REMARK ON MAPPINGS NOT RAISING DIMENSION OF CURVES

JOZEF KRASINKIEWICZ
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J. KRASINKIEWICZ

The purpose of this note is to prove three theorems on dimension raising ability of certain classes of maps defined on 1-dimensional continua. In particular we obtain a generalization of a recent result of J. Jobe concerning dimension raising ability of inverse arc functions defined on dendrites.

By a continuum we mean a compact connected metric space. A 1-dimensional continuum is called a curve. If each point of a continuum $X$ has arbitrary small neighborhood with finite boundary, then $X$ is said to be regular. $X$ is suslinian provided any collection of mutually disjoint nondegenerate subcontinua of $X$ is at most countable [6]. For a nondegenerate continuum we have the following implications:

(i) (regular) $\Rightarrow$ (suslinian) $\Rightarrow$ (1-dimensional).

Let $f$ be a mapping of a continuum $X$ into a continuum $Y$. We shall consider the following properties of $f$:

$\alpha)$ for every arc $L \subseteq Y$ there exists an arc $M \subseteq X$ which is mapped by $f$ onto $L$, i.e., $f(M) = L$.

$\beta)$ for every arc $L \subseteq Y$ there exists a continuum $M \subseteq X$ which is mapped by $f$ onto $L$.

$\gamma)$ for every continuum $L \subseteq Y$ there exists a continuum $M \subseteq X$ which is mapped by $f$ onto $L$.

**Theorem 1.** If $f$ is a mapping with property $\beta$ of a suslinian continuum $X$ onto a locally connected continuum $Y$, then $Y$ is suslinian.

**Proof.** Suppose it is not true. Then there is an uncountable collection $\{B\}$ of nondegenerate mutually disjoint subcontinua of $Y$. Consider a member $B \in \{B\}$. Let $a$ and $b$ be distinct points of $B$. Let $U_1, U_2, \cdots$ be a decreasing sequence of neighborhoods of $B$ (in $Y$) which limits on $B$, i.e.,

\[(1) \quad \bigcap_n U_n = B.\]

For each positive integer $n$ there is a locally connected continuum $C_n$ such that

\[(2) \quad B \subseteq C_n \subseteq U_n \quad \text{(see [5], p. 260).}\]
Let $L_n$ be an arc in $C \%$ joining $a$ and $b$. We may assume that $\{L_n\}$ is a convergent sequence (otherwise we take a convergent subsequence). Let $B'$ denote the limit of this sequence. Hence by (1) and (2) we have

\[(3) \quad B' \text{ is a nondegenerate subcontinuum of } B \text{ (because } a, b \in B').\]

For each integer $n$ there is a continuum $A_n \subset X$ which is mapped by $f$ onto $L_n$. Choose a convergent subsequence of $\{A_n\}$ and let $A_B$ be its limit. It is clear that

\[(4) \quad f(A_B) = B'.\]

According to (3) and (4) we see that for each $B \in \{B\}$ we can construct a nondegenerate continuum $A_B \subset X$ which is mapped by $f$ onto a subcontinuum of $B$. It follows that $\{A_B : B \in \{B\}\}$ constitute an uncountable collection of nondegenerate mutually disjoint subcontinua of $X$, contrary to our assumption on $X$. This proves the theorem.

Mappings with property $\alpha$ were considered by J. Jobe in [3] (where they are called inverse arc functions). There was shown that if $f$ is a mapping with property $\alpha$ from a dendrite $X$ with countably number of endpoints onto $Y$, then $\dim X \leq 1$ (dendrite = locally connected continuum containing no simple closed curve). J. Jobe asks if the above result can be extended onto all dendrites. Since $\alpha \Rightarrow \beta$, then the following corollary to Theorem 1 answers this question in the affirmative.

**COROLLARY.** If $f$ is a mapping with property $\beta$ defined on a dendrite $X$, then $f(X)$ is at most 1-dimensional.

**Proof.** Clearly, $f(X)$ is a locally connected continuum. Since each dendrite is regular ([5], p. 301), the corollary is an immediate consequence of (i) and Theorem 1.

We are now going to prove two theorems related to the above corollary.

Let $D$ be the unit disk in the complex plane and let $S$ denote the boundary of $D$. A mapping $f : X \to D$ is called essential in the sense of Alexandroff-Hopf, briefly: $AH$-essential, provided the partial mapping

$$f \mid f^{-1}(S) : f^{-1}(S) \to S$$

can not be extended onto $X$. It is known that

\[(ii) \quad \text{If } X \text{ is compact and } \dim X \leq 2, \text{ then there exists an } AH-\text{essential map of } X \text{ onto } D \text{ (see [7]).}\]
By a classical result of Mazurkiewicz [7] we have

(iii) An $AH$-essential map has property ($\gamma$).

A space $X$ is said to be contractible with respect to $S$, briefly: $cr\ S$, if each map $f: X\rightarrow S$ is nullhomotopic. It is well known that

(iv) Each closed subset of a $cr\ S$ curve is $cr\ S$ ([2], p. 83).

It has been proved by M. K. Fort, Jr. [1] that there exists a continuum $K\subset D$ such that

(v) No continuum $cr\ S$ can be mapped onto $K$.

Using these facts we shall prove the following

**Theorem 2.** If $X$ is a $cr\ S$ curve and $f: X\rightarrow Y$ has property ($\gamma$), then $\dim Y \leq 1$.

**Proof.** Suppose $\dim Y \geq 2$. Hence by (ii) there is an $AH$-essential map $g: Y\rightarrow D$. Since the composition of two maps having property ($\gamma$) is a map with property ($\gamma$), then by (iii) the map $h = gf$ has property ($\gamma$). Let $K\subset D$ be the Fort continuum. There exists a continuum $L\subset X$ such that $h(L) = K$. By (iv), $L$ is $cr\ S$. Hence $K$ can be obtained as a continuous image of a $cr\ S$ continuum, contrary to (v). This contradiction completes the proof.

A continuum $X$ is tree-like if for each $\varepsilon > 0$ there exist a finite tree $T$ and a continuous map $f: X\rightarrow T$ onto $T$ such that $\text{diam} f^{-1}(t) < \varepsilon$ for every $t \in T$. It is known that every tree-like continuum is $cr\ S$. Recently the author has proved that if $Y$ is a $cr\ S$ curve and if there exists a tree-like curve which can be mapped onto $Y$, then $Y$ is tree-like [4]. Combining these results with Theorem 2 we obtain

**Theorem 3.** Let $f$ be a mapping from a tree-like curve onto a continuum $Y$. If $f$ has property ($\gamma$) and $Y$ is $cr\ S$, then $Y$ is tree-like.

**References**


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