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**LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES**

M. H. LIM

## LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES

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Let  $U$  be an  $n$ -dimensional vector space over an algebraically closed field  $F$  of characteristic zero, and let  $\mathbf{V}^r U$  denote the  $r$ th symmetric product space of  $U$ . Let  $T$  be a linear transformation on  $\mathbf{V}^r U$  which sends nonzero decomposable elements to nonzero decomposable elements. We prove the following:

(i) If  $n = r + 1$  then  $T$  is induced by a nonsingular transformation on  $U$ .

(ii) If  $2 < n < r + 1$  then either  $T$  is induced by a nonsingular transformation on  $U$  or  $T(\mathbf{V}^r U) = \mathbf{V}^r W$  for some two dimensional subspace  $W$  of  $U$ .

The result for  $n > r + 1$  was recently obtained by L. J. Cummings.

1. Preliminaries. Let  $U$  be a finite dimensional vector space over an algebraically closed field  $F$ . Let  $\mathbf{V}^r U$  denote the  $r$ th symmetric product space over  $U$  where  $r \geq 2$ . Unless otherwise stated, the characteristic of  $F$  is assumed to be zero or greater than  $r$ .

A decomposable subspace of  $\mathbf{V}^r U$  is a subspace consisting of decomposable elements. Let  $x_1, \dots, x_{r-1}$  be  $r - 1$  nonzero vectors in  $U$ . Then the set  $\{x_1 \vee \dots \vee x_{r-1} \vee u : u \in U\}$ , denoted by  $x_1 \vee \dots \vee x_{r-1} \vee U$ , is a decomposable subspace of  $\mathbf{V}^r U$  and is called a *type 1 subspace* of  $\mathbf{V}^r U$ . Let  $W$  be a two dimensional subspace of  $U$ . It is shown in [2] that  $\mathbf{V}^r W$  is decomposable and is called a *type  $r$  subspace* of  $\mathbf{V}^r U$ . If  $y_1, \dots, y_{r-k}$  are vectors in  $U - W$  where  $1 < k < r$ , then the set  $\{y_1 \vee \dots \vee y_{r-k} \vee w_1 \vee \dots \vee w_k : w_i \in W, i = 1, \dots, k\}$ , denoted by  $y_1 \vee \dots \vee y_{r-k} \vee W \vee \dots \vee W$ , is also decomposable and is called a *type  $k$  subspace* of  $\mathbf{V}^r U$ . In [2] Cummings showed that every maximal decomposable subspace of  $\mathbf{V}^r U$  is of type  $i$  for some  $1 \leq i \leq r$ .

A linear transformation on  $\mathbf{V}^r U$  is called a *decomposable mapping* if it maps nonzero decomposable elements to nonzero decomposable elements. In [3] Cummings proved that if  $\dim U > r + 1$  then every decomposable mapping  $T$  on  $\mathbf{V}^r U$  is induced by a nonsingular linear transformation  $f$  on  $U$ ; that is,  $T(y_1 \vee \dots \vee y_r) = f(y_1) \vee \dots \vee f(y_r)$ . In this paper we consider the case when  $3 \leq \dim U \leq r + 1$ .

2. The case when  $\dim U = r + 1$ . Two type 1 subspaces  $M_1$  and  $M_2$  of  $\mathbf{V}^r U$  are called *adjacent* if

$$\begin{aligned} M_1 &= x_1 \vee \dots \vee x_{r-2} \vee y_1 \vee U \\ M_2 &= x_1 \vee \dots \vee x_{r-2} \vee y_2 \vee U \end{aligned}$$

for some  $x_1, \dots, x_{r-2}, y_1, y_2$  where  $y_1$  and  $y_2$  are linearly independent.

The proof of the following lemma is contained in that of Proposition 4 of [3].

LEMMA 1. *The images of two adjacent type 1 subspaces under a decomposable mapping are distinct.*

THEOREM 1. *If  $\dim U = r + 1$  then every decomposable mapping  $T$  of  $\mathbf{V}^r U$  is induced by a nonsingular mapping of  $U$ .*

*Proof.* Let  $M$  be a type 1 subspace of  $\mathbf{V}^r U$ . Then  $T(M)$  is a decomposable subspace of  $\mathbf{V}^r U$ . Moreover  $\dim M = \dim T(M) = r + 1$ . Let  $T(M) \subseteq N$  where  $N$  is a maximal decomposable subspace. If  $N$  is of type  $k$  where  $1 < k < r$ , then  $\dim N = k + 1 < r + 1$  which is a contradiction. Hence  $N$  is of type 1 or type  $r$ . Since  $\dim N = r + 1$ , it follows that  $T(M) = N$ .

Suppose that some type 1 subspace  $x_1 \vee \dots \vee x_{r-2} \vee y \vee U$  is mapped onto a type  $r$  subspace  $\mathbf{V}^r W$  where  $W$  is a two dimensional subspace of  $U$ . We shall show that this leads to a contradiction.

Let  $\mathcal{E} = \{T(M_u) : u \in U, u \neq 0\}$  where  $M_u = x_1 \vee \dots \vee x_{r-2} \vee u \vee U$ . We shall show that  $\mathbf{V}^r W$  is the only type  $r$  subspace in  $\mathcal{E}$ . Suppose there is another type  $r$  subspace  $\mathbf{V}^r W^*$  in  $\mathcal{E}$ . Since  $\mathbf{V}^r W \cap \mathbf{V}^r W^* \neq 0$ ,  $W \cap W^*$  is 1-dimensional. Choose a nonzero vector  $z$  in  $U$  such that

$$T(x_1 \vee \dots \vee x_{r-2} \vee y \vee z) = w_1 \vee \dots \vee w_r$$

where  $\dim \langle w_1, \dots, w_r \rangle = 2$ ,  $\langle y \rangle \neq \langle z \rangle$ , and  $W \cap W^* \neq \langle w_i \rangle$  for all  $i = 1, \dots, r$ . If

$$T(M_z) = z_1 \vee \dots \vee z_{r-1} \vee U$$

for some  $z_i$  in  $U$  then

$$T(M_z) \cap \mathbf{V}^r W \neq 0$$

and

$$T(M_z) \cap \mathbf{V}^r W^* \neq 0$$

imply that  $z_1, \dots, z_{r-1} \in W \cap W^*$  and hence  $\langle z_1 \rangle = \dots = \langle z_{r-1} \rangle = W \cap W^*$ . Since  $w_1 \vee \dots \vee w_r \in z_1 \vee \dots \vee z_{r-1} \vee U$ , it follows that  $\langle w_i \rangle = W \cap W^*$  for some  $i$ , a contradiction. Hence

$$T(M_z) = \mathbf{V}^r S$$

for some two dimensional subspace  $S$  of  $U$ . Note that  $x_1 \vee \dots \vee x_{r-2} \vee y \vee z \in M_z \cap M_y$ . Thus  $w_1, \dots, w_r \in W \cap S$ . This implies that  $\langle w_1, \dots, w_r \rangle = W = S$ , a contradiction to Lemma 1 since  $M_z$  and  $M_y$

are adjacent type 1 subspaces. This proves that  $\mathbf{V}^r W$  is the only type  $r$  subspace in  $\mathcal{E}$ .

Since  $\{T(M_x): \langle x \rangle \neq \langle y \rangle, x \neq 0\}$  is an infinite family of type 1 subspaces (Lemma 1) it follows from Proposition 4 of [3] that there exist vectors  $u_1, \dots, u_{r-2}$  such that for any  $x \in U - \{0\}$  and  $\langle x \rangle \neq \langle y \rangle$ ,

$$T(M_x) = u_1 \vee \dots \vee u_{r-2} \vee x' \vee U$$

for some  $x' \in U$ . Since  $T(M_x) \cap \mathbf{V}^r W \neq 0$  we have  $x' \in W$ . Let  $g$  be a fixed nonzero vector such that  $\langle g \rangle \neq \langle y \rangle$ . Then for any  $x \in U - \{0\}$  such that  $\langle x \rangle \neq \langle g \rangle$ ,  $\langle x \rangle \neq \langle y \rangle$ ,

$$T(x_1 \vee \dots \vee x_{r-2} \vee x \vee g) = u_1 \vee \dots \vee u_{r-2} \vee x' \vee g_x$$

for some  $g_x$ . Since  $u_1 \vee \dots \vee u_{r-2} \vee x' \vee g_x \in u_1 \vee \dots \vee u_{r-2} \vee g' \vee U$  and  $\langle x' \rangle \neq \langle g' \rangle$  we have  $\langle g_x \rangle = \langle g' \rangle$ . Therefore

$$\begin{aligned} T(M_g) &\subseteq u_1 \vee \dots \vee u_{r-2} \vee g' \vee W \\ &\cup \langle T(x_1 \vee \dots \vee x_{r-2} \vee g \vee y) \rangle \\ &\cup \langle T(x_1 \vee \dots \vee x_{r-2} \vee g \vee g) \rangle. \end{aligned}$$

This is impossible since  $\dim T(M_g) = \dim U > 2$ .

Therefore,  $T$  maps type 1 subspaces to type 1 subspaces. By Theorem 2 of [3]  $T$  is induced by a nonsingular linear transformation on  $U$ .

3. The case when  $3 \leq \dim U < r + 1$ . In this section we assume that  $\text{char } F = 0$ .

LEMMA 2. Let  $x_1, \dots, x_k$  be  $k$  nonzero vectors of  $U$ . Let  $r > k + 1$  and  $x_1 \vee \dots \vee x_k \vee A = z_1 \vee \dots \vee z_r \neq 0$  in  $\mathbf{V}^r U$  where  $A \in \mathbf{V}^{r-k} U$  and  $z_i \in U$ . Then  $\langle x_i \rangle = \langle z_{j_i} \rangle$  for some  $j_i$  where  $j_s \neq j_t$  for distinct  $s$  and  $t$ .

Proof. Let  $u_1, \dots, u_n$  be a basis of  $U$ . Let  $\phi$  be the isomorphism from the symmetric algebra  $\mathbf{V} U$  over  $U$  onto the polynomial algebra  $F[\xi_1, \dots, \xi_n]$  in  $n$  indeterminates  $\xi_1, \dots, \xi_n$  over  $F$  such that  $\phi(u_i) = \xi_i$ ,  $i = 1, \dots, n$  [4, p. 428]. Then

$$\phi(x_1) \dots \phi(x_k) \phi(A) = \phi(z_1) \dots \phi(z_r) \neq 0.$$

Since  $F[\xi_1, \dots, \xi_n]$  is a Gaussian domain and since  $\phi(x_1), \dots, \phi(x_k), \phi(z_1), \dots, \phi(z_r)$  are linear homogeneous polynomials, it follows that for each  $i = 1, \dots, k$ ,  $\langle \phi(x_i) \rangle = \langle \phi(z_{j_i}) \rangle$  for some  $j_i$  where  $j_t \neq j_s$  if  $s \neq t$ . This implies that  $\langle x_i \rangle = \langle z_{j_i} \rangle$ . Hence the lemma is proved.

The following result is proved in [1, p. 131] under the assumption that  $\text{char } F = 0$ .

LEMMA 3.  $\mathbf{V}^r U$  is spanned by  $\{u^r = \underbrace{u \vee \dots \vee u}_{r\text{-times}} : u \in U\}$ .

Hereafter we will assume that  $3 \leq \dim U < r + 1$  and  $T$  is a decomposable mapping on  $\mathbf{V}^r U$ . Since every type  $k$  subspace has dimension  $< r + 1$  where  $1 \leq k < r$  we see that every type  $r$  subspace of  $\mathbf{V}^r U$  is mapped onto a type  $r$  subspace under  $T$ .

LEMMA 4. If there are two distinct type  $r$  subspaces  $M$  and  $N$  of  $\mathbf{V}^r U$  such that  $M \cap N \neq 0$  and  $T(M) = T(N)$ , then  $T(\mathbf{V}^r U) = T(M)$ .

*Proof.* Let  $M = \mathbf{V}^r S_1$ ,  $N = \mathbf{V}^r S_2$  and  $T(M) = T(N) = \mathbf{V}^r S$  where  $S, S_1, S_2$  are two dimensional subspaces of  $U$ . By hypothesis,

$$M \cap N = \mathbf{V}^r S_1 \cap \mathbf{V}^r S_2 = \mathbf{V}^r (S_1 \cap S_2) \neq 0.$$

Hence  $S_1 \cap S_2$  is one dimensional. Let  $S_1 = \langle y_1, y_2 \rangle$ ,  $S_2 = \langle y_1, y_3 \rangle$ . Consider  $S_3 = \langle y_2, y_3 \rangle$ . Then

$$\mathbf{V}^r S_3 \cap \mathbf{V}^r S_2 = \langle y_3^r \rangle, \quad \mathbf{V}^r S_3 \cap \mathbf{V}^r S_1 = \langle y_2^r \rangle.$$

Hence  $T(\mathbf{V}^r S_3) \cap \mathbf{V}^r S \cong \langle T(y_3^r), T(y_2^r) \rangle$ . Since  $T$  is a decomposable mapping and  $\langle y_2^r, y_3^r \rangle$  is a two dimensional decomposable subspace, it follows that  $\langle T(y_2^r), T(y_3^r) \rangle$  is two dimensional. Hence  $T(\mathbf{V}^r S_3) = \mathbf{V}^r S$  because any two distinct type  $r$  subspaces of  $\mathbf{V}^r U$  have at most one dimension in common.

Let  $z = \alpha y_1 + \beta y_2 + \gamma y_3$  where  $\alpha, \beta, \gamma$  are all nonzero scalars. Consider  $S_4 = \langle y_1, z \rangle = \langle y_1, \beta y_2 + \gamma y_3 \rangle$ . Since

$$\begin{aligned} \mathbf{V}^r S_4 \cap \mathbf{V}^r S_3 &\cong \langle (\beta y_2 + \gamma y_3)^r \rangle, \\ \mathbf{V}^r S_4 \cap \mathbf{V}^r S_1 &\cong \langle y_1^r \rangle, \end{aligned}$$

we have  $T(\mathbf{V}^r S_4) \cap \mathbf{V}^r S \cong \langle T(y_1^r), T((\beta y_2 + \gamma y_3)^r) \rangle$  which is two dimensional. Hence  $T(\mathbf{V}^r S_4) = \mathbf{V}^r S$ . Consequently by Lemma 3,  $T(\mathbf{V}^r \langle y_1, y_2, y_3 \rangle) = \mathbf{V}^r S$ .

Now, let  $w \in U$  such that  $w \notin \langle y_1, y_2, y_3 \rangle$ . Let  $W = \langle y_1, w \rangle$ . Consider the type 1 subspace  $P = y_1 \vee \dots \vee y_1 \vee U$ . Since

$$\dim(P \cap \mathbf{V}^r \langle y_1, y_2, y_3 \rangle) = 3,$$

we have  $\dim(T(P) \cap \mathbf{V}^r S) \geq 3$ . Since the maximal dimension of the intersection of two distinct maximal decomposable subspaces is 2, we conclude that  $T(P) \cong \mathbf{V}^r S$ . This shows that

$$T(\mathbf{V}^r W) \cap \mathbf{V}^r S \cong \langle T(y_1^r), T(y_1 \vee \dots \vee y_1 \vee w) \rangle.$$

Since  $\langle y_1^r, y_1^{r-1} \vee w \rangle$  is a two dimensional decomposable subspace,  $\langle T(y_1^r), T(y_1^{r-1} \vee w) \rangle$  is also two dimensional. Hence  $T(\mathbf{V}^r W) = \mathbf{V}^r S$ . By Lemma 3, we conclude that  $T(\mathbf{V}^r U) = \mathbf{V}^r S$ . This completes the proof.

LEMMA 5. Suppose that for any two distinct type  $r$  subspaces  $M, N$  such that  $M \cap N \neq 0$ , we have  $T(M) \neq T(N)$ . Then  $T$  is induced by a nonsingular transformation on  $U$ .

*Proof.* Let  $y, y_1, y_2$  be linearly independent vectors. Let  $S_1 = \langle y, y_1 \rangle, S_2 = \langle y, y_2 \rangle$ . Then  $T(\mathbf{V}^r S_1) = \mathbf{V}^r S'_1$  and  $T(\mathbf{V}^r S_2) = \mathbf{V}^r S'_2$  for some two dimensional subspaces  $S'_1, S'_2$  of  $U$ . By hypothesis  $\mathbf{V}^r S'_1 \neq \mathbf{V}^r S'_2$ . Hence

$$\mathbf{V}^r S'_1 \cap \mathbf{V}^r S'_2 = T(\mathbf{V}^r S_1 \cap \mathbf{V}^r S_2) = \langle y'^r \rangle$$

for some  $y' \in U$ . Therefore  $T(y^r) = \lambda y'^r$  for some  $\lambda$  in  $F$ .

Let  $H = y \vee \dots \vee y \vee U$ . We claim that  $T(H) = y' \vee \dots \vee y' \vee U$ . Since  $T(H)$  is a decomposable subspace, it is contained in a maximal decomposable subspace. If  $T(H)$  is contained in a type  $k$  subspace  $g_1 \vee \dots \vee g_{r-k} \vee W \vee \dots \vee W$  where  $2 \leq k < r$ , then  $y'^r \in g_1 \vee \dots \vee g_{r-k} \vee W \vee \dots \vee W$  and hence  $\langle g_1 \rangle = \langle y' \rangle, y' \in W$ . This implies  $g_1 \in W$ , a contradiction. If  $T(H)$  is contained in a type  $r$  subspace  $\mathbf{V}^r W$ , then

$$\begin{aligned} \dim(\mathbf{V}^r S_1 \cap H) = 2 &\implies \dim(T(\mathbf{V}^r S_1) \cap \mathbf{V}^r W) \geq 2, \\ \dim(\mathbf{V}^r S_2 \cap H) = 2 &\implies \dim(T(\mathbf{V}^r S_2) \cap \mathbf{V}^r W) \geq 2. \end{aligned}$$

Since  $T(\mathbf{V}^r S_1)$  and  $T(\mathbf{V}^r S_2)$  are both type  $r$  subspaces, it follows that  $T(\mathbf{V}^r S_1) = \mathbf{V}^r W = T(\mathbf{V}^r S_2)$ , a contradiction to our hypothesis. Hence  $T(H)$  is a type 1 subspace of  $\mathbf{V}^r U$ . Since  $y'^r \in T(H)$ , it follows that

$$T(H) = y' \vee \dots \vee y' \vee U.$$

By Lemma 3, let  $x_1^{r-1}, \dots, x_t^{r-1}$  be a basis of  $\mathbf{V}^{r-1} U$ . Note that  $3 \leq \dim U < r + 1$  implies that  $r \geq 3$ . Clearly if  $i \neq j$  then  $x_i$  and  $x_j$  are linearly independent. Consider any type one subspace  $D = z_1 \vee \dots \vee z_{r-1} \vee U$ . Let  $z_1 \vee \dots \vee z_{r-1} = \sum_{i=1}^t \lambda_i x_i^{r-1}$  where  $\lambda_i \in F$  and  $i = 1, \dots, t$ . We shall show that  $T(D)$  is a type 1 subspace. Suppose to the contrary that

$$(i) \quad T(D) \subseteq \mathbf{V}^r S$$

or

$$(ii) \quad T(D) \subseteq w_1 \vee \dots \vee w_{r-k} \vee S \vee \dots \vee S, \quad 2 \leq k < r,$$

for some two dimensional subspace  $S$  of  $U$  and some  $w_1, \dots, w_{r-k} \in U - S$ .

Let  $T(x_i \vee \dots \vee x_i \vee U) = x'_i \vee \dots \vee x'_i \vee U, i = 1, \dots, t$ . Note that  $T(x_i^r) = \eta_i x_i'^r$  for some  $\eta_i \in F, i = 1, \dots, t$ . For  $i \neq j, \langle x_i^r, x_j^r \rangle$  is a two dimensional subspace of  $\mathbf{V}^r U$  implies that  $T(\langle x_i^r, x_j^r \rangle) = \langle x_i'^r, x_j'^r \rangle$  is a two dimensional subspace of  $\mathbf{V}^r U$ . Hence  $x'_i$  and  $x'_j$  are linearly independent if  $i \neq j$ .

Consider case (ii). Choose a vector  $w$  of  $U$  such that

$$w \notin \langle w_1 \rangle \cup \dots \cup \langle w_{r-k} \rangle \cup S \cup \left( \bigcup_{i \neq j} \langle x'_i, x'_j \rangle \right).$$

Let  $u \in U$  such that  $T(x_1^{r-1} \vee u) = x_1'^{r-1} \vee w$ . For each  $i \geq 2$ , let  $T(x_i^{r-1} \vee u) = x_i'^{r-1} \vee u_i$ . We shall show that  $\langle u_i \rangle = \langle w \rangle$  for  $i \geq 2$ .

Since  $\langle x_i^{r-1} \vee u, x_i'^{r-1} \vee u \rangle$  is a decomposable subspace for  $i \geq 2$ ,  $\langle x_1'^{r-1} \vee w, x_i'^{r-1} \vee u_i \rangle$  is also a decomposable subspace. By our choice of  $w$ ,  $\langle x_1', w, x'_i \rangle$  is three dimensional. Hence  $\langle x_1'^{r-1} \vee w, x_i'^{r-1} \vee u_i \rangle$  is contained in a type  $k$  subspace  $A$  for some  $1 \leq k < r$ . If  $A$  is of type  $k$  where  $1 \leq k \leq r-2$ , then we have  $\langle x'_i \rangle = \langle w \rangle$  or  $\langle x'_i \rangle = \langle x_1' \rangle$  which is a contradiction. Hence  $A$  is of type  $r-1$ . This implies that  $\langle u_i \rangle = \langle w \rangle$ ,  $i \geq 2$ .

Let  $u_i = a_i w$  where  $a_i \in F$ ,  $i \geq 2$ . Then

$$\begin{aligned} T(z_1 \vee \dots \vee z_{r-1} \vee u) &= T\left(\sum_{i=1}^t \lambda_i x_i^{r-1} \vee u\right) \\ &= \lambda_1 x_1'^{r-1} \vee w + \sum_{i=2}^t \lambda_i x_i'^{r-1} \vee (a_i w) \\ &= \left(\lambda_1 x_1'^{r-1} + \sum_{i=2}^t \lambda_i a_i x_i'^{r-1}\right) \vee w \\ &= g_1 \vee \dots \vee g_r \neq 0 \end{aligned}$$

for some  $g_i \in U$ ,  $i = 1, \dots, r$ . In view of Lemma 2,  $\langle g_j \rangle = \langle w \rangle$  for some  $j$ ,  $1 \leq j \leq r$ . Since

$$g_1 \vee \dots \vee g_r \in w_1 \vee \dots \vee w_{r-k} \vee S \vee \dots \vee S,$$

we have  $\langle w \rangle = \langle w_i \rangle$  for some  $i$  or  $w \in S$ . This contradicts our choice of  $w$ . Hence

$$T(D) \not\subseteq w_1 \vee \dots \vee w_{r-k} \vee S \vee \dots \vee S.$$

Similarly  $T(D) \not\subseteq \mathbf{V}^r S$ . Therefore  $T(D)$  is a type 1 subspace. In view of Theorem 2 of [3],  $T$  is induced by a nonsingular linear transformation on  $U$ .

Combining Lemmas 4 and 5 we have the following main result:

**THEOREM 2.** *Let  $T: \mathbf{V}^r U \rightarrow \mathbf{V}^r U$  be a decomposable mapping. If  $3 \leq \dim U < r + 1$  then either  $T$  is induced by a nonsingular transformation on  $U$  or  $T(\mathbf{V}^r U)$  is a type  $r$  subspace. In particular, if  $T$  is nonsingular, then  $T$  is induced by a nonsingular transformation on  $U$ .*

We have so far not been able to determine whether there does in fact exist a decomposable mapping on  $\mathbf{V}^r U$  such that its image is a type  $r$  subspace when  $3 \leq \dim U < r + 1$ .

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#### REFERENCES

1. H. Boerner, *Representations of Groups*, North-Holland, Amsterdam, 1963.
2. L. J. Cummings, *Decomposable symmetric tensors*, Pacific J. Math., **35** (1970), 65-77.
3. ———, *Transformations of symmetric tensors*, Pacific J. Math., **42** (1972), 603-613.
4. S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.

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