LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES

M. H. Lim
LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES

M. H. LIM

Let $U$ be an $n$-dimensional vector space over an algebraically closed field $F$ of characteristic zero, and let $V^r U$ denote the $r$th symmetric product space of $U$. Let $T$ be a linear transformation on $V^r U$ which sends nonzero decomposable elements to nonzero decomposable elements. We prove the following:

(i) If $n = r + 1$ then $T$ is induced by a nonsingular transformation on $U$.

(ii) If $2 < n < r + 1$ then either $T$ is induced by a nonsingular transformation on $U$ or $T(V^r U) = V^r W$ for some two dimensional subspace $W$ of $U$.

The result for $n > r + 1$ was recently obtained by L. J. Cummings.

1. Preliminaries. Let $U$ be a finite dimensional vector space over an algebraically closed field $F$. Let $V^r U$ denote the $r$th symmetric product space over $U$ where $r \geq 2$. Unless otherwise stated, the characteristic of $F$ is assumed to be zero or greater than $r$.

A decomposable subspace of $V^r U$ is a subspace consisting of decomposable elements. Let $x_1, \ldots, x_{r-1}$ be $r - 1$ nonzero vectors in $U$. Then the set $\{x_1 \vee \cdots \vee x_{r-1} \vee u : u \in U\}$, denoted by $x_1 \vee \cdots \vee x_{r-1} \vee U$, is a decomposable subspace of $V^r U$ and is called a type 1 subspace of $V^r U$. Let $W$ be a two dimensional subspace of $U$. It is shown in [2] that $V^r W$ is decomposable and is called a type $r$ subspace of $V^r U$. If $y_1, \ldots, y_{r-k}$ are vectors in $U - W$ where $1 < k < r$, then the set $\{y_1 \vee \cdots \vee y_{r-k} \vee w_1 \vee \cdots \vee w_i : w_i \in W, i = 1, \ldots, k\}$, denoted by $y_1 \vee \cdots \vee y_{r-k} \vee W \vee \cdots \vee W$, is also decomposable and is called a type $k$ subspace of $V^r U$. In [2] Cummings showed that every maximal decomposable subspace of $V^r U$ is of type $i$ for some $1 \leq i \leq r$.

A linear transformation on $V^r U$ is called a decomposable mapping if it maps nonzero decomposable elements to nonzero decomposable elements. In [3] Cummings proved that if dim $U > r + 1$ then every decomposable mapping $T$ on $V^r U$ is induced by a nonsingular linear transformation $t$ on $U$; that is, $T(y_1 \vee \cdots \vee y_r) = t(y_1) \vee \cdots \vee t(y_r)$. In this paper we consider the case when $3 \leq \text{dim } U \leq r + 1$.

2. The case when dim $U = r + 1$. Two type 1 subspaces $M_1$ and $M_2$ of $V^r U$ are called adjacent if

$$M_1 = x_1 \vee \cdots \vee x_{r-2} \vee y_1 \vee U$$
$$M_2 = x_1 \vee \cdots \vee x_{r-2} \vee y_2 \vee U$$
The proof of the following lemma is contained in that of Proposition 4 of [3].

**Lemma 1.** The images of two adjacent type 1 subspaces under a decomposable mapping are distinct.

**Theorem 1.** If \( \dim U = r + 1 \) then every decomposable mapping \( T \) of \( V^r U \) is induced by a nonsingular mapping of \( U \).

**Proof.** Let \( M \) be a type 1 subspace of \( V^r U \). Then \( T(M) \) is a decomposable subspace of \( V^r U \). Moreover \( \dim M = \dim T(M) = r + 1 \). Let \( T(M) \subseteq N \) where \( N \) is a maximal decomposable subspace. If \( N \) is of type \( k \) where \( 1 < k < r \), then \( \dim N = k + 1 < r + 1 \) which is a contradiction. Hence \( N \) is of type 1 or type \( r \). Since \( \dim N = r + 1 \), it follows that \( T(M) = N \).

Suppose that some type 1 subspace \( x_1 \vee \cdots \vee x_{r-2} \vee y \vee U \) is mapped onto a type \( r \) subspace \( V^r W \) where \( W \) is a two dimensional subspace of \( U \). We shall show that this leads to a contradiction.

Let \( \mathcal{C} = \{ T(M_u) : u \in U, u \neq 0 \} \) where \( M_u = x_1 \vee \cdots \vee x_{r-2} \vee u \vee U \). We shall show that \( V^r W \) is the only type \( r \) subspace in \( \mathcal{C} \). Suppose there is another type \( r \) subspace \( V^r W^* \) in \( \mathcal{C} \). Since \( V^r W \cap V^r W^* \neq 0 \), \( W \cap W^* \) is 1-dimensional. Choose a nonzero vector \( z \) in \( U \) such that

\[
T(x_1 \vee \cdots \vee x_{r-2} \vee y \vee z) = w_1 \vee \cdots \vee w_r
\]

where \( \dim \langle w_1, \ldots, w_r \rangle = 2 \), \( \langle y \rangle \neq \langle z \rangle \), and \( W \cap W^* \neq \langle w_i \rangle \) for all \( i = 1, \ldots, r \). If

\[
T(M_u) = z_1 \vee \cdots \vee z_{r-1} \vee U
\]

for some \( z_i \) in \( U \) then

\[
T(M_u) \cap V^r W \neq 0
\]

and

\[
T(M_u) \cap V^r W^* \neq 0
\]

imply that \( z_1, \ldots, z_{r-1} \in W \cap W^* \) and hence \( \langle z_i \rangle = \cdots = \langle z_{r-1} \rangle = W \cap W^* \). Since \( w_1 \vee \cdots \vee w_r \in z_1 \vee \cdots \vee z_{r-1} \vee U \), it follows that \( \langle w_i \rangle = W \cap W^* \) for some \( i \), a contradiction. Hence

\[
T(M_u) = V^r S
\]

for some two dimensional subspace \( S \) of \( U \). Note that \( x_1 \vee \cdots \vee x_{r-2} \vee y \vee z \in M_u \cap M_u^* \). Thus \( w_1, \ldots, w_r \in W \cap S \). This implies that \( \langle w_1, \ldots, w_r \rangle = W = S \), a contradiction to Lemma 1 since \( M_u \) and \( M_u^* \).
are adjacent type 1 subspaces. This proves that \( V^r W \) is the only
type \( r \) subspace in \( V \).

Since \( \{ T(M_x) : \langle x \rangle \neq \langle y \rangle, x \neq 0 \} \) is an infinite family of type 1
subspaces (Lemma 1) it follows from Proposition 4 of [3] that there
exist vectors \( u_i, \ldots, u_{r-2} \) such that for any \( x \in U - \{0\} \) and \( \langle x \rangle \neq \langle y \rangle \),
\[
T(M_x) = u_1 \vee \cdots \vee u_{r-2} \vee x' \vee U
\]
for some \( x' \in U \). Since \( T(M_x) \cap V' W \neq 0 \) we have \( x' \in W \). Let \( g \) be a fixed nonzero vector such that \( \langle g \rangle \neq \langle y \rangle \). Then for any \( x \in U - \{0\} \) such that \( \langle x \rangle \neq \langle g \rangle \), \( \langle x \rangle \neq \langle y \rangle \),
\[
T(x_1 \vee \cdots \vee x_{r-2} \vee x \vee g) = u_1 \vee \cdots \vee u_{r-2} \vee x' \vee g_x
\]
for some \( g_x \). Since \( u_1 \vee \cdots \vee u_{r-2} \vee x' \vee g_x \in U \) and \( \langle x' \rangle \neq \langle g' \rangle \) we have \( \langle g_x \rangle = \langle g' \rangle \). Therefore
\[
T(M_g) \subseteq u_1 \vee \cdots \vee u_{r-2} \vee g' \vee W
\]
\[
\cup \langle T(x_1 \vee \cdots \vee x_{r-2} \vee g \vee y) \rangle
\]
\[
\cup \langle T(x_1 \vee \cdots \vee x_{r-2} \vee g \vee g) \rangle .
\]
This is impossible since \( \dim T(M_g) = \dim U > 2 \).

Therefore, \( T \) maps type 1 subspaces to type 1 subspaces. By
Theorem 2 of [3] \( T \) is induced by a nonsingular linear transformation
on \( U \).

3. The case when \( 3 \leq \dim U < r + 1 \). In this section we assume
that \( \text{char } F = 0 \).

**Lemma 2.** Let \( x_1, \ldots, x_k \) be \( k \) nonzero vectors of \( U \). Let \( r > k + 1 \) and \( x_1 \vee \cdots \vee x_k \vee A = z_1 \vee \cdots \vee z_r \neq 0 \) in \( V' U \) where \( A \in \lambda V^{r-k} U \) and \( z_t \in U \). Then \( \langle x_i \rangle = \langle z_{j_i} \rangle \) for some \( j_i \) where \( j_i \neq j \), for
distinct \( s \) and \( t \).

**Proof.** Let \( u_1, \ldots, u_n \) be a basis of \( U \). Let \( \phi \) be the isomorphism
from the symmetric algebra \( V U \) over \( U \) onto the polynomial algebra
\( F[\xi_1, \ldots, \xi_n] \) in \( n \) indeterminates \( \xi_1, \ldots, \xi_n \) over \( F \) such that \( \phi(u_i) = \xi_i \),
\( i = 1, \ldots, n \) [4, p. 428]. Then
\[
\phi(x_1) \cdots \phi(x_k) \phi(A) = \phi(z_1) \cdots \phi(z_r) \neq 0 .
\]
Since \( F[\xi_1, \ldots, \xi_n] \) is a Gaussian domain and since \( \phi(x_1), \ldots, \phi(x_k), \phi(z_1), \ldots, \phi(z_r) \) are linear homogeneous polynomials, it follows that for
each \( i = 1, \ldots, k \), \( \langle \phi(x_i) \rangle = \langle \phi(z_{j_i}) \rangle \) for some \( j_i \) where \( j_i \neq j \), if \( s \neq t \).
This implies that \( \langle x_i \rangle = \langle z_{j_i} \rangle \). Hence the lemma is proved.

The following result is proved in [1, p. 131] under the assumption
that \( \text{char } F = 0 \).
LEMMA 3. \( V^r U \) is spanned by \( \{ u^r = u \lor \cdots \lor u : u \in U \} \).

Hereafter we will assume that \( 3 \leq \dim U < r + 1 \) and \( T \) is a decomposable mapping on \( V^r U \). Since every type \( k \) subspace has dimension \( < r + 1 \) where \( 1 \leq k < r \) we see that every type \( r \) subspace of \( V^r U \) is mapped onto a type \( r \) subspace under \( T \).

LEMMA 4. If there are two distinct type \( r \) subspaces \( M \) and \( N \) of \( V^r U \) such that \( M \cap N \neq 0 \) and \( T(M) = T(N) \), then \( T(V^r U) = T(M) \).

Proof. Let \( M = V^r S_1 \), \( N = V^r S_2 \) and \( T(M) = T(N) = V^r S \) where \( S, S_1, S_2 \) are two dimensional subspaces of \( U \). By hypothesis,
\[
M \cap N = V^r S_1 \cap V^r S_2 = V^r (S_1 \cap S_2) \neq 0.
\]
Hence \( S_1 \cap S_2 \) is one dimensional. Let \( S_1 = \langle y_1, y_2 \rangle \), \( S_2 = \langle y_1, y_3 \rangle \). Consider \( S_3 = \langle y_2, y_3 \rangle \). Then
\[
V^r S_3 \cap V^r S_2 = \langle y_3 \rangle, \quad V^r S_3 \cap V^r S_1 = \langle y_2 \rangle.
\]
Hence \( T(V^r S_1) \cap V^r S \supseteq \langle T(y_1), T(y_2) \rangle \). Since \( T \) is a decomposable mapping and \( \langle y_2, y_3 \rangle \) is a two dimensional decomposable subspace, it follows that \( \langle T(y_1), T(y_3) \rangle \) is two dimensional. Hence \( T(V^r S_3) = V^r S \) because any two distinct type \( r \) subspaces of \( V^r U \) have at most one dimension in common.

Let \( z = \alpha y_1 + \beta y_2 + \gamma y_3 \) where \( \alpha, \beta, \gamma \) are all nonzero scalars. Consider \( S_4 = \langle y_1, z \rangle = \langle y_1, \beta y_2 + \gamma y_3 \rangle \). Since
\[
V^r S_4 \cap V^r S_2 \supseteq \langle (\beta y_2 + \gamma y_3)^r \rangle, \\
V^r S_4 \cap V^r S_1 \supseteq \langle y_1^r \rangle,
\]
we have \( T(V^r S_4) \cap V^r S \supseteq \langle T(y_1), T((\beta y_2 + \gamma y_3)^r) \rangle \) which is two dimensional. Hence \( T(V^r S_4) = V^r S \). Consequently by Lemma 3, \( T(V^r \langle y_1, y_2, y_3 \rangle) = V^r S \).

Now, let \( w \in U \) such that \( w \in \langle y_1, y_2, y_3 \rangle \). Let \( W = \langle y_1, w \rangle \). Consider the type 1 subspace \( P = y_1 \lor \cdots \lor y_1 \lor U \). Since
\[
\dim (P \cap V^r \langle y_1, y_2, y_3 \rangle) = 3,
\]
we have \( \dim (T(P) \cap V^r S) \geq 3 \). Since the maximal dimension of the intersection of two distinct maximal decomposable subspaces is 2, we conclude that \( T(P) \subseteq V^r S \). This shows that
\[
T(V^r W) \cap V^r S \supseteq \langle T(y_i), T(y_1 \lor \cdots \lor y_1 \lor w) \rangle.
\]
Since \( \langle y_i^r, y_i^{-1} \lor w \rangle \) is a two dimensional decomposable subspace, \( \langle T(y_i^r), T(y_i^{-1} \lor w) \rangle \) is also two dimensional. Hence \( T(V^r W) = V^r S \). By Lemma 3, we conclude that \( T(V^r U) = V^r S \). This completes the proof.
LEMMA 5. Suppose that for any two distinct type $r$ subspaces $M, N$ such that $M \cap N \neq 0$, we have $T(M) \neq T(N)$. Then $T$ is induced by a nonsingular transformation on $U$.

Proof. Let $y_1, y_2, y_3$ be linearly independent vectors. Let $S_1 = \langle y_1, y_2 \rangle$, $S_2 = \langle y_1, y_3 \rangle$. Then $T(V^r S_1) = V^r S'_1$ and $T(V^r S_2) = V^r S'_2$ for some two dimensional subspaces $S'_1, S'_2$ of $U$. By hypothesis $V^r S'_1 \neq V^r S'_2$. Hence

$$V^r S'_1 \cap V^r S'_2 = T(V^r S_1 \cap V^r S_2) = \langle y' \rangle$$

for some $y' \in U$. Therefore $T(y^r) = \lambda y'^r$ for some $\lambda$ in $F$.

Let $H = y \vee \ldots \vee y \vee U$. We claim that $T(H) = y' \vee \ldots \vee y' \vee U$. Since $T(H)$ is a decomposable subspace, it is contained in a maximal decomposable subspace. If $T(H)$ is contained in a type $k$ subspace $z_1 \vee \ldots \vee z_{r-k} \vee W \vee \ldots \vee W$ where $2 \leq k < r$, then $y^r \in z_1 \vee \ldots \vee z_{r-k} \vee W \vee \ldots \vee W$ and hence $\langle z_1 \rangle = \langle y^r \rangle$, $y^r \in W$. This implies $g_1 \in W$, a contradiction. If $T(H)$ is contained in a type $r$ subspace $V^r W$, then

$$\dim (V^r S_1 \cap H) = 2 \implies \dim (T(V^r S_1) \cap V^r W) \geq 2,$$

$$\dim (V^r S_2 \cap H) = 2 \implies \dim (T(V^r S_2) \cap V^r W) \geq 2.$$

Since $T(V^r S_1)$ and $T(V^r S_2)$ are both type $r$ subspaces, it follows that $T(V^r S_1) = V^r W = T(V^r S_2)$, a contradiction to our hypothesis. Hence $T(H)$ is a type 1 subspace of $V^r U$. Since $y^r \in T(H)$, it follows that

$$T(H) = y' \vee \ldots \vee y' \vee U.$$

By Lemma 3, let $x_i^{r-i}, \ldots, x_i^{r-1}$ be a basis of $V^{r-1} U$. Note that $3 \leq \dim U < r + 1$ implies that $r \geq 3$. Clearly if $i \neq j$ then $x_i$ and $x_j$ are linearly independent. Consider any type one subspace $D = z_1 \vee \ldots \vee z_{r-1} \vee U$. Let $z_1 \vee \ldots \vee z_{r-1} = \sum_{i=1}^{r-1} \lambda_i x_i^{r-1}$ where $\lambda_i \in F$ and $i = 1, \ldots, t$. We shall show that $T(D)$ is a type 1 subspace. Suppose to the contrary that

(i) $T(D) \subseteq V^r S$

or

(ii) $T(D) \subseteq w_1 \vee \ldots \vee w_{r-k} \vee S \vee \ldots \vee S, 2 \leq k < r$,

for some two dimensional subspace $S$ of $U$ and some $w_1, \ldots, w_{r-k} \in U - S$.

Let $T(x_i \vee \ldots \vee x_i \vee U) = x_i' \vee \ldots \vee x_i' \vee U, i = 1, \ldots, t$. Note that $T(x_i') = \eta_i x_i'^r$ for some $\eta_i \in F$, $i = 1, \ldots, t$. For $i \neq j$, $\langle x_i', x_j' \rangle$ is a two dimensional subspace of $V^r U$ implies that $T(\langle x_i', x_j' \rangle) = \langle x_i'^r, x_j'^r \rangle$ is a two dimensional subspace of $V^r U$. Hence $x_i'$ and $x_j'$ are linearly independent if $i \neq j$.

Consider case (ii). Choose a vector $w$ of $U$ such that
Let \( w \in \langle w_i \rangle \cup \cdots \cup \langle w_{r-k} \rangle \cup S \cup \left( \bigcup_{i \neq j} \langle x_i, x_j \rangle \right) \).

For each \( i \geq 2 \), let \( T(x_i^{r-1} \vee w) = x_i^{r-1} \vee u_i \). We shall show that \( \langle u_i \rangle = \langle w \rangle \) for \( i \geq 2 \).

Since \( \langle x_i^{r-1} \vee u, x_i^{r-1} \vee u \rangle \) is a decomposable subspace for \( i \geq 2 \), \( \langle x_i^{r-1} \vee w, x_i^{r-1} \vee u_i \rangle \) is also a decomposable subspace. By our choice of \( w \), \( \langle x_i, w, x_i \rangle \) is three dimensional. Hence \( \langle x_i^{r-1} \vee w, x_i^{r-1} \vee u_i \rangle \) is contained in a type \( k \) subspace \( A \) for some \( 1 \leq k < r \). If \( A \) is of type \( k \) where \( 1 \leq k \leq r - 2 \), then we have \( \langle x_i \rangle = \langle w \rangle \) or \( \langle x_i \rangle = \langle x_i \rangle \) which is a contradiction. Hence \( A \) is of type \( r - 1 \). This implies that \( \langle u_i \rangle = \langle w \rangle, \ i \geq 2 \).

Let \( u_i = a_i w \) where \( a_i \in F, \ i \geq 2 \). Then

\[
T(z_1 \vee \cdots \vee z_{r-1} \vee u) = T\left( \sum_{i=1}^{r} \lambda_i x_i^{r-1} \vee u \right)
= \lambda_i x_i^{r-1} \vee w + \sum_{i=2}^{r} \lambda_i x_i^{r-1} \vee (a_i w)
= \left( \lambda_i x_i^{r-1} + \sum_{i=2}^{r} \lambda_i a_i x_i^{r-1} \right) \vee w
= g_1 \vee \cdots \vee g_r \neq 0
\]

for some \( g_i \in U, \ i = 1, \cdots, r \). In view of Lemma 2, \( \langle g_j \rangle = \langle w \rangle \) for some \( j, 1 \leq j \leq r \). Since

\[
g_1 \vee \cdots \vee g_r \in w_1 \vee \cdots \vee w_{r-k} \vee S \vee \cdots \vee S,
\]

we have \( \langle w \rangle = \langle w_i \rangle \) for some \( i \) or \( w \in S \). This contradicts our choice of \( w \). Hence

\[
T(D) \not\subseteq w_1 \vee \cdots \vee w_{r-k} \vee S \vee \cdots \vee S.
\]

Similarly \( T(D) \not\subseteq V^r S \). Therefore \( T(D) \) is a type 1 subspace. In view of Theorem 2 of [3], \( T \) is induced by a nonsingular linear transformation on \( U \).

Combining Lemmas 4 and 5 we have the following main result:

**Theorem 2.** Let \( T: V^r U \rightarrow V^r U \) be a decomposable mapping. If \( 3 \leq \dim U < r + 1 \) then either \( T \) is induced by a nonsingular transformation on \( U \) or \( T(V^r U) \) is a type \( r \) subspace. In particular, if \( T \) is nonsingular, then \( T \) is induced by a nonsingular transformation on \( U \).

We have so far not been able to determine whether there does in fact exist a decomposable mapping on \( V^r U \) such that its image is a type \( r \) subspace when \( 3 \leq \dim U < r + 1 \).
The author is indebted to Professor R. Westwick for his encouragement and suggestions. Thanks are also due to the referee for his suggestions.

REFERENCES


Received October 3, 1973 and in revised form September 18, 1974.

UNIVERSITY OF MALAYA, KUALA LUMPUR, MALAYSIA
Pacific Journal of Mathematics
Vol. 55, No. 2 October, 1974

Walter Allegretto, *On the equivalence of two types of oscillation for elliptic operators* ................................................................. 319
Edward Arthur Bertram, *A density theorem on the number of conjugacy classes in finite groups* .................................................. 329
Arne Brøndsted, *On a lemma of Bishop and Phelps* .......................... 335
Jacob Burbea, *Total positivity and reproducing kernels* ...................... 343
Ed Dubinsky, *Linear Pincherle sequences* ........................................ 361
Benny Dan Evans, *Cyclic amalgamations of residually finite groups* ........ 371
Barry J. Gardner and Patrick Noble Stewart, *A “going down” theorem for certain reflected radicals* .............................................. 381
Sav Roman Harasymiv, *Groups of matrices acting on distribution spaces* 403
Robert Winship Heath and David John Lutzer, *Dugundji extension theorems for linearly ordered spaces* ......................................... 419
Chung-Wu Ho, *Deforming p. l. homeomorphisms on a convex polygon 2-disk* ................................................................................. 427
Richard Earl Hodel, *Metrizability of topological spaces* ...................... 441
Wilfried Imrich and Mark E. Watkins, *On graphical regular representations of cyclic extensions of groups* ........................................ 461
Jozef Krasinkiewicz, *Remark on mappings not raising dimension of curves* 479
Melven Robert Krom, *Infinite games and special Baire space extensions* 483
S. Leela, *Stability of measure differential equations* ............................. 489
M. H. Lim, *Linear transformations on symmetric spaces* ....................... 499
Teng-Sun Liu, Arnoud C. M. van Rooij and Ju-Kwei Wang, *On some group algebra modules related to Wiener’s algebra M1* ............... 507
Dale Wayne Myers, *The back-and-forth isomorphism construction* ........ 521
Donovan Harold Van Osdol, *Extensions of sheaves of commutative algebras by nontrivial kernels* ...................................................... 531
Alan Rahilly, *Generalized Hall planes of even order* ............................. 543
Joylyn Newberry Reed, *On completeness and semicompleteness of first countable spaces* ............................................................. 553
Alan Schwartz, *Generalized convolutions and positive definite functions associated with general orthogonal series* ................................. 565
Thomas Jerome Scott, *Monotonic permutations of chains* ...................... 583
Eivind Stensholt, *An application of Steinberg’s construction of twisted groups* ............................................................ 595
Yasuji Takeuchi, *On strongly radical extensions* .................................... 619
William P. Ziemer, *Some remarks on harmonic measure in space* .......... 629
John Grant, *Corrections to: “Automorphisms definable by formulas”* .... 639
Peter Michael Rosenthal, *Corrections to: “On an inversion for the general Mehler-Fock transform pair”* ....................................... 640
Carl Clifton Faith, *Corrections to: “When are proper cyclics injective”* .... 640