Pacific Journal of Mathematics

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Vol. 55, No. 2 October 1974

PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

EXTENSIONS OF SHEAVES OF COMMUTATIVE ALGEBRAS BY NONTRIVIAL KERNELS

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Let A, M, and R be sheaves of commutative algebras on a topological space. Given a surjection from R to M there is associated a cohomology class in $H^2(R, ZA)$, the second bicohomology group of R with coefficients in the center of A. This cohomology class is zero if and only if the original surjection arises from an extension of R by A.

Introduction. Let X be a topological space, R a sheaf of commutative algebras on X, and A a sheaf of R-modules considered as an algebra with trivial multiplication. It was shown in [5] that the group of equivalence classes of commutative algebra extensions of R with A as kernel is isomorphic to $H^1(R, A)$, the first bicohomology group of R with coefficients in A. In this paper we will not assume that A has trivial multiplication; we will find that, if ZA is the center of A, then $H^2(R, ZA)$ contains all of the obstructions to the existence of extensions of R by A which "realize" a given morphism. This will generalize the results of [1] to the category of sheaves, and of [4] in that no assumptions need be made on X or R.

In order to keep this paper as short as possible, we shall follow the format of [1]. We shall not, however, generalize §4 of [1]. There are two reasons for this: first, we do not know how to globalize Barr's theory, although we can do his §4 locally using only triple-theoretic techniques (and then the underlying set of A is $Z \times K$ where K is the kernel of R's structure morphism); secondly, the correct setting for completely characterizing the bicohomology $H^n, n > 1$, will not be known until Duskin writes up his results [3].

Let *Sets* be the category of pointed sets. The distinguished point of a set will be the zero of any corresponding algebra. Let Λ be a sheaf of commutative rings on X, $\mathscr{F}(X, Alg)$ the category of sheaves of commutative Λ-algebras on X, $\Pi\Lambda_x\text{-}alg$ the product over $x \in X$ of the categories of Λ_x-algebras ($\Lambda_x =$ stalk of Λ at $x \in X$), and $\mathscr{F}(X, Sets)$ the category of sheaves of pointed sets. We should stress that our algebras need not have unit elements. It is easy to verify that we have a bicohomology situation [5]:

$$\begin{array}{ccc} \mathscr{F}(X, Alg) & \underset{Q}{\overset{S}{\rightleftarrows}} & \Pi\Lambda_x\text{-}alg \\ F\big\uparrow\big\downarrow U & & F\big\uparrow\big\downarrow U \\ \mathscr{F}(X, Sets) & \underset{Q}{\overset{S}{\rightleftarrows}} & Sets^{|X|} \end{array}$$

where the horizontal arrows are adjoint resolutions of the Godement standard construction, and the vertical ones are the obvious free and forgetful functors. Given a sheaf R of Λ-algebras and a sheaf Z of R-modules, the bicohomology theory we use is that arising from the above picture and the functor Der_Λ. Hence we take a "free" simplicial resolution of R, a Godement cosimplicial resolution of Z, and examine the cohomology groups of the double complex gotten by looking at Λ-derivations of the resolution over R into the resolution under Z.

I. **The Class** E. There is no problem in globalizing §1 of [1], but we will give a brief outline in order to fix notation. Let A be a sheaf of ideals in C and for each $x \in X$ let $Z(A_x, C_x) = \{c \in C_x \mid cA_x = 0\}$. Define the centralizer of A in C to be the pullback

$$\begin{array}{ccc} Z(A, C) & \longrightarrow & Q\{Z(A_x, C_x) \mid x \in X\} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\eta C} & QSC\,, \end{array}$$

and the center of A to be $ZA = Z(A, A)$. Then $Z(A, C)$ is a sheaf of ideals in C and we let $E(A)$ denote the set of equivalence classes of exact sequences of sheaves of commutative algebras

$$0 \longrightarrow ZA \longrightarrow A \longrightarrow C/Z(A, C) \longrightarrow C/A + Z(A, C) \longrightarrow 0\,.$$

Here equivalence is by isomorphisms which fix ZA and A.

On the other hand, let E be any sheaf of subalgebras of the sheaf of germs of endomorphisms of A such that E contains the image of $\omega\colon A \to \mathrm{Hom}_A(A, A)$. For each $a \in A$ and open U in X, $\omega U(a)\colon A|_U \longrightarrow A|_U$ is defined by $[\omega U(a)]V(a') = [A(i)a]\cdot a'$ where i is the inclusion of V in U, $a' \in A(V)$, and $''\cdot''$ represents multiplication. Let E' be the set of all such E.

PROPOSITION 1.1. *There is a natural one-one correspondence* $E(A) \cong E'$.

Proof. As in [1]. Here we also construct the truncated simplicial algebra

$$B \underset{d^2}{\overset{d^0}{\Rrightarrow}} P \underset{d^1}{\overset{d^0}{\rightrightarrows}} E \xrightarrow{\pi} M\,, \qquad s^0\colon E \to P\,.$$

PROPOSITION 1.2. *The above simplicial algebra is exact.*

PROPOSITION 1.3. *There is a derivation* $\partial\colon B \to ZA$ *given by* $\partial =$

$(P - s^0\cdot d^0)\cdot(d^0 - d^1 + d^2)$.

II. The obstruction to a morphism. Let R be a sheaf of commutative algebras, $p\colon R \to M$ a surjection, and $0 \to A \to C \to R \to 0$ an exact sequence (extension) of commutative algebras. We say that p arises from this extension if there is a commutative diagram:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & R & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle \nu_0} & & \Big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & ZA & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & M & \longrightarrow & 0\,. \end{array}$$

Given a surjection p, we wish to determine if there are any extensions from which it arises.

Since $\pi\colon E \to M$ is surjective, there is a map $s\colon SUM \to SUE$ such that $SU\pi\cdot s = SUM$. By adjointness we get $s'\colon FUM \to QSE$ such that the diagram

$$\begin{array}{ccc} FUM & \xrightarrow{s'} & QSE \\ {\scriptstyle \varepsilon M}\Big\downarrow & & \Big\downarrow{\scriptstyle QS\pi} \\ M & \xrightarrow{\eta M} & QSM \end{array}$$

commutes. Let $p_0 = s'\cdot FUp$. Then

$$\begin{aligned} QS\pi\cdot p_0\cdot \varepsilon FUR &= \eta M\cdot p\cdot \varepsilon R\cdot \varepsilon FUR \\ &= \eta M\cdot p\cdot \varepsilon R\cdot FU\varepsilon R \\ &= QS\pi\cdot p_0\cdot FU\varepsilon R \end{aligned}$$

so there exists a unique $\tilde{p}_1\colon (FU)^2R \to QS\tilde{P}$ such that

$$QS\tilde{d}^0\cdot\tilde{p}_1 = p_0\cdot\varepsilon FUR,\ QS\tilde{d}^1\cdot\tilde{p}_1 = p_0\cdot FU\varepsilon R\,.$$

Here $(\tilde{P}, \tilde{d}^i)$ is the kernel pair of π, and QS preserves finite limits. Now the unique map $u\colon P \to \tilde{P}$ such that $\tilde{d}^i\cdot u = d^i$ is surjective, so there is $t\colon SU\tilde{P} \to SUP$ splitting it. Using this map and adjointness we produce $t'\colon FUQS\tilde{P} \to (QS)^2P$ such that $(QS)^2u\cdot t' = \eta QS\tilde{P}\cdot\varepsilon QS\tilde{P}$.

Define $\bar{p}_1\colon (FU)^3R \to (QS)^2P$ by $\bar{p}_1 = t'\cdot FU\tilde{p}_1$ and then

$$p_1 = \mu P\cdot\bar{p}_1\cdot\delta' GR$$

where $\mu =$ multiplication for QS, $\delta' =$ comultiplication for FU. One computes that $QSu\cdot p_1 = \tilde{p}_1$, from which it follows that there is a unique $p_2\colon (FU)^3R \to QSB$ such that $d^i\cdot p_2 = p_1\cdot\varepsilon^i$, $0 \leqq i \leqq 2$ where $\varepsilon^i = (FU)^i\varepsilon(FU)^{2-i}R$. By the naturality of ε, $T\partial\cdot p_2\cdot\sum_{i=0}^{3}(-1)^i\varepsilon^i = 0$.

On the other hand,

$$\begin{aligned}(QS)^2\pi\cdot\eta QSE\cdot p_0 &= \eta QSM\cdot QS\pi\cdot p_0\\ &= \eta QSM\cdot\eta M\cdot p\cdot\varepsilon R\\ &= QS\eta M\cdot\eta M\cdot p\cdot\varepsilon R\\ &= QS\eta M\cdot QS\pi\cdot p_0\\ &= (QS)^2\pi\cdot QS\eta E\cdot p_0\end{aligned}$$

so there is a unique $\tilde{q}_1\colon FUR\to(QS)^2\tilde{P}$ such that $(QS)^2\tilde{d}^i\cdot\tilde{q}_1=\eta^iE\cdot p_0$, $i=0,1$, where η^iE is defined as was ε^i above. Let as before $t''\colon FU(QS)^2\tilde{P}\to(QS)^3P$ be such that $(QS)^3u\cdot t''=\eta(QS)^2\tilde{P}\cdot\varepsilon(QS)^2\tilde{P}$. Define $\bar{q}_1=t''\cdot FU\tilde{q}_1$ and $q_1=\mu QSP\cdot\bar{q}_1\cdot\delta' R$. Then $(QS)^2u\cdot q_1=\tilde{q}_1$ and q_1 induces $q_2\colon FUR\to(QS)^3B$ such that $(QS)^3d^i=\eta^iP\cdot q_1$, $0\leqq i\leqq 2$. The induced derivation $(QS)^3\delta\cdot q_2$ has the property that $\sum_{i=0}^3(-1)^i\eta^i\cdot T^3\partial\cdot q_2=0$.

Finally, for $i=0,1$ consider $(QS)^2\tilde{d}^i\cdot\eta^i\cdot\tilde{p}_1\colon(FU)^2R\to(QS)^2E$. One computes that $(QS)^2\pi\cdot(QS)^2\tilde{d}^0\cdot\eta QS\tilde{P}\cdot\tilde{p}_1=(QS)^2\pi\cdot(QS)^2\tilde{d}^1\cdot QS\eta\tilde{P}\cdot\tilde{p}_1$ and concludes that there exists $\tilde{v}\colon(FU)^2R\to(QS)^2\tilde{P}$ such that $(QS)^2\tilde{d}^i\cdot\tilde{v}=(QS)^2\tilde{d}^i\cdot\eta^i\cdot\tilde{p}_1$ for $i=0,1$. As before, the fact that $u\colon P\to\tilde{P}$ is surjective allows us to define $v\colon(FU)^2R\to(QS)^2P$ such that $(QS)^2u\cdot v=\tilde{v}$. Let $r_1\colon(FU)^2R\to(QS)^2B$ be the unique map such that $(QS)^2d^0\cdot r_1=\eta QSP\cdot p_1$, $(QS)^2d^1\cdot r_1=v$, $(QS)^2d^2\cdot r_1=q_1\cdot FU\varepsilon R$ (it is easy to see that such r_1 exists, because $(QS)^2B$ is the kernel triple of $(QS)^2d^0$ and $(QS)^2d^1$). Similarly let $(QS)^2d^0\cdot r_2=q_1\cdot\varepsilon FUR$, $(QS)^2d^1\cdot r_2=v$, and $(QS)^2d^2\cdot r_2=QS\eta P\cdot p_1$. Now we have:

$$\begin{aligned}&((QS)^2\partial\cdot r_2-(QS)^2\partial\cdot r_1)\cdot\sum_{i=0}^{2}(-1)^i\varepsilon^i\\ &=(QS)^2(P-s^0\cdot d^0)\cdot(q_1\cdot\varepsilon^0-v+QS\eta P\cdot p_1)\cdot\sum_{i=0}^{2}(-1)^i\varepsilon^i\\ &\quad-(QS)^2(P-s^0\cdot d^0)\cdot(\eta QSP\cdot p_1-v+q_1\cdot\varepsilon^1)\cdot\sum_{i=0}^{2}(-1)^i\varepsilon^i\\ &=-(QS)^2(P-s^0\cdot d^0)\cdot\left(\sum_{j=0}^{1}(-1)^j\eta^j\right)\cdot p_1\cdot\left(\sum_{i=0}^{2}(-1)^i\varepsilon^i\right)\\ &=\left(\sum_{j=0}^{1}(-1)^j\eta^j\right)\cdot QS\delta\cdot p_2\,,\end{aligned}$$

and similarly

$$\left(\sum_{j=0}^{2}(-1)^j\eta^j\right)\cdot((QS)^2\delta\cdot r_2-(QS)^2\delta\cdot r_1)=(QS)^3\delta\cdot q_2\cdot\sum_{i=0}^{1}(-1)^i\varepsilon^i\,.$$

Hence $(QS\partial\cdot p_2,(QS)^2\partial\cdot r_2-(QS)^2\partial\cdot r_1,(QS)^3\partial\cdot q_2)$ is a cocycle in the bicohomology double complex; we will denote its cohomology class by $[p]$ and call $[p]$ the obstruction of p. We say p is *unobstructed* if $[p]=0$. This terminology is justified by the next two results.

PROPOSITION 2.1. *The cohomology class of* $(QS\partial\cdot p_2,(QS)^2\partial\cdot r_2-(QS)^2\partial\cdot r_1,(QS)^3\partial\cdot q_2)$ *is independent of the choices of* $s\colon SUM\to SUE$

and $t: SU\tilde{P} \to SUP$.

Proof. Once we have p_1, q_1, and v the maps p_2, q_2, r_1, and r_2 are uniquely determined. So suppose σ_0, σ_1, τ_1, ρ_1, ρ_2 are different choices of p_0, p_1, q_1, r_1, r_2 and construct simplicial homotopies as in [1]. Specifically let $QS\tilde{d}^0 \cdot \tilde{h}^0 = p_0$, $QS\tilde{d}^1 \cdot \tilde{h}^0 = \sigma_0$, $Tu \cdot h^0 = \tilde{h}^0$, and

$$QSd^0 \cdot v' = QSd^0 \cdot p_1,\ QSd^1 \cdot v' = QSd^1 \cdot \sigma_1 .$$

Considering the maps p_1, v', and $h^0 \cdot \varepsilon^1$ from $(FU)^2R$ to QSP we see that there exists $h^0: (FU)^2R \to QSB$ such that $QSd^0 \cdot h^0 = p_1$, $QSd^1 \cdot h^0 = v'$, and $QSd^2 \cdot h^0 = h^0 \cdot \varepsilon^1$. Similarly there exists $h^1: (FU)^2R \to QSB$ such that $QSd^0 \cdot h^1 = h^0 \cdot \varepsilon^0$, $QSd^1 \cdot h^1 = v'$, and $QSd^2 \cdot h^1 = \sigma_1$. From these relations it is easy to compute that $(QS\partial \cdot h_0 - QS\partial \cdot h_1) \cdot \sum_{i=0}^{2} (-1)^i \varepsilon^i = QS\partial \cdot p_2 - QS\partial \cdot \sigma_2$.

Now let $w: FUR \to (QS)^2P$ be such that $(QS)^2d^0 \cdot w = (QS)^2d^0 \cdot q_1$ and $(QS)^2d^1 \cdot w = (QS)^2d^1 \cdot \tau_1$ where τ_1 "lifts" σ_0. As above let k^0, k^1: $FUR \to (QS)^2B$ be determined by the conditions

$$(QS)^2d^0 \cdot k^0 = q_1,\ (QS)^2d^1 \cdot k^0 = w,\ (QS)^2d^2 \cdot k^0 = QS\eta P \cdot h^0 ,$$
$$(QS)^2d^0 \cdot k^1 = \eta QSP \cdot h^0,\ (QS)^2d^1 \cdot k^1 = w ,$$

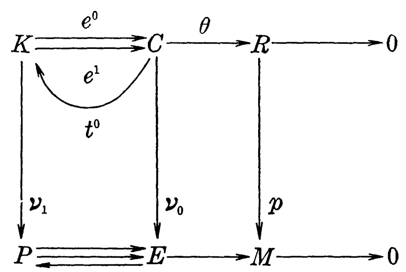
and $(QS)^2d^2 \cdot k^1 = \tau_1$. Again one finds that $(\sum_{j=0}^{2} (-1)^j \cdot \eta^j) \cdot ((QS)^2\partial \cdot k^0 - (QS)^2\partial \cdot k^1) = (QS)^3\partial \cdot q_2 - (QS)^3\partial \cdot \tau_2$. Finally,

$$((QS)^2\partial \cdot k^0 - (QS)^2\partial \cdot k_1) \cdot \sum_{i=0}^{1} (-1)^i \varepsilon^i - \left(\sum_{j=0}^{1} (-1)^j \eta^j\right) \cdot (QS\partial \cdot h^0 - QS\partial \cdot h^1)$$
$$= (QS)^2\partial \cdot \rho_1 - (QS)^2\partial \cdot \rho_2 - (QS)^2\partial \cdot r_1 + (QS)^2\partial \cdot r_2 .$$

Hence the cohomology class of $(QS\partial \cdot p_2, (QS)^2\partial \cdot r_2 - (QS)^2\partial \cdot r_1, (QS)^3\partial \cdot q_2)$ agrees with that of $(QS\partial \cdot \partial_2, (QS)^2\partial \cdot \rho_2 - (QS)^2\partial \cdot \rho_1, (QS)^3\partial \cdot \tau_2)$, as was to be shown.

THEOREM 2.2. *A surjection* $p: R \to M$ *arises from an extension if and only if* p *is unobstructed.*

Proof. Suppose p arises from an extension $0 \to A \to C \xrightarrow{\theta} R \to 0$ and let K be the kernel pair of θ. Then we have a commutative diagram:



Moreover we can find $\sigma_0: FUR \to QSC$ such that $QS\theta\cdot\sigma_0 = \eta R\cdot\varepsilon R$. If we let $\sigma_1: (FU)^2R \to QSK$ be such that $QSe^i\cdot\sigma_1 = \sigma_0\cdot\varepsilon^i$ and $\tau_1: FUR \to (QS)^2K$ such that $(QS)^2e^i\cdot\tau_1 = \eta^i\cdot\sigma_0$ for $i = 0, 1$ then $QS\nu_0\cdot\sigma_0$ serves as p_0, $QS\nu_1\cdot\sigma_1$ as p_1, and $(QS)^2\nu_1\cdot\tau_1$ as q_1. By 2.1 we can assume that things have been so arranged. But then using the fact that $(QS)^je^0$, $(QS)^je^1$ is a kernel pair for each $j \geqq 0$, one can show that

$$QS(K - t^0\cdot e^0)\cdot\sigma_1\cdot\sum_{i=0}^{2}(-1)^i\varepsilon^i = 0\ ,$$

$(QS)^2(K - t^0\cdot e^0)\cdot[(\sum_{j=0}^{1}(-1)^j\eta^j)\cdot\sigma_1 - \tau_1\cdot(\sum_{j=0}^{1}(-1)^i\varepsilon^i)] = 0$, and

$$(QS)^3(K - t^0\cdot e^0)\cdot\left(\sum_{j=0}^{2}(-1)^j\eta^j\right)\cdot\tau_1 = 0\ .$$

From this it follows that $QS\partial\cdot p_2 = 0$, $(QS)^2\partial\cdot r_2 - (QS)^2\partial\cdot r_1 = 0$, and $(QS)^3\partial\cdot q_2 = 0$. Thus $[p] = 0$.

Conversely, suppose $[p] = 0$. Then there exist $\tau: (FU)^2R \to QS(ZA)$, $\rho: FUR \to (QS)^2ZA$ with $\tau\cdot\varepsilon = QS\partial\cdot p_2$, $\eta\cdot\rho = (QS)^3\partial\cdot q_2$, and $\rho\cdot\varepsilon - \eta\cdot\tau = (QS)^2\partial\cdot r_2 - (QS)^2\partial\cdot r_1$. Here we abbreviate $\sum_{i=0}^{n}(-1)^i\varepsilon^i = \varepsilon$ and similarly for η. Now $\bar{p}_1 = p_1 - \tau$, $\bar{q}_1 = q_1 - \rho$ serve as new p_1, q_1 and also give $\bar{p}_2$, $\bar{q}_2$, $\bar{r}_1$, $\bar{r}_2$. We have

$$\begin{aligned} QS(P - s^0\cdot d^0)\cdot QSd\cdot\bar{p}_2 &= QS(P - s^0\cdot d^0)\cdot\bar{p}_1\cdot\varepsilon \\ &= QS(P - s^0\cdot d^0)\cdot p_1\cdot\varepsilon - QS(P - s^0\cdot d^0)\cdot\tau\cdot\varepsilon \\ &= QS(P - s^0\cdot d^0)\cdot p_1\cdot\varepsilon \\ &\quad -\tau\cdot\varepsilon + QSs^0\cdot QSd^0\cdot\tau\cdot\varepsilon \\ &= QS\partial\cdot p_2 - \tau\cdot\varepsilon \\ &= 0 \end{aligned}$$

because the kernel of QSd^0 is $QS(Z(A, P))$ which contains $QS(ZA)$. Similar computations yield $(QS)^3(P - s^0\cdot d^0)\cdot(QS)^3d\cdot\bar{q}_2 = 0$ and

$$(QS)^2(P - s^0\cdot d^0)\cdot(QS)^2d\cdot\bar{r}_2 - (QS)^2(P - s^0\cdot d^0)\cdot(QS)^2d\cdot\bar{r}_1 = 0\ .$$

Hence we can assume that $(QS)^3\partial\cdot q_2$, $QS\partial\cdot p_2$, $(QS)^2\partial\cdot r_2 - (QS)^2\partial\cdot r_1$ are all zero (by Prorosition 2.1). We now go over to the equivalent category $(Sets^{|X|})_G^T$ where $T = UF$, $G = SQ$. The reader is referred to [6] for a clarification of what this means, and to [5] for an introduction to the techniques to be used below. Let R, M, E, $\tilde{P}$, P, B, A, ZA be translated respectively into $\{R_x, \xi_1, \xi_2\}$, $\{M_x, \beta_1, \beta_2\}$, $\{E_x, \gamma_1, \gamma_2\}$, $\{\tilde{P}_x, \bar{\nu}_1, \bar{\nu}_2\}$, $\{A_x \times E_x, \nu_1, \nu_2\}$, $\{B_x, -, -\}$, $\{A_x, \alpha_1, \alpha_2\}$, $\{ZA_x, -, -\}$. Since we want to use the symbols p_i for projections from a product, we let our old p_i be u_i, $0 \leqq i \leqq 2$.

For notational convenience, we drop all subscripts x and say once and for all that an equation will stand for the same equation with subscripts adjoined. For example, $\theta\cdot s = M$ means $\theta_x\cdot s_x = M_x$ for each x in X. Our assumption that $(QS)^3\partial\cdot q_2$ e.t.c. are all zero translates into the following three equations in $(Sets^{|X|})^T_G$:

(i) $p_1\cdot u_1\cdot T\xi_1 - p_1\cdot u_1\cdot \mu R + p_1\cdot \nu_1\cdot Tu_1 = 0$

(ii) $G^2p_1\cdot G\nu_2\cdot q_1 - G^2p_1\cdot\delta'(A\times E)\cdot q_1 + G^2p_1\cdot Gq_1\cdot\xi_2 = 0$

(iii) $Gp_1\cdot q_1\cdot\xi_1 - Gp_1\cdot G\nu_1\cdot\lambda P\cdot Tq_1 = Gp_1\cdot Gu_1\cdot\lambda R\cdot T\xi_2 - Gp_1\cdot\nu_2\cdot u_1$.

Here $\lambda\colon TG\to GT$ is the distributive law (see [5]), and p_1 (or p_2) is the first (or second) projection from the appropriate product. Since our presentation has now begun to differ significantly from that of Barr [1], we will provide more detail than earlier in the paper. Let $C = A\times R$, and define $\zeta_1\colon TC\to C$, $\zeta_2\colon C\to GC$ by the conditions $p_1\cdot\zeta_1 = p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot Tp_2$, $p_2\cdot\zeta_1 = \xi_1\cdot Tp_2$, $Gp_1\cdot\zeta_2 = \alpha_2\cdot p_1 + Gp_1\cdot q_1\cdot p_2$, $Gp_2\cdot\zeta_2 = \xi_2\cdot p_2$. We claim that (C, ζ_1, ζ_2) is in $(Sets^{|X|})^T_G$. Besides the "cocycle identities" listed above, the only fact we need is that

$$\nu_1\colon T(A\times E)\longrightarrow A\times E$$

has the following property: For each $g\colon X\to A$ and $f\colon X\to A\times E$ we have

(iv) $p_1\cdot\nu_1\cdot T([g+p_1\cdot f]\times d^1\cdot f) = p_1\cdot\nu_1\cdot T(g\times d^0\cdot f) + p_1\cdot\nu_1\cdot Tf$. Since this amounts to a combinatorial identity, we relegate its proof to the Appendix. Using (i) and (iv) we can prove that ζ_1 is associative:

$$\begin{aligned}
p_1\cdot\zeta_1\cdot T\zeta_1 &= [p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot Tp_2]\cdot T\zeta_1\\
&= p_1\cdot\nu_1\cdot T([p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot Tp_2]\times s\cdot p\cdot\xi_1\cdot Tp_2)\\
&\quad + p_1\cdot u_1\cdot T\xi_1\cdot T^2p_2\\
&= p_1\cdot\nu_1\cdot T([p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)]\times\gamma_1\cdot Ts\cdot Tp\cdot Tp_2)\\
&\quad + p_1\cdot\nu_1\cdot Tu_1\cdot T^2p_2 + p_1\cdot u_1\cdot T\xi_1\cdot T^2p_2\\
&= p_1\cdot\nu_1\cdot T(\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)) + p_1\cdot u_1\cdot\mu R\cdot T^2p_2\\
&= p_1\cdot\nu_1\cdot\mu(A\times E)\cdot T^2(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot\mu R\cdot T^2p_2\\
&= p_1\cdot\zeta_1\cdot\mu(A\times R);
\end{aligned}$$

the fact that $p_2\cdot\zeta_1\cdot T\zeta_1 = p_2\cdot\zeta_1\cdot\mu(A\times R)$ is an easy computation. Notice that in the above computation we have taken

$$g = p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)$$

and $f = u_1\cdot Tp_2$ in (iv). Before proving that ζ_1 is unitary, we show that u_1 is "normalized":

$$\begin{aligned}
0 &= (p_1\cdot u_1\cdot T\xi_1 - p_1\cdot u_1\cdot\mu R + p_1\cdot\nu_1\cdot Tu_1)\cdot\eta TR\\
&= p_1\cdot u_1\cdot\eta R\cdot\xi_1 - p_1\cdot u_1 + p_1\cdot\nu_1\cdot\eta R\cdot u_1\\
&= p_1\cdot u_1\cdot\eta R\cdot\xi_1\,.
\end{aligned}$$

But composing this equation with ηR gives $p_1\cdot u_1\cdot\eta R = 0$, and from this it follows that ζ_1 is unitary:

$$\begin{aligned}\zeta_1\cdot\eta(A\times R) &= [p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)\cdot\eta(A\times R)\\ &\quad + p_1\cdot u_1\cdot Tp_2\cdot\eta(A\times R)]\times\xi_1\cdot Tp_2\cdot\eta(A\times R)\\ &= [p_1\cdot([p_1\times s\cdot p\cdot p_2]) + p_1\cdot u_1\cdot\eta R\cdot T^2p_2]\times\xi_1\cdot\eta R\cdot T^2p_2\\ &= p_1\times p_2\,.\end{aligned}$$

The computations which show that ζ_2 is counitary and coassociative use only (ii) above, and will be omitted. The "compatibility" of ζ_1 and ζ_2 uses (iii) and (iv) above, and proceeds as follows:

$$\begin{aligned}&Gp_1\cdot G\zeta_1\cdot\lambda(A\times R)\cdot T\zeta_2\\ &\quad = G(p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot Tp_2)\cdot\lambda(A\times R)\cdot T\zeta_2\\ &\quad = Gp_1\cdot G\nu_1\cdot GT(p_1\times s\cdot p\cdot p_2)\cdot\lambda(A\times R)\cdot T\zeta_2\\ &\qquad + Gp_1\cdot Gu_1\cdot GTp_2\cdot\lambda(A\times R)\cdot T\zeta_2\\ &\quad = Gp_1\cdot G\nu_1\cdot\lambda(A\times E)\cdot TG(p_1\times s\cdot p\cdot p_2)\cdot T\zeta_2\\ &\qquad + Gp_1\cdot Gu_1\cdot\lambda R\cdot TGp_2\cdot T\zeta_2\\ &\quad = Gp_1\cdot G\nu_1\cdot\lambda(A\times E)\cdot T([\alpha_2\cdot p_1 + Gp_1\cdot q_1\cdot p_2]\times Gs\cdot Gp\cdot\xi_2\cdot p_2)\\ &\qquad + Gp_1\cdot Gu_1\cdot\lambda R\cdot T(\xi_2\cdot p_2)\\ &\quad = Gp_1\cdot G\nu_1\cdot\lambda(A\times E)\cdot T(\alpha_2\cdot p_1\times\gamma_2\cdot s\cdot p\cdot p_2)\\ &\qquad + Gp_1\cdot G\nu_1\cdot\lambda(A\times E)\cdot Tq_1\cdot Tp_2 + Gp_1\cdot Gu_1\cdot\lambda R\cdot T\xi_2\cdot Tp_2\\ &\quad = Gp_1\cdot G\nu_1\cdot\lambda(A\times E)\cdot T\nu_2\cdot T(p_1\times s\cdot p\cdot p_2)\\ &\qquad + Gp_1\cdot q_1\cdot\xi_1\cdot Tp_2 + Gp_1\cdot\nu_2\cdot u_1\cdot Tp_2\\ &\quad = Gp_1\cdot\nu_2\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + \alpha_2\cdot p_1\cdot u_1\cdot Tp_2 + Gp_1\cdot q_1\cdot\xi_1\cdot Tp_2\\ &\quad = \alpha_2\cdot p_1\cdot\zeta_1 + Gp_1\cdot q_1\cdot p_2\cdot\zeta_1\\ &\quad = Gp_1\cdot\zeta_2\cdot\zeta_1;\end{aligned}$$

here, again, that $Gp_2\cdot G\zeta_1\cdot\lambda(A\times R)\cdot T\zeta_2 = Gp_2\cdot\zeta_2\cdot\zeta_1$ is obvious. Notice that we have not used (iv) as it stands, but rather the analog of (iv) for $GP = G(A\times E)$. We have taken $g = \alpha_2\cdot p_1$ and $f = q_1\cdot p_2$. At any rate, (C, ζ_1, ζ_2) is in $(Sets^{|X|})^T_G$ and the first injection, second projection give us an exact sequence $0\longrightarrow A\xrightarrow{\ i\ } A\times R = C\xrightarrow{p_2} R\longrightarrow 0$ in $(Sets^{|X|})^T_G$. Define $h\colon C\longrightarrow E$ by $h = \omega\cdot p_1 + s\cdot p\cdot p_2$. Clearly $\pi\cdot h = p\cdot p_2$ and $h\cdot i = \omega$, so that if h is a morphism in $(Sets^{|X|})^T_G$ then we will have produced an extension from which p arises, and the proof will be complete. But we have:

$$\begin{aligned}h\cdot\zeta_1 &= \omega\cdot(p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_1\cdot u_1\cdot Tp_2) + s\cdot p\cdot\xi_1\cdot Tp_2\\ &= \omega\cdot p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)\\ &\quad + \gamma_1\cdot Ts\cdot Tp\cdot Tp_2 - s\cdot p\cdot\xi_1\cdot Tp_2 + s\cdot p\cdot\xi_1\cdot Tp_2\\ &= \omega\cdot p_1\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2) + p_2\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)\\ &= d^0\cdot\nu_1\cdot T(p_1\times s\cdot p\cdot p_2)\end{aligned}$$

$$= \gamma_1 \cdot Td^0 \cdot T(p_1 \times s \cdot p \cdot p_2)$$
$$= \gamma_1 \cdot T(\omega \cdot p_1 + s \cdot p \cdot p_2)$$
$$= \gamma_1 \cdot Th \ ,$$

and

$$Gh \cdot \zeta_2 = G\omega \cdot (\alpha_2 \cdot p_1 + Gp_1 \cdot q_1 \cdot p_2) + Gs \cdot Gp \cdot \xi_2 \cdot p_2$$
$$= \gamma_2 \cdot \omega \cdot p_1 + \gamma_2 \cdot s \cdot p \cdot p_2 - Gs \cdot Gp \cdot \xi_2 \cdot p_2 + Gs \cdot Gp \cdot \xi_2 \cdot p_2$$
$$= \gamma_2 \cdot h \ .$$

III. The Action of H^1.

THEOREM 3.1. *Let $p: R \to M$ be unobstructed, and let Σ denote the equivalence classes of extensions of R by A which induce p. Then the group $H^1(R, ZA)$ acts on Σ as a principal homogeneous representation.*

Proof. It is shown in [5] that $H^1(R, ZA)$ is in one-one correspondence with the set of equivalence classes of singular extensions of R by ZA. Once this is known, Barr's proof of this proposition [1] translates almost verbatum into a proof for sheaves.

APPENDIX. In this appendix we give a proof of equation (iv) above (§II), and compare Barr's constructions [1] to our own. To dispose of equation (iv), recall that given a commutative algebra A, its structure map $\alpha: TA \to A$ takes a polynomial in elements of A to the "value" of the polynomial. That is, α remembers that A is an algebra and uses the algebra operations in A to compute the polynomial. Now multiplication in $P = A \times E$ is defined by $(a_1, x_1)(a_2, x_2) = (a_1a_2 + x_1a_2 + a_1x_2, x_1x_2)$ where x_1a_2 and a_1x_2 denote the value of x on a.

PROPOSITION A.1. *Given $a_i \in A$, $x_i \in E$ for $1 \leqq i \leqq n$ we have $\prod_{i=1}^n (a_i, x_i) = (\Sigma f(1)_1 \cdots f(n)_n, x_1 \cdots x_n)$ where the sum is taken over all functions $f: n = \{1, 2, \cdots, n\} \to \{a, x\}$ such that f is not identically equal to x.*

Proof. By induction on n. We have

$$\prod_{i=1}^n (a_i, x_i)$$
$$= (\Sigma f(1)_1 \cdots f(n-1)_{n-1}, x_1 \cdots x_{n-1})(a_n, x_n)$$
$$= (\Sigma f(1)_1 \cdots f(n-1)_{n-1}a_n + \Sigma f(1)_1 \cdots f(n-1)_{n-1}x_n$$
$$+ x_1 \cdots x_{n-1}a_n, x_1 \cdots x_n)$$
$$= (\Sigma f(1)_1 \cdots f(n)_n, x_1 \cdots x_n)$$

where the indexing sets for the sums are clear.

PROPOSITION A.2. *Given $a_i, b_i \in A$, $x_i \in E$ for $1 \leqq i \leqq n$ we have that $\prod_{i=1}^{n}(a_i + b_i, x_i)$ and $\prod_{i=1}^{n}(b_i, \omega a_i + x_i) + \prod_{i=1}^{n}(a_i, x_i)$ have the same first coordinates.*

Proof. Induction on n and Proposition A.1.

$$\prod_{i=1}^{n}(a_i + b_i, x_i) = (\Sigma g(1)_1 \cdots g(n-1)_{n-1} + \Sigma h(1)_1 \cdots h(n-1)_{n-1}, x_1 \cdots x_{n-1})(a_n + b_n, x_n)$$

where the g's run through the set of functions from $n - 1 \rightarrow \{b, \omega a + x\}$ which are not identically $\omega a + x$ and the h's through all $n - 1 \rightarrow \{a, x\}$ which are not identically x. Hence we get as first coordinate

$$\begin{aligned}&\Sigma g(1)_1 \cdots g(n-1)_{n-1}a_n + \Sigma g(1)_1 \cdots g(n-1)_{n-1}b_n \\ &\quad + \Sigma h(1)_1 \cdots h(n-1)_{n-1}a_n + \Sigma h(1)_1 \cdots h(n-1)_{n-1}b_n \\ &\quad + \Sigma g(1)_1 \cdots g(n-1)_{n-1}x_n + \Sigma h(1)_1 \cdots h(n-1)_{n-1}x_n \\ &\quad + x_1 \cdots x_{n-1}a_n + x_1 \cdots x_{n-1}b_n .\end{aligned}$$

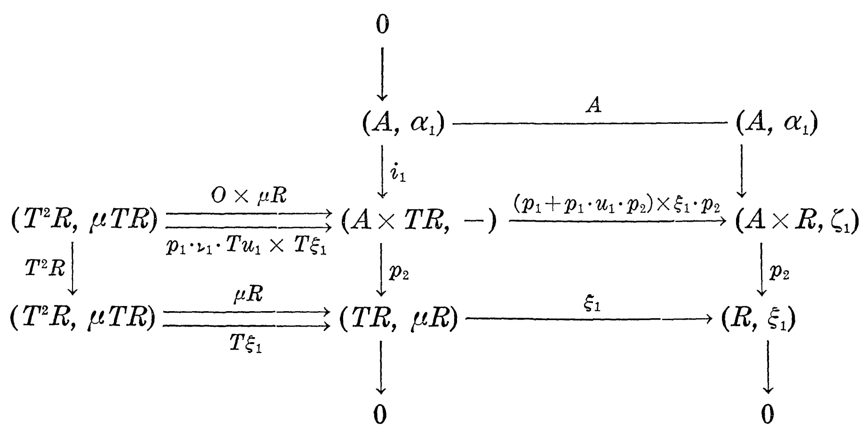
The third, sixth, and seventh terms of this sum give us

$$\Sigma h(1)_1 \cdots h(n)_n .$$

Since $\Sigma h(1)_1 \cdots h(n-1)_{n-1}b_n = \prod_{i=1}^{n-1}(\omega a_i + x_i)b_n - x_1 \cdots x_{n-1}b_n$ the remaining terms give us $\Sigma g(1)_1 \cdots g(n)_n$. This completes the proof.

Taking into account the remarks preceding Proposition A.1, equation (iv) follows immediately from A.2.

In [1] Barr constructs the extension which realizes an unobstructed p as a certain coequalizer. In the notation of our §II, his diagram on page 365 would look like:



He uses the coequalizer $(p_1 + p_1 \cdot u_1 \cdot p_2) \times \xi_1 \cdot p_2$ to define the algebra which gives the extension, and then must make some rather tedious computations to verify that all requirements are met. One knows that if an extension exists, then its underlying set will have to be $A \times R$: the only question is how the algebra structure on $A \times R$ is "twisted". By passing to the equivalent category of T-algebras it becomes clear exactly how the cocycle should be used to produce this twisted structure. All of this was first noticed by Beck in the case of singular extensions [2]. At any rate, the "globalization" of Barr's results seems to require that we pass to $(Sets^{|X|})^T_G$.

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Received January 16, 1973. Research supported by NSF Grant GP-29067. The author would like to thank the University of Oslo for providing the pleasant atmosphere under which part of this work was carried out.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 55, No. 2 October, 1974