

# Pacific Journal of Mathematics

**REMARK ON MAPPINGS NOT RAISING DIMENSION OF  
CURVES**

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## REMARK ON MAPPINGS NOT RAISING DIMENSION OF CURVES

J. KRASINKIEWICZ

**The purpose of this note is to prove three theorems on dimension raising ability of certain classes of maps defined on 1-dimensional continua. In particular we obtain a generalization of a recent result of J. Jobe concerning dimension raising ability of inverse arc functions defined on dendrites.**

By a continuum we mean a compact connected metric space. A 1-dimensional continuum is called a curve. If each point of a continuum  $X$  has arbitrary small neighborhood with finite boundary, then  $X$  is said to be regular.  $X$  is suslinian provided any collection of mutually disjoint nondegenerate subcontinua of  $X$  is at most countable [6]. For a nondegenerate continuum we have the following implications:

(i) (regular)  $\Rightarrow$  (suslinian)  $\Rightarrow$  (1-dimensional).

Let  $f$  be a mapping of a continuum  $X$  into a continuum  $Y$ . We shall consider the following properties of  $f$ :

( $\alpha$ ) for every arc  $L \subset Y$  there exists an arc  $M \subset X$  which is mapped by  $f$  onto  $L$ , i.e.,  $f(M) = L$ .

( $\beta$ ) for every arc  $L \subset Y$  there exists a continuum  $M \subset X$  which is mapped by  $f$  onto  $L$ .

( $\gamma$ ) for every continuum  $L \subset Y$  there exists a continuum  $M \subset X$  which is mapped by  $f$  onto  $L$ .

**THEOREM 1.** *If  $f$  is a mapping with property ( $\beta$ ) of a suslinian continuum  $X$  onto a locally connected continuum  $Y$ , then  $Y$  is suslinian.*

*Proof.* Suppose it is not true. Then there is an uncountable collection  $\{B\}$  of nondegenerate mutually disjoint subcontinua of  $Y$ . Consider a member  $B \in \{B\}$ . Let  $a$  and  $b$  be distinct points of  $B$ . Let  $U_1, U_2, \dots$  be a decreasing sequence of neighborhoods of  $B$  (in  $Y$ ) which limits on  $B$ , i.e.,

$$(1) \quad \bigcap_n U_n = B.$$

For each positive integer  $n$  there is a locally connected continuum  $C_n$  such that

$$(2) \quad B \subset C_n \subset U_n \quad (\text{see [5], p. 260}).$$

Let  $L_n$  be an arc in  $C_n$  joining  $a$  and  $b$ . We may assume that  $\{L_n\}$  is a convergent sequence (otherwise we take a convergent subsequence). Let  $B'$  denote the limit of this sequence. Hence by (1) and (2) we have

(3)  $B'$  is a nondegenerate subcontinuum of  $B$  (because  $a, b \in B'$ ).

For each integer  $n$  there is a continuum  $A_n \subset X$  which is mapped by  $f$  onto  $L_n$ . Choose a convergent subsequence of  $\{A_n\}$  and let  $A_B$  be its limit. It is clear that

(4)  $f(A_B) = B'$ .

According to (3) and (4) we see that for each  $B \in \{B\}$  we can construct a nondegenerate continuum  $A_B \subset X$  which is mapped by  $f$  onto a subcontinuum of  $B$ . It follows that  $\{A_B: B \in \{B\}\}$  constitute an uncountable collection of nondegenerate mutually disjoint subcontinua of  $X$ , contrary to our assumption on  $X$ . This proves the theorem.

Mappings with property  $(\alpha)$  were considered by J. Jobe in [3] (where they are called inverse arc functions). There was shown that if  $f$  is a mapping with property  $(\alpha)$  from a dendrite  $X$  with countably number of endpoints onto  $Y$ , then  $\dim Y \leq 1$  (dendrite = locally connected continuum containing no simple closed curve). J. Jobe asks if the above result can be extended onto all dendrites. Since  $(\alpha) \Rightarrow (\beta)$ , then the following corollary to Theorem 1 answers this question in the affirmative.

**COROLLARY.** *If  $f$  is a mapping with property  $(\beta)$  defined on a dendrite  $X$ , then  $f(X)$  is at most 1-dimensional.*

*Proof.* Clearly,  $f(X)$  is a locally connected continuum. Since each dendrite is regular ([5], p. 301), the corollary is an immediate consequence of (i) and Theorem 1.

We are now going to prove two theorems related to the above corollary.

Let  $D$  be the unit disk in the complex plane and let  $S$  denote the boundary of  $D$ . A mapping  $f: X \rightarrow D$  is called essential in the sense of Alexandroff-Hopff, briefly: *AH-essential*, provided the partial mapping

$$f|f^{-1}(S): f^{-1}(S) \longrightarrow S$$

can not be extended onto  $X$ . It is known that

(ii) If  $X$  is compact and  $\dim X \geq 2$ , then there exists an *AH-essential* map of  $X$  onto  $D$  (see [7]).

By a classical result of Mazurkiewicz [7] we have

(iii) An  $AH$ -essential map has property  $(\gamma)$ .

A space  $X$  is said to be contractible with respect to  $S$ , briefly:  $cr S$ , if each map  $f: X \rightarrow S$  is nullhomotopic. It is well known that

(iv) Each closed subset of a  $cr S$  curve is  $cr S$  ([2], p. 83).

It has been proved by M. K. Fort, Jr. [1] that there exists a continuum  $K \subset D$  such that

(v) No continuum  $cr S$  can be mapped onto  $K$ .

Using these facts we shall prove the following

**THEOREM 2.** *If  $X$  is a  $cr S$  curve and  $f: X \rightarrow Y$  has property  $(\gamma)$ , then  $\dim Y \leq 1$ .*

*Proof.* Suppose  $\dim Y \geq 2$ . Hence by (ii) there is an  $AH$ -essential map  $g: Y \rightarrow D$ . Since the composition of two maps having property  $(\gamma)$  is a map with property  $(\gamma)$ , then by (iii) the map  $h = gf$  has property  $(\gamma)$ . Let  $K \subset D$  be the Fort continuum. There exists a continuum  $L \subset X$  such that  $h(L) = K$ . By (iv),  $L$  is  $cr S$ . Hence  $K$  can be obtained as a continuous image of a  $cr S$  continuum, contrary to (v). This contradiction completes the proof.

A continuum  $X$  is tree-like if for each  $\varepsilon > 0$  there exist a finite tree  $T$  and a continuous map  $f: X \rightarrow T$  onto  $T$  such that  $\text{diam } f^{-1}(t) < \varepsilon$  for every  $t \in T$ . It is known that every tree-like continuum is  $cr S$ . Recently the author has proved that if  $Y$  is a  $cr S$  curve and if there exists a tree-like curve which can be mapped onto  $Y$ , then  $Y$  is tree-like [4]. Combining these results with Theorem 2 we obtain

**THEOREM 3.** *Let  $f$  be a mapping from a tree-like curve onto a continuum  $Y$ . If  $f$  has property  $(\gamma)$  and  $Y$  is  $cr S$ , then  $Y$  is tree-like.*

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