DEFORMING P. L. HOMEOMORPHISMS ON A CONVEX POLYGONAL 2-DISK

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It is shown that for each p. l. homeomorphism $f$ on a convex polygonal disk which is pointwise fixed on the boundary of the disk, there exists a triangulation $K$ of the disk such that $f$ may be obtained by successively moving the vertices of $K$ (with the motion being extended linearly to each triangle of $K$) in a finite number of steps such that no triangles will be collapsed in the process of motion. An algebraic interpretation of this result is also given.

1. Introduction. By a convex polygonal disk in $\mathbb{R}^2$, we mean a closed region $D$ in $\mathbb{R}^2$ enclosed by a convex polygon $P$. The vertices of $P$ will also be called the vertices of the disk $D$. We shall always assume that no three consecutive vertices of $D$ lie on a straight line. By a triangulation of $D$, we mean a (rectilinear) simplicial complex with $D$ as its underlying space. If $K$ is a triangulation of $D$, we shall let $L(K)$ denote the set of all homeomorphisms from $D$ onto $D$ which are linear on each simplex of $K$ and are pointwise fixed on $\text{Bd}(D)$. Elements of $L(K)$ will be called linear homeomorphisms of $D$ with respect to $K$. We shall consider $L(K)$ as a topological space with the compact open topology. A map $f: D \to D$ will be called a p. l. homeomorphism of $D$ if $f \in L(K)$ for some triangulation $K$ of $D$. A triangulation $K$ of $D$ will be called a proper triangulation if the only 0-simplices of $K$ lying on $\text{Bd}(D)$ are those which are the vertices of $D$. Finally, a p. l. homeomorphism $f$ of $D$ is called proper if $f \in L(K)$ for some proper triangulation $K$ of $D$. In the next section, we shall establish the following:

**Theorem A.** Let $D$ be a convex polygonal disk in $\mathbb{R}^2$. For any proper triangulation $K$ of $D$, the space $L(K)$ is pathwise connected.

Note that a path in the space $L(K)$ with the initial point $f$ and terminal point $g$ corresponds to a deformation from $f$ to $g$ through a family of homeomorphisms of $D$ onto $D$ which are linear with respect to $K$ and are pointwise fixed on $\text{Bd}(D)$. When $D$ is a triangle in $\mathbb{R}^2$, Theorem A is a consequence of S. S. Cairns' result on the deformation of isomorphically imbedded triangulations in the plane [2], [3] also [4, Proposition 2.19]. However, if $D$ is not a triangle,
Cairns' technique, unlike ours, will in general carry the deformation outside the disk $D$. Using Theorem A, we shall show in §3 that a general (i.e., not necessarily proper) $p.l.$ homeomorphism $f$ of $D$ may still be deformed into the identity map of $D$ (or vice versa) in some space $L(K)$. Finally in §4, we shall improve our result of §3 to the following statement, which together with Theorem A form the main results of this paper.

**Theorem B.** For each $p.l.$ homeomorphism $f$ of $D$, there exists a triangulation $K$ of $D$ such that $f$ may be obtained in a finite number of steps by successively moving the vertices of $K$ (with the motion being extended linearly to each simplex of $K$) such that none of the simplices is collapsed in the process.

The problem of deforming a prescribed map of a space into the identity map, or vice versa, in a specific manner has been studied by many mathematicians. We shall mention some results which are more directly related to the problems considered in this paper. H. Tietze [9] and H. L. Smith [8] showed in 1914 and 1917 respectively that a homeomorphism of a 2-dimensional disk which leaves the boundary pointwise fixed is deformable into the identity map through a family of homeomorphisms which leave the boundary pointwise fixed. O. Veblen in 1917 [10] and J. W. Alexander in 1923 [1] extended the result for homeomorphisms of $n$-cells. The technique used by Alexander ("The Alexander Trick") can in fact be used to show that each $p.l.$ homeomorphism $f$ on a polyhedral $n$-cell in $R^n$ which leaves the boundary fixed can be deformed into the identity map through a family of such $p.l.$ homeomorphisms $f_t$. However, each of the homeomorphisms $f_t$ in this process requires a different triangulation of the domain space. In fact, as $t$ approaches 1, the triangulation for $f_t$ requires triangles which are arbitrarily small. It is therefore natural to ask whether the given $p.l.$ homeomorphism on the polyhedral $n$-cell may be deformed into the identity map through a family of $p.l.$ homeomorphisms which are all linear with respect to a fixed triangulation of the $n$-cell. Our Theorem B clearly answers this question in affirmative for a convex 2-dimensional polyhedral disk. In fact, when the disk is a triangle, we have shown that an entire loop of $p.l.$ homeomorphisms of the disk can be deformed into the constant loop at the identity with respect to a fixed triangulation of the disk [5].

2. Deforming proper linear homeomorphisms. In this section, we shall first collect some of the basic properties of the spaces $L(K)$ and then carry out a proof of Theorem A. We shall use a process
similar to that used by the author in [4] to resolve a special case which will appear in our proof of Theorem A. For this reason, we found it necessary to restate some of the definitions and theorems of [4] in the present framework. The proofs of the theorems will be omitted. In the following, if $K$ is a triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$ and if $v$ is a vertex of $K$, we shall let $St(v, K)$ and $Lk(v, K)$ to denote respectively the star and link of $v$ with respect to $K$. A vertex $v$ of $K$ will be called an inner vertex of $K$ if $v \in \text{Bd}(D)$. We shall start with the following observations.

**Remark 2.1.** Let $D$ be a convex polygonal disk in $\mathbb{R}^2$ and $K$ be any triangulation of $D$. For each $f \in L(K)$, the set $\{f(\sigma) | \sigma \in K\}$ also forms a triangulation of $D$. This triangulation will be denoted by $f(K)$. Observe that for each $f \in L(K)$, the spaces $L(K)$ and $L(f(K))$ are homeomorphic; the map: $L(f(K)) \rightarrow L(K)$ carrying each $g \in L(f(K))$ onto $gf$ is clearly a homeomorphism.

**Definition 2.2.** A triangulation $K$ of a polygonal disk $D$ is said to be decomposable if there are vertices $v_a, v_b, v_c$ of $K$ such that the 1-simplices $\langle v_a, v_b \rangle$, $\langle v_b, v_c \rangle$, and $\langle v_c, v_a \rangle$ all belong to $K$ but the 2-simplex $\langle v_a, v_b, v_c \rangle$ does not belong to $K$. If $D$ is a triangle, we shall require in addition that at least one of the open simplices $\langle v_a, v_b \rangle$, $\langle v_b, v_c \rangle$, and $\langle v_c, v_a \rangle$ lies in the interior of $D$.

A triangulation $K$ of $D$ is called indecomposable if it is not decomposable.

The following two propositions may be proved essentially the same way as [4, Propositions 1.4 and 1.5].

**Proposition 2.3.** If $K$ is a decomposable triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$, the space $L(K)$ is homeomorphic to a Cartesian product $L(K_1) \times L(K_2)$ where $K_1$ is a triangulation of $D$ and $K_2$ is a triangulation of a triangle in $\mathbb{R}^2$ such that each of $K_1$ and $K_2$ has a fewer number of vertices than $K$.

**Proposition 2.4.** Let $K$ be a proper, indecomposable triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$. For each inner vertex $v_0$ of $K$, there is an $f \in L(K)$ such that $St(f(v_0), f(K))$ is strictly convex (i.e., $St(f(v_0), f(k))$ is convex and no three consecutive vertices of $Lk(f(v_0), f(K))$ lie on a straight line).

Let $K$ be a triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$. To study the topological problems of the space $L(K)$, it turns out to be convenient to study some nice subspaces of it. We shall now describe these nice subspaces of $L(K)$.
DEFINITION 2.5. Let $P$ be a polygonal circle in $\mathbb{R}^2$ (i.e., $P$ is a simplicial complex in $\mathbb{R}^2$ with $|P|$ homeomorphic to the 1-sphere $S^1$). Let $[P]$ be the union of $|P|$ with the bounded component of $\mathbb{R}^2 - |P|$. For each 1-simplex $\langle v_1, v_2 \rangle$ of $P$, we shall let $H_P(v_1, v_2)$ to denote the open half-plane of $\mathbb{R}^2$ such that:

1. $\langle v_1, v_2 \rangle$ lies on $\text{Bd}(H_P(v_1, v_2))$.
2. $\bar{H}_P(v_1, v_2)$ (the closed half-plane) contains a neighborhood of $\text{Int}(\langle v_1, v_2 \rangle)$ in $[P]$.

Now, for each polygonal circle $P$ in $\mathbb{R}^2$, we define the core of $P$, $\text{cor}(P)$, to be the set

$$\text{cor}(P) = \bigcap \{H_P(v_i, v_j) \mid \langle v_i, v_j \rangle \text{ is a 1-simplex of } P\}.$$ 

REMARK 2.6. Let $K$ be a triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$. Let $v_0$ be an inner vertex of $K$. Observe that for each $f \in L(K)$, the vertex $f(v_0)$ must lie in $\text{cor}(Lk(f(v_0), f(K)))$. Conversely, for each point $x \in \text{cor}(Lk(f(v_0), f(K)))$, there is a unique element $g \in L(K)$ such that $g(v) = f(v)$ for each vertex $v \neq v_0$ of $K$ and $g(v_0) = x$.

DEFINITION 2.7. Let $K$ be a triangulation of a convex polygonal disk $D$. For each inner vertex $v_i$ of $K$, we shall let $L_{v_i}(K)$ or simply $L_i(K)$ denote the subspace of $L(K)$ consisting of all elements $f \in L(K)$ such that $f(v_i)$ is located at the centroid of the set $\text{cor}(Lk(f(v_i), f(K)))$.

The space $L_i(K)$ is clearly of the same homotopy type as $L(K)$ for each inner vertex $v_i$ of $K$ (cf. [4, Proposition 2.6]). The reason for introducing these subspaces $L_i(K)$ is that under suitable conditions, they may be decomposed into a finite union of the spaces $L(K)$ where each $K_i$ is a triangulation of the disk $D$ with one fewer inner vertex than $K$. This makes it possible to carry out an induction argument on the number of vertices of $K$ to prove the pathwise connectedness of $L(K)$. We shall now describe conditions which make such a decomposition of $L_i(K)$ possible.

DEFINITION 2.8. Let $K$ be a triangulation of a convex polygonal disk $D$. For each vertex $v_i$ of $K$, the incidence number of $v_i$ in $K$, denoted by $m_{v_i}$ or simply $m_i$, is defined as the number of vertices of $K$ lying on $Lk(v_i, K)$ (hence, the number of 1-simplices of $K$ incident on $v_i$).

PROPOSITION 2.9. Let $K$ be a triangulation of a convex polygonal disk $D$ with $n$ inner vertices. Let $v_0$ be an inner vertex of $K$ such that
1. The incidence number $m_0$ of $v_0$ is $\leq 5$.
2. $St(v_0, K)$ is strictly convex.

Then the space $L_0(K)$ may be written as the union of at most five subspaces $\{L(K_i)\}$ where each $K_i$ is a triangulation of $D$ with exactly $n - 1$ inner vertices. Furthermore, if $K$ is a proper triangulation of $D$, each of the $K_i$'s is also a proper triangulation.

The proof of this proposition, though needing more work to establish, is still similar to that given for [4, Proposition 2.16], hence, will be omitted. Finally, we shall quote a combinatorial lemma [4, Proposition 2.17] which will be used to establish the existence of an inner vertex $v_0$ with the incidence number $\leq 5$ when the triangulation is sufficiently nice.

**Proposition 2.10.** Let $K$ be a triangulation of a polygonal disk $D$ in $\mathbb{R}^2$. Then

$$\sum_{v_i = \text{inner vertex of } D} (6 - m_i) = 6 + \sum_{v_i \in \text{Bd}(D)} (m_i - 4).$$

**Remark 2.11.** Let $K$ be a triangulation of a polygonal disk $D$ in $\mathbb{R}^2$ with at least one inner vertex. It follows immediately from Proposition 2.10 that if $D$ is a triangle or if $D$ is a general polygonal disk with $m_i \geq 4$ for each vertex $v_i \in \text{Bd}(D)$, then there is at least one inner vertex $v_0$ of $K$ with $m_0 \leq 5$.

We can now prove Theorem A by induction on the number of vertices of the triangulation $K$. The theorem is clearly true if the number of vertices of $K$ is equal to three, since in that case, $D$ must be a triangle and $K$ has no inner vertex. Hence, $L(K)$ consists only of a single element. Assuming the theorem to be true for any convex polygonal disk in $\mathbb{R}^2$ with a proper triangulation consisting of less than $n$ vertices, we now consider a convex polygonal disk in $D$ in $\mathbb{R}^2$ and a proper triangulation $K$ of $D$ of $n$ vertices. We assume that $K$ has at least one inner vertex, for otherwise the theorem is trivially true. By Proposition 2.3, we may also assume that $K$ is an indecomposable triangulation.

We now consider the following special case: For each vertex $v_i$ of $K$ lying on $\text{Bd}(D)$, the incidence number $m_i \geq 4$. The theorem may be proved very easily for this case as follows: By Remark 2.11, there is at least one inner vertex $v_0$ of $K$ with $m_0 \leq 5$. Using Proposition 2.4 and then Remark 2.1 if necessary, we may assume that $St(v_0, K)$ is strictly convex. Applying Proposition 2.9, we see that the space $L_0(K)$, which is of the same homotopy type as $L(K)$, may be written as a finite union of the spaces $L(K_i)$, where each $K_i$ is a triangulation of $D$ with one fewer vertex than $K$. We may therefore
apply the induction hypothesis to conclude that each $L(K_i)$ is pathwise connected. Since the identity map of the disk $D$ clearly belongs to all the $L(K_i)$'s, these pathwise connected spaces $L(K_i)$ have a nonempty intersection. Therefore, $L(K)$ is pathwise connected. With the above special case being taken care of, we need only consider the following case in our inductive step:

Prove that space $L(K)$ is pathwise connected provided that $K$ has a vertex $v_0 \in \text{Bd}(D)$ with $m_0 \leq 3$. Here, as before, $K$ is a proper, indecomposable triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$, and has $n(>3)$ vertices.

For any vertex on $\text{Bd}(D)$, the incidence number is clearly at least 2. We first show that the case for $m_0 = 2$ is trivial. Let $v_1, v_2$ be the two vertices adjacent to $v_0$ in either sense along $\text{Bd}(D)$. The fact that $m_0 = 2$ implies that the triangle $\langle v_0, v_1, v_2 \rangle$ is a 2-simplex of $K$. Since $v_0, v_1, v_2$ are all on $\text{Bd}(D)$, each $f \in L(K)$ must be pointwise fixed on this triangle $\langle v_0, v_1, v_2 \rangle$. Hence, the space $L(K)$ is homeomorphic to a space $L(K')$ where $K'$ is the triangulation inherited from $K$ on a smaller convex disk $D'$ obtained from $D$ by cutting off the triangle $\langle v_0, v_1, v_2 \rangle$. Our induction hypothesis then guarantees the pathwise connectedness of $L(K')$, and hence, of $L(K)$.

Henceforth, we may assume that $m_0 = 3$. Again let $v_1, v_2$ be the two vertices adjacent to $v_0$ in either sense along $\text{Bd}(D)$. Let $v_3$ be the third vertex such that $\langle v_0, v_3 \rangle$ is a 1-simplex of $K$. Note that we may assume $v_3$ to be an inner vertex. For otherwise, the 1-simplex $\langle v_0, v_3 \rangle$ would cut the disk into two convex disks $D_1, D_2$, hence, the space $L(K)$ would be homeomorphic to the Cartesian product $L(K_1) \times L(K_2)$ where $K_i$ is the triangulation inherited from $K$ on the disk $D_i(i = 1, 2)$. Pathwise connectedness of $L(K)$ would then be immediate from the induction hypothesis applied to $L(K_1)$ and $L(K_2)$.

Now by Proposition 2.4 and then by Remark 2.1 if necessary, we may assume that $\text{St}(v_3, K)$ is strictly convex. Let an arbitrary $f \in L(K)$ be given. We wish to show that $f$ may be connected to the identity map by a path in the space $L(K)$. We shall consider three cases.

Case 1. The vertex $f(v_3)$ lies in the open triangular region $\langle v_0, v_1, v_2 \rangle$. In this case, we shall first consider a map $g$ defined by $g(v) = v$ for each vertex $v \neq v_3$ of $K$ and $g(v_3) = f(v_3)$. Since $\text{St}(v_3, K)$ is convex and $f(v_3)$ lies in $\text{St}(v_3, K)$, $g$ is a well-defined element in $L(K)$. Note that $g$ agrees with the identity map outside the region $\text{St}(v_3, K)$ in $D$ and $g$ agrees with $f$ on the triangles $\langle v_0, v_1, v_2 \rangle$ and $\langle v_0, v_3, v_2 \rangle$. Also note that the identity map may be connected to $g$ by a straight line homotopy which moves the vertex $v_3$ to the point $f(v_3)$ along a straight line and keeps all the other
vertices of $K$ fixed. Let $\gamma$ be the path in $L(K)$ corresponding to this homotopy. $\gamma$ is a path from the identity map to $g$. Observe that $f \circ g^{-1}$ is an element in the space $L(g(K))$ which is pointwise fixed on the triangles $\langle g(v_0), g(v_1), g(v_2) \rangle$ and $\langle g(v_3), g(v_4), g(v_5) \rangle$. We may therefore view $f \circ g^{-1}$ as a map on the smaller disk $D'$ obtained from $D$ by cutting off the triangles $\langle g(v_0), g(v_1), g(v_2) \rangle$ and $\langle g(v_3), g(v_4), g(v_5) \rangle$. The map $f \circ g^{-1}$ is clearly linear with respect to the triangulation $K_r$ on $\Pi$ inherited from the triangulation $g(K)$ of $D$. The fact that $g(v_3) = f(v_3)$ lies in the interior of the triangular region $\langle v_0, v_1, v_2 \rangle$ implies that the disk $D'$ is still strictly convex. Since the triangulation $K'$ has a fewer number of vertices than $K$ ($v_0$ belongs to $K$ but not $K'$), we may apply the induction hypothesis on the space $L(K')$ to get a path $\tau'$ from the identity map of $D'$ to the map $f \circ g^{-1}$ of $D'$. This path $\tau'$ then gives rise to a path $\tau$ in the space $L(K)$ from the identity map to the map $f \circ g^{-1}$. Note that $\tau \circ g$ is then a path in the space $L(K)$ from the map $g$ to the map $f$. We may then connect the identity map to the map $f$ in the space $L(K)$ by the path $\gamma$ followed by $\tau \circ g$.

Case 2. The vertex $f(v_3)$ lies on the line segment $\langle v_1, v_2 \rangle$. In §4 (Corollary 4.3), we shall prove that the vertex $f(v_3)$ may be moved slightly off the line segment $\langle v_1, v_2 \rangle$ to produce a path $\gamma_1$ in $L(K)$ connecting the element $f$ with an element $f' \in L(K)$ where $f'(v) = f(v)$ for each $v \neq v_3$ in $K$ and $f'(v_3)$ lies inside the triangular region $\langle v_0, v_1, v_2 \rangle$. Then by Case 1, $f'$ may be connected to the identity element of $L(K)$ by a path $\gamma_2$. Then $\gamma_1$ followed by $\gamma_2$ will connect $f$ to the identity element.

Case 3. The vertex $f(v_3)$ lies outside the closed triangular region $\langle v_0, v_1, v_2 \rangle$. By assumption, $St(v_3, K)$ is a convex open set in the plane. It therefore contains a neighborhood of the open line segment $(v_1, v_2)$, and hence, contains points outside the triangle $\langle v_0, v_1, v_2 \rangle$. Moving the vertex $v_3$ to any such point if necessary, we may assume that the vertex $v_3$ is outside the triangular region $\langle v_0, v_1, v_2 \rangle$.

Now, let $M$ denote the set of all maps in $L(K)$ which carry $v_3$ outside the triangular region $\langle v_0, v_1, v_2 \rangle$. By the above assumptions, both $f$ and the identity element of $L(K)$ are in $M$. Therefore, it suffices to prove that $M$ is pathwise connected. We shall do this by showing that $M$ is the homeomorphic image of a map $j$ from some pathwise connected space into the space $L(K)$. Such a pathwise connected space and map $j$ are defined as follows: Let $D'$ be the polygonal disk obtained from $D$ by cutting off the triangle $\langle v_0, v_1, v_2 \rangle$. Note that $D'$ is still a convex polygonal disk. We now describe a triangulation $K'$ of $D'$. Outside the triangular region $\langle v_0, v_1, v_2 \rangle$, we
let $K'$ be the same as the triangulation $K$. While on that triangular region, we let \( \langle v_1, v_2, v_3 \rangle \) be a 2-simplex of $K'$. Note that this determines a proper triangulation $K'$ of $D'$ which has one fewer vertex $K$ (for $v_0$ belongs to $K$ but not to $K'$). By the induction hypothesis, $L(K')$ is pathwise connected. We now describe a map $j: L(K') \to L(K)$. Consider $D'$ as a subset of $D$. Since the disk $D$ is convex, for each element $g \in L(K')$, observe that the open line segment $(g(v_0), g(v_3))$ lies completely in the quadrilateral $[v_0, v_1, v_2, v_3]$. We may therefore define an element $f \in L(K)$ by $f(v) = g(v)$ for all vertices $v \in K'$ and $f(v_0) = v_0$ (i.e., $f$, as a map on the whole disk $D$, is obtained from $g$ by deleting the 1-simplex $\langle v_1, v_2 \rangle$ from both $K'$ and the image $g(K')$, and then inserting the 1-simplex $\langle v_0, g(v_3) \rangle$ to the domain space and the 1-simplex $\langle v_0, v_3 \rangle$ to the image). The map $f$ is a well-defined element in $L(K)$. We let $f = j(g)$. Since for each vertex $v$ of $K'$ and for any two elements $g_1, g_2 \in L(K')$, dist $(g_1(v), g_2(v)) = \text{dist} (j(g_1)(v), j(g_2)(v))$, the map $j$ is continuous (in fact, it is an isometry). Also note that for each $g \in L(K')$, the vertex $j(g)(v_0)$ is outside the triangular region $\langle v_0, v_1, v_2 \rangle$. Hence, $j(g)$ belongs to the subset $M$. Conversely, we may easily show that each element $f \in M$ is the image $j(g)$ of some $g \in L(K')$. This shows that $M$ is the homeomorphic image of the pathwise connected space $L(K')$ under the map $j$. $M$ must be pathwise connected. This finishes the proof of Theorem A.

3. Deforming arbitrary p.l. homeomorphisms. In this section, we shall show that each arbitrary p.l. homeomorphism on a convex polygonal disk $D$ is deformable into the identity map in the space $L(K)$ for some triangulation $K$ of $D$. We do this by first showing that each p.l. homeomorphism on $D$ is deformable into a proper one. The deformation process may then be completed by using Theorem A. We start with a trivial observation which will be needed in the deformation process.

**Lemma 3.1.** Let $D$ be a convex polygonal disk in $R^2$ and $K$ be an arbitrary triangulation of $D$. For each subdivision $K'$ of $K$, the space $L(K)$ may be considered in a natural way as a subspace of the space $L(K')$.

**Proof.** Observe that each element $f$ in $L(K)$ is also linear with respect to the triangulation $K'$, and hence, may be considered as an element of $L(K')$.

**Proposition 3.2.** Let $K$ be an arbitrary triangulation of a convex polygonal disk $D$ in $R^2$. For each element $f \in L(K)$, there exist triangulations $K_1, K_2$ of $D$ such that
1. $K_1$ is a proper triangulation of $D$. $K_2$ is a common subdivision of $K$ and $K_1$. (Hence, $L(K_1)$ and $L(K)$ may both be considered as subspaces of $L(K_2)$.)

2. The element $f$ may be deformed in $L(K_2)$ into an element $f'$ contained in $L(K)$.

Outline of the proof. When $D$ is a triangle, this theorem is proved in detail in [5, Theorem 1.3]. In fact, we showed there how to deform, instead of a single element $f$ in $L(K)$, a compact set of elements in $L(K)$. The same proof, with obvious modifications, also works here. We shall therefore only give a brief outline of the proof.

For a sufficiently small positive number $\delta$, we let $D_\delta$ be the disk lying inside $D$ and concentric to $D$ (for some point in Int $(D)$) such that the perpendicular distances between the corresponding parallel sides of $D$ and $D_\delta$ are $\delta$. We also let $B_\delta$ be the annular strip $Cl (D - D_\delta)$ where $Cl (X)$ means the closure of $X$ with respect to $D$. Each $D_\delta$ gives rise to a decomposition of $D$ into a rectilinear cell complex (in the sense of [7, p. 74] or [6, p. 5]) which is obtained by letting the disk $D_\delta$ be a 2-cell and by cutting the region $B_\delta$ into 2-cells by connecting each vertex of $D$ to the closest vertex of $D_\delta$ by a 1-cell. We denote this cell complex by $R(\delta)$.

Let $K$ be an arbitrary triangulation of the disk $D$ and consider an arbitrary element $f \in L(K)$. If $K$ is proper, we simply let $K_1$ be $K$ and $f'$ be $f$ and there is nothing to prove. Hence, we may assume that $K$ is not proper. We first fix a number $\alpha > 0$ such that

1. All the inner vertices of $K$ are contained in $\text{Int} (D_\alpha)$.
2. If $q$ is an inner 1-simplex of $K$ with both vertices on $\text{Bd} (D)$, then $q \cap \text{Int} (D_\delta) \neq \emptyset$.

We then fix a sufficiently small number $\delta (0 < \delta < \alpha)$ and let $J(\delta)$ be the rectilinear cell complex obtained by imposing the two cell complexes $R(\alpha)$ and $R(\delta)$ on $K$ (i.e., each cell in $J(\delta)$ is an intersection of cells of $R(\alpha)$, $R(\delta)$, and $K$). Finally, we let $K(\delta)$ be a simplicial subdivision of $J(\delta)$ without adding any more vertices (i.e., we get $K(\delta)$ by adding a number of diagonals to each 2-cell of $J(\delta)$ which is not a triangle). $K(\delta)$ is then a simplicial subdivision of $K$, hence, $f \in L(K(\delta))$.

The idea of the proof is this. We first deform the element $f$ in the space $L(K(\delta))$ into an element $f'$ such that $f'|B_\delta$ is the identity map of $B_\delta$. We then observe that any such $f'$ may be considered as a linear homeomorphism with respect to a proper triangulation $K_1$ of $D$. If we let $K_2$ be a common simplicial subdivision of $K$ and $K(\delta)$, the triangulations $K_1$ and $K_2$ will clearly satisfy all the conditions of the proposition.

A deformation $F$ carrying $f$ into an $f'$ described above can be
defined as follows: For each vertex $v$ of $K(\delta)$ which does not lie on the polygon $Bd(D_0)$, we let $F(v, t) = f(v)$ for all $t$ (i.e., we fix the image of $f$ on the set $f(D_0)$ and on $Bd(D)$). For all the vertices $v$'s of $K(\delta)$ which lie on $Bd(D_0)$, we pull all the $f(v)$'s simultaneously back to the $v$'s. It can be shown that if $\delta$ is sufficiently small (e.g., $\delta$ satisfies the conditions given in [5, Definition 3.5]) this will give us a deformation in the space $K(\delta)$. The resulting map $f'$ then agrees with $f$ on the set $f(D_0)$ and $f'|B_\delta = \text{identity map}$. The reader is referred to [5] for details.

We then observe that $f'$ may be considered as an element in $L(K_1)$ for some proper triangulation $K_1$ of $D$. We now describe such a proper triangulation. We let $K_1$ be identical with $K(\delta)$ on the closed region $D_0$ of $D$. While on the annular strip $B_\delta$, we first cut the strip into a number of 2-cells by the cell decomposition $B(\delta)$ on this strip (each of these 2-cells will then be a quadrilateral). We then triangulate the strip by cutting each of these 2-cells into triangles in the following manner: Pick a vertex $v$ of the 2-cell which is also a vertex of $K(\delta)$. Note that exactly one of the sides of this 2-cell lies in the polygon $Bd(D_0)$. Let $v, v_2, \ldots, v_k$ be vertices of $K(\delta)$ lying on this side of the 2-cell. We shall then cut the 2-cell into triangles by inserting a diagonal from $v$ to each vertex $v_i$ for $i = 1, 2, \ldots, k$. This gives us a desired proper triangulation $K_1$.

We contend that the map $f'$ belongs to $K_1$. Observe that $f'$ is linear on each simplex $\sigma$ of $K_1$ lying in $D_0$, for $\sigma$ is also a simplex of $K(\delta)$. $f'$ is also linear on each simplex $\delta$ of $K_1$ lying on $B_\delta$, for $f'$ is the identity map there. Hence $f' \in L(K_1)$. The proposition then follows by letting $K_2$ be a common simplicial subdivision of $K_1$ and $K(\delta)$.

**Theorem 3.3.** Let $K$ be an arbitrary triangulation of a convex polygonal disk $D$ in $\mathbb{R}^2$. For each linear homeomorphism $f \in L(K)$, there exists a simplicial subdivision $K'$ of $K$ and path $\gamma: I \rightarrow L(K')$ such that $\gamma(0) = f$ and $\gamma(1) = \text{the identity map of } D$.

**Proof.** Consider any triangulation $K$ and any $f \in L(K)$. By Proposition 3.2, we may find a proper triangulation $K_1$ of $D$ and a triangulation $K_2$ which is a common subdivision of $K$ and $K_1$ such that there is a path $\gamma_1$ in $L(K_2)$ connecting $f$ to some element $f'$ of $L(K_1)$. By Theorem A, we may also get a path $\gamma_2$ in the space $L(K_1)$ from the element $f'$ to the identity map of $D$. Since $K_2$ is a subdivision of $K_1$, the path $\gamma_2$ is also a path in the space $L(K_2)$. We may therefore set $K' = K_2$. Using the path $\gamma_1$ followed by $\gamma_2$, we get a path in the space $L(K')$ connecting $f$ to the identity map of $D$. 
COROLLARY 3.4. Each p.l. homeomorphism of a convex polygonal
disk $D$ in $\mathbb{R}^2$ is deformable to the identity map of $D$ in the space
$L(K)$ for some triangulation $K$ of $D$.

4. Decomposing a deformation into a finite sequence of single
moves. In this section, we shall prove our Theorem B of §1. Let
$D$ be a convex polygonal disk $D$ in $\mathbb{R}^2$. We first make some observa-
tions on the topology of the space $L(K)$ for an arbitrary triangulation
$K$ of $D$.

REMARK 4.1. For each triangulation $K$ of $D$, observe that the
space $L(K)$ is metrizable, say with a metric $\gamma(f, g) = \max_{v \in S} \{d(f(v),
g(v))\}$, where $S$ is the set of inner vertices of $K$ and $d$ is the sup-
metric of the Euclidean plane i.e., $d((x_1, y_1), (x_2, y_2)) = \max_{i=1,2} |x_i - y_i|$.

PROPOSITION 4.2. Let $K$ be any triangulation of $D$ with $k$ inner
vertices. The space $L(K)$ may be identified with an open subset of
$\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ ($k$ copies).

Proof. In the following, all the maps from $D$ into $\mathbb{R}^2$ are assumed
to be pointwise fixed on $\text{Bd}(D)$. By a linear map $f: K \to \mathbb{R}^2$, we
mean a map $f: D \to \mathbb{R}^2$ (not necessarily a homeomorphism into) which
is linear with respect to each simplex of $K$. First observe that each
linear map from $K$ into $\mathbb{R}^2$ is completely determined by its images
of the inner vertices of $K$. Suppose that an ordering is assigned to the
inner vertices, say $v_1, v_2, \ldots, v_k$ of $K$. The correspondence between
a linear map $f: K \to D$ and the point $(f(v_1), f(v_2), \ldots, f(v_k)) \in \mathbb{R}^2 \times
\mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ ($k$ copies) gives rise to a one-to-one correspondence
between the set of all such linear maps and $\mathbb{R}^{2k}$. Under this corre-
spondence, the space $L(K)$ is identified with a subset of $\mathbb{R}^{2k}$. This
identification is indeed a homeomorphism of $L(K)$ into $\mathbb{R}^{2k}$. In fact,
it is an isometry into $\mathbb{R}^{2k}$ if $L(K)$ is given the metric $\gamma$ and $\mathbb{R}^{2k}$ is
given the sup-metric of the Euclidean space.

To see that the image of $L(K)$ in $\mathbb{R}^{2k}$ is an open subset, we need
only show that any linear map $g$ from $K$ to $\mathbb{R}^2$ belongs to $L(K)$ if
it is sufficiently close to some element of $L(K)$ under the metric $\gamma$.
This can be done by means of a result of J. H. C. Whitehead, that
a sufficiently close $C^1$ approximation to an immersion from a complex
into a manifold is itself an immersion [11] (also see [7, Theorem 8.8]).
However, we shall sketch a more elementary argument here. Observe
that a linear map $g$ from $K$ into $\mathbb{R}^2$ belongs to $L(K)$ if and only if
the image under $g$ of each 2-simplex has positive area (i.e., nonzero,
nonnegative area) (cf. 5, Lemma 2.1]). If $g$ has this property, any
linear map: $K \to \mathbb{K}$ sufficiently close to $g$ must also have this property.
This shows that \( L(K) \) must correspond to an open subset of \( \mathbb{R}^{2k} \). This finishes the proof.

The following corollary was needed in §1. It follows immediately from the Proposition 4.2 and the fact that Euclidean spaces are locally pathwise connected.

**Corollary 4.3.** For each triangulation \( K \) of \( D \), the space \( L(K) \) is locally pathwise connected. In fact, for each \( f \in L(K) \), there exists a number \( \delta > 0 \) such that for each linear map \( g : K \to \mathbb{R}^2 \) with \( g \mid \text{Bd}(D) = \text{identity} \) and \( d(g(v), f(v)) < \delta \) for all inner vertices \( v \) of \( K \), the map \( g \) belongs to \( L(K) \) and may be connected to \( f \) by a path in \( L(K) \).

**Proof of Theorem B.** Consider any p.l. homeomorphism \( f \) of \( D \). By Corollary 3.4, there exists a triangulation \( K \) of \( D \) and a path \( \gamma : I \to L(K) \) such that \( \gamma \) connects \( f \) to the identity map of \( D \). Let \( k \) be the number of inner vertices of \( K \). Using Proposition 4.2, we may find, for each point \( x \in \gamma(I) \) an open rectangular box \( U_x = V_1 \times V_2 \times \cdots \times V_k \) of \( \mathbb{R}^{2k} \) containing \( x \) such that \( U_x \subset L(K) \) and each \( V_i \) is an open rectangle in \( \mathbb{R}^2 \). Let \( \varepsilon \) be the Lebesgue number of the open covering \( \{ \gamma^{-1}(U_i) \mid x \in \gamma(I) \} \) of the unit interval \( I \) and let \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) be a partition of the unit interval such that the length of each closed interval \( [t_{i-1}, t_i] \) is less than \( \varepsilon \).

We now show that for each \( i = 1, \ldots, n \), the element \( \gamma(t_{i-1}) \) may be deformed into the element \( \gamma(t_i) \) in the space \( L(K) \) in a finite number of steps by successively moving the vertices of \( \gamma(t_{i-1}) \). Consider any particular \( i (1 \leq i \leq n) \). Let \( \gamma(t_{i-1}) = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^{2k} \) and \( \gamma(t_i) = (b_1, b_2, \ldots, b_k) \in \mathbb{R}^{2k} \). We may choose an open rectangular box \( U = V_1 \times V_2 \times \cdots \times V_k \) in \( L(K) (\subset \mathbb{R}^{2k}) \) containing both \( \gamma(t_{i-1}) \) and \( \gamma(t_i) \). Since the whole rectangular box \( U \) is contained in \( L(K) \), we may move \( \gamma(t_{i-1}) \) to \( \gamma(t_i) \) (considered as points of \( \mathbb{R}^{2k} \)) within the set \( L(K) \) by a sequence of \( k \) moves such that for each \( j = 1, 2, \ldots, k \), the \( j \)-th move carries the point \( (b_1, b_2, \ldots, b_{j-1}, a_j, \ldots, a_k) \) to the point \( (b_1, \ldots, b_{j-1}, a_j, \ldots, a_k) \). Under the identification of \( L(K) \) as an open subset of \( \mathbb{R}^{2k} \), each of these \( k \)-moves clearly corresponds to a deformation from an element \( g \) to some other element in \( L(K) \) by moving a single vertex of \( g(K) \). Hence, \( \gamma(t_i) \) may be reached from \( \gamma(t_{i-1}) \) by finitely many such single moves. This finishes the proof of Theorem B.

Theorem B also allows an algebraic interpretation as follows: Let \( D \) be a convex polygonal disk in \( \mathbb{R}^2 \) and \( \text{PL}(D) \) be the set of all p.l. homeomorphisms of \( D \) onto \( D \). Observe that \( \text{PL}(D) \) forms a group with respect to the composition of functions. For each triangulation \( K \) of \( D \), we let \( S(K) = \{ f \in L(K) \mid \text{there is a vertex} \)
v \in K$ such that $f \mid (D - St(v, K)) = \text{identity}$. It can be shown with the help of Remark 2.6 that for each triangulation $K$ of $D$, the set $S(K)$ consists of all elements of $L(K)$ which may be obtained by moving a single vertex of $K$. Now let $S(D) = \bigcup \{S(K) \mid K$ a triangulation of $D \}$. The set $S(D)$ may be called the set of single moves. Theorem B says that for each p.l. homeomorphism $f$ of $D$, there exists a triangulation $K$ of $D$ and a finite sequence of elements $f_1, f_2, \ldots, f_m \in S(D)$ such that

1. $f_1 \in S(K)$ and $f_{i+1} \in S(f_i(\cdots (f_1(K)) \cdots))$ for each $i = 1, \ldots, m - 1$.
2. $f = f_m \circ f_{m-1} \circ \cdots \circ f_1$.

In particular, each element of $\text{PL}(D)$ is a finite product of elements of $S(D)$. Hence, we have the following.

**Theorem 4.4.** For any convex polygonal disk $D$ in $\mathbb{R}^2$, the group $\text{PL}(D)$ is generated by the subset $S(D)$ of single moves.

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