suppose $\{m_1, m_2, m_3, \cdots\}$ is a subset of M^+ and $h = \bigvee_{i=1}^{\infty} m_i$. Let $r_p = \bigvee_{i=1}^{p} m_i$. Then $\theta \leqq r_1 \leqq r_2 \leqq r_3 \leqq \cdots$ and $\bigvee_{i=1}^{\infty} r_i = h$. Let j be a positive integer, $f_1 = P_{r_{j+1}} h$, $f_2 = h - f_1$, $g_1 = P_{r_j} h$, $g_2 = h - g_1$, and $d_j = f_1 - g_1$. Note that d_j is in M. Since $f_1 + f_2 = g_1 + g_2$, $d_j = g_2 - f_2$. As each of g_2 and f_2 is in $\{r_j\}^d$, d_j is in $\{r_j\}^d$ and $d_j \wedge g_1 = \theta$. Thus $d_j \vee g_1 = f_1$.

Therefore, there exists a countable pairwise disjoint subset $\{d_1, d_2,$

$d_3, \cdots\}$ of M such that $h = \bigvee_{i=1}^{\infty} d_i$. Now the sequence d_1, $d_1 + (1/2)d_2$, $d_1 + (1/2)d_2 + (1/3)d_3$, $d_1 + (1/2)d_2 + (1/3)d_3 + (1/4)d_4, \cdots$ converges relatively uniformly to a point m of M. Then h belongs to the band generated by m, $P_m h = h$, and it follows that h is in M.

An example of a uniformly complete space with the quasi principal projection property in which property 2 does not imply property 1 is the lexiographically ordered plane. The vertical axis is a σ-ideal but does not have property 1 (a).

Suppose L is a Riesz space and $e \geqq \theta$ is a point of L. Then e will be called a *weak unit* if $e \wedge |f| = \theta$ only in case $f = \theta$.

When necessary, it will be assumed that L is a subspace of the set of all almost finite extended real valued continuous functions on an extremally disconnected compact Hausdorff space S. Further if L has a weak unit e, this subspace may be chosen so that e is the function identically to 1.

Suppose e is a weak unit of the Riesz space L. The pair (L, e) will be said to be a *Vulikh algebra* if a multiplication can be defined on L which makes it an associative, commutative algebra with multiplicative unit e which is positive in the sense that if $f \geqq \theta$ and $g \geqq \theta$ then $fg \geqq \theta$. For some properties of Vulikh algebras see Rice [4], Tucker [5], or Vulikh [9], [10].

Suppose that it is assumed that L is a subspace of the set of all almost finite extended real valued continuous functions on an extremally disconnected compact Hausdorff space S and that e is the function identically equal to 1. If each of f and g belong to L their pointwise product will be defined as follows: Both f and g are finite on a dense subset Q of S. Their pointwise product on Q is a continuous function on Q and can be extended uniquely to a continuous function on S, since S is extremally disconnected.

There is at most one multiplication which makes (L, e) a Vulikh algebra (Kantorovitch, Vulikh, and Pinsker [1]). If (L, e) is a Vulikh algebra and it is represented as a Riesz space as a subspace of the set of all almost finite extended real valued continuous functions on an extremally disconnected compact Hausdorff space with e the constant function 1, then the Vulikh algebra multiplication will be the same as the pointwise multiplication described above.

THEOREM 7. *Suppose L is a Riesz space with the principal projection property, M is a uniformly closed ideal of L, π is the natural map of L onto L/M and for each m in M^+, if K is the principal band generated by m, (K, m) is a Vulikh algebra. Then L/M has the principal projection property and $\pi P_f = P_{\pi f}\pi$ for each point f of L.*

Proof. By Theorem 4 it is sufficient to show that for each point m of M^+ and f of L^+ that $\bigvee (nm \wedge f)$ belongs to M. Let K be the principal band generated by m.

By the representation theorem for Riesz spaces K can be assumed to consist of almost finite continuous extended real valued functions on a compact Hausdorff space S, where m is the constant function with value 1 everywhere.

Let $h = \bigvee (nm \wedge f)$. The point h belongs to K. By hypothesis (K, m) is a Vulikh algebra. Thus h^2 belongs to K.

Suppose x is a point of S. If $h(x) \geqq n$, then

$$(h - (nm \wedge f))(x) \leqq h(x) \leqq \frac{1}{n} h^2(x) .$$

If $h(x) < n$, then

$$(h - (nm \wedge f))(x) = 0 \leqq \frac{1}{n} h^2(x) .$$

Thus $m \wedge f,\ 2m \wedge f,\ 3m \wedge f, \cdots$ converges relatively uniformly to h with regulator h^2. As M is uniformly closed, h is in M.

If α is a subset of L^+ with the property that for each two points f and g of α, $f \wedge g = \theta$, then α is said to be *orthogonal.*

THEOREM 8. *Suppose L is a Riesz space with the principal projection property, M is a uniformly closed ideal of L with the property that if $\{f_1, f_2, f_3, \cdots\}$ is a bounded countable orthogonal subset of M^+ there is an unbounded nondecreasing positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1 f_1, c_2 f_2, c_3 f_3, \cdots\}$ is bounded, and π is the natural map of L onto L/M. Then L/M has the principal projection property and $\pi P_f = P_{\pi f}\pi$ for each point f of L.*

Proof. By Theorem 4 it is sufficient to show that for each point m of M^+ and f of L^+ that $\bigvee (nm \wedge f)$ belongs to M.

Let K be the principal band generated by m. By hypothesis K is a projection band, let $h = \bigvee (nm \wedge f)$. The point h belongs to K. Also $\bigvee (nm \wedge f) = \bigvee (nm \wedge h)$.

If k is in K^+, let $\chi(k) = \bigvee (nk \wedge m)$. This supremum exists as K has the principal projection property. Let

$$d_n = \chi((nm \wedge h - (n-1)m)^+) - \chi(((n+1)m \wedge h - nm)^+) .$$

By the representation theorem for Riesz spaces K can be assumed to consist of almost finite continuous extended real valued functions on a compact Hausdorff space S, where m is the constant function with value 1 everywhere.

Suppose x is a point of S. If $h(x) > n$, then $d_n(x) = 0$, if

$n \geqq h(x) > n - 1$, then $d_n(x) = 1$, and if $h(x) \leqq n - 1$, then $d_n(x) = 0$. Let $h_n = (nm \wedge h - (n-1)m)^+ - \chi((h - nm)^+) + (n-1)d_n$. If $h(x) > n$, then $h_n(x) = 0$, if $n \geqq h(x) > n - 1$, then $h_n(x) = h(x)$, and if $h(x) \leqq n - 1$, then $h_n(x) = 0$.

Therefore $\{h_1, h_2, h_3, \cdots\}$ is an orthogonal subset of M^+ bounded above by h. By hypothesis there is an unbounded nondecreasing positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1h_1, c_2h_2, c_3h_3, \cdots\}$ is bounded above by a point b of L. Then if i is a positive integer, $h - (h_1 + h_2 + \cdots + h_i) \leqq (1/c_{i+1})b$, and the sequence $h_1, h_1 + h_2, h_1 + h_2 + h_3, \cdots$ converges relatively uniformly to h. As M is uniformly closed, h is in M.

COROLLARY 9. *Suppose L is a Riesz space which is σ-complete and with the property that if $\{f_1, f_2, f_3, \cdots\}$ is a bounded countable orthogonal subset of L^+ there is an unbounded nondecreasing positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1f_1, c_2f_2, c_3f_3, \cdots\}$ is bounded then every Riesz homomorphism φ from L onto an Archimedean Riesz space is a Riesz σ-homomorphism.*

EXAMPLE 10. Suppose L is one of the space L_p, $1 \leqq p < \infty$; l_p, $1 \leqq p < \infty$; or C_0 in which order convergence is stable or L is one of the spaces R^X or $B[0, 1]$ which has a point with property c as described in Example 2. Then L satisfies the conditions of Corollary 9. On the other hand, let L be the space of all functions defined on the x-axis with compact support. In this case L satisfies the hypothesis of Corollary 9, while L neither contains a point with property c nor is order convergence stable in L.

By what has just been shown, if L is a σ-complete Riesz space with the property that if $\{f_1, f_2, f_3, \cdots\}$ is a bounded countable orthogonal subset of L^+ then there is an unbounded nondecreasing positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1f_1, c_2f_2, c_3f_3, \cdots\}$ is bounded is sufficient to imply that every uniformly closed ideal is a σ-ideal, but this condition is not necessary, as the following example shows.

EXAMPLE 11. Let S be the set of all ordered pairs of positive integers. Let L be the collection to which f belongs only in case f is a real valued function on S with the property that there is a set ω which includes all but at most a finite number of positive integers such that if k is a positive integer in ω, $f(1, k)$, $f(2, k)$, $f(3, k)$, $\cdots$ is a bounded number sequence.

The space L is a complete Riesz space.

Suppose M is an ideal which is uniformly closed. Let f be the l.u.b. of a countable subset α of M. Let β be the collection to which

g belongs only in case there is a positive integer k and a member h of α such that $g(k, p) = h(k, p)$ for each positive integer p and if i is a positive integer not k then $g(i, p) = 0$ for each positive integer p. Then f is the l.u.b. of β. For each positive integer k, let f_k be the function such that $f_k(k, p) = f(k, p)$ for each positive integer p and if i is a positive integer not k then $f_k(i, p) = 0$ for each positive integer p.

The function which is equal to $f(i, j)$ at (i, j) and zero elsewhere is in M. Then since the function which is $pf_k(i, p)$ at (i, p) is in L, f_k is in M. Since the function which is $if(i, j)$ at (i, j) is in L, f is in M.

Thus each uniformly closed ideal of M is a σ-ideal. For each positive integer i let g_i be the function such that $g_i(p, q) = 1$ if $p = i$ and $g_i(p, q) = 0$ if $i \neq p$. Then $\{g_1, g_2, g_3, \cdots\}$ is an orthogonal subset of L which is bounded above by the constant function 1 but there is no nondecreasing unbounded positive number sequence $c_1, c_2, c_3, \cdots$ such that $\{c_1g_1, c_2g_2, c_3g_3, \cdots\}$ is bounded above.

The Riesz space L has the *projection property* if every band in L is a projection band. Suppose L has the projection property, ω is a subset of L, H is the band generated by ω, and f is a point of L. There is a unique point f_1 of H^d and a unique point f_2 of H such that $f = f_1 + f_2$. Denote f_2 by $P_\omega(f)$.

The analogous question of when can the projection property be preserved in a natural manner can be answered easily.

THEOREM 12. *Suppose L is a Riesz space with the projection property, M is an ideal of L, and π is the natural map of L onto L/M. Then the following two properties are equivalent*:

(1) *π is a normal Riesz homomorphism, and*

(2) (a) *L/M has the projection property, and*

(b) *$\pi P_\omega = P_{\pi\omega}\pi$ for each subset ω of L.*

Proof. If (1) is true then the kernel of π, M, is a projection band and 2 (a) and (b) clearly hold. If (2) is true and ω is a subset of M with the point f as least upper bound, then $\pi P_\omega f = \pi f$, but $P_{\pi\omega}\pi f = P_\theta \pi f = \theta$.

Also, several answers to the question of when is every Riesz σ-homomorphism from an Archimedean Riesz space L onto a Riesz space K a normal Riesz homomorphism are given in Theorem 29.3 of Luxemburg and Zaanen [3].

REFERENCES

1. L. Kantorovitch, B. Vulikh, and A. Pinsker, *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, Moscow, 1950, (Russian).
2. W. A. J. Luxemburg and L. C. Moore, Jr., *Archimedean quotient Riesz spaces*, Duke Math. J., **34** (1967), 725-740.
3. W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam, 1971.
4. Norman M. Rice, *Multiplication in vector lattices*, Canad. J. Math., **20** (1968), 1136-1149.
5. C. T. Tucker, *Sequentially relatively uniformly complete Riesz spaces and Vulikh algebras*, Canad. J. Math., **34** (1972), 1110-1113.
6. ———, *On a theorem concerning Baire functions*, Proc. Amer. Math. Soc., **41** (1973), 173-178.
7. A. I. Veksler, *On factor lineals in vector lattices*, A.I. Gercen Ped. Inst., Leningrad, **183** (1958), 107-127.
8. ———, *The Archimedean principle in homomorphic images of l-groups and of vector lattices*, lzv. Vysš. Učebn Zaved. Mathematika 1966, **5** (53), 33-38.
9. B. Z. Vulikh, *The product in linear partially ordered spaces and its applications to the theory of operators*, Mat. Sb. (N.S.), **22** (64) (1948); I, 27-78; II, 267-317, (Russian).
10. ———, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff, Groningen, 1967.

Received March 5, 1973.

UNIVERSITY OF HOUSTON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 55, No. 1 September, 1974