EXTENSIONS OF SHEAVES OF COMMUTATIVE ALGEBRAS BY NONTRIVIAL KERNELS

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Let $A, M$, and $R$ be sheaves of commutative algebras on a topological space. Given a surjection from $R$ to $M$ there is associated a cohomology class in $H^2(R, ZA)$, the second bicohomology group of $R$ with coefficients in the center of $A$. This cohomology class is zero if and only if the original surjection arises from an extension of $R$ by $A$.

Introduction. Let $X$ be a topological space, $R$ a sheaf of commutative algebras on $X$, and $A$ a sheaf of $R$-modules considered as an algebra with trivial multiplication. It was shown in [5] that the group of equivalence classes of commutative algebra extensions of $R$ with $A$ as kernel is isomorphic to $H^1(R, A)$, the first bicohomology group of $R$ with coefficients in $A$. In this paper we will not assume that $A$ has trivial multiplication; we will find that, if $ZA$ is the center of $A$, then $H^3(R, ZA)$ contains all of the obstructions to the existence of extensions of $R$ by $A$ which "realize" a given morphism. This will generalize the results of [1] to the category of sheaves, and of [4] in that no assumptions need be made on $X$ or $R$.

In order to keep this paper as short as possible, we shall follow the format of [1]. We shall not, however, generalize §4 of [1]. There are two reasons for this: first, we do not know how to globalize Barr’s theory, although we can do his §4 locally using only triple-theoretic techniques (and then the underlying set of $A$ is $Z \times K$ where $K$ is the kernel of $R$’s structure morphism); secondly, the correct setting for completely characterizing the bicohomology $H^n, n > 1$, will not be known until Duskin writes up his results [3].

Let $\mathcal{S}$ be the category of pointed sets. The distinguished point of a set will be the zero of any corresponding algebra. Let $A$ be a sheaf of commutative rings on $X$, $\mathcal{F}(X, Alg)$ the category of sheaves of commutative $A$-algebras on $X$, $\Pi A_x$-alg the product over $x \in X$ of the categories of $A_x$-algebras ($A_x$ = stalk of $A$ at $x \in X$), and $\mathcal{F}(X, \mathcal{S})$ the category of sheaves of pointed sets. We should stress that our algebras need not have unit elements. It is easy to verify that we have a bicohomology situation [5]:

\[
\begin{array}{ccc}
\mathcal{F}(X, Alg) & \xrightarrow{S} & \Pi A_x\text{-alg} \\
F & \xrightarrow{U} & Q \\
\mathcal{F}(X, \mathcal{S}) & \xleftarrow{S} & \mathcal{S}^{[X]}
\end{array}
\]
where the horizontal arrows are adjoint resolutions of the Godement standard construction, and the vertical ones are the obvious free and forgetful functors. Given a sheaf $R$ of $A$-algebras and a sheaf $Z$ of $R$-modules, the bicohomology theory we use is that arising from the above picture and the functor $\text{Der}_A$. Hence we take a “free” simplicial resolution of $R$, a Godement cosimplicial resolution of $Z$, and examine the cohomology groups of the double complex gotten by looking at $A$-derivations of the resolution over $R$ into the resolution under $Z$.

I. The Class $E$. There is no problem in globalizing §1 of [1], but we will give a brief outline in order to fix notation. Let $A$ be a sheaf of ideals in $C$ and for each $x \in X$ let $Z(A_x, C_x) = \{c \in C_x | cA_x = 0\}$. Define the centralizer of $A$ in $C$ to be the pullback

\[
\begin{array}{c}
Z(A, C) \\
\downarrow \gamma_C \\
C \\
\downarrow \\
QSC,
\end{array}
\]

and the center of $A$ to be $ZA = Z(A, A)$. Then $Z(A, C)$ is a sheaf of ideals in $C$ and we let $E(A)$ denote the set of equivalence classes of exact sequences of sheaves of commutative algebras

\[0 \longrightarrow ZA \longrightarrow A \longrightarrow C/Z(A, C) \longrightarrow C/A + Z(A, C) \longrightarrow 0 .\]

Here equivalence is by isomorphisms which fix $ZA$ and $A$.

On the other hand, let $E$ be any sheaf of subalgebras of the sheaf of germs of endomorphisms of $A$ such that $E$ contains the image of $\omega: A \rightarrow \text{Hom}_A (A, A)$. For each $a \in A$ and open $U$ in $X$, $\omega U(a): A|_U \longrightarrow A|_U$ is defined by $[\omega U(a)]V(a') = [A(i)a]\cdot a'$ where $i$ is the inclusion of $V$ in $U$, $a' \in A(V)$, and "\cdot" represents multiplication. Let $E'$ be the set of all such $E$.

**Proposition 1.1.** There is a natural one-one correspondence $E(A) \cong E'$.

*Proof.* As in [1]. Here we also construct the truncated simplicial algebra

\[
\begin{array}{c}
B \\
\downarrow d^0 \\
P \\
\downarrow d^0 \\
E \\
\downarrow \gamma \\
M,
\end{array}
\]

**Proposition 1.2.** The above simplicial algebra is exact.

**Proposition 1.3.** There is a derivation $\partial: B \rightarrow ZA$ given by $\partial = \ldots$
II. The obstruction to a morphism. Let \( R \) be a sheaf of commutative algebras, \( p: R \to M \) a surjection, and \( 0 \to A \to C \to R \to 0 \) an exact sequence (extension) of commutative algebras. We say that \( p \) arises from this extension if there is a commutative diagram:

\[
\begin{array}{c}
0 \\
\downarrow A \\
0
\end{array}
\quad\begin{array}{c}
A \\
\downarrow \eta_0 \\
A
\end{array}
\quad\begin{array}{c}
C \\
\downarrow p \\
R
\end{array}
\quad\begin{array}{c}
R \\
\downarrow \pi \\
M
\end{array}
\quad\begin{array}{c}
0 \end{array}
\]

Given a surjection \( p \), we wish to determine if there are any extensions from which it arises.

Since \( \pi: E \to M \) is surjective, there is a map \( s: SUM \to SUE \) such that \( S\pi \cdot s = SUM \). By adjointness we get \( s': FUM \to QSE \) such that the diagram

\[
\begin{array}{c}
FUM \\
\downarrow \varepsilon_M \\
M
\end{array}
\quad\begin{array}{c}
s' \\
\downarrow QS\pi \\
QSE
\end{array}
\quad\begin{array}{c}
\gamma_M \\
\downarrow \eta_M \\
QSM
\end{array}
\]

commutes. Let \( p_0 = s' \cdot FUp \). Then

\[
QS\pi \cdot p_0 \cdot \varepsilon FUR = \eta M \cdot p \cdot \varepsilon R \cdot \varepsilon FUR
\]
\[
= \eta M \cdot p \cdot \varepsilon R \cdot FUzR
\]
\[
= QS\pi \cdot p_0 \cdot FUzR
\]

so there exists a unique \( \tilde{p}_i: (FU)^iR \to QS\tilde{P} \) such that

\[
QS\tilde{d}^i \cdot \tilde{p}_i = p_0 \cdot \varepsilon FUR, \quad QS\tilde{d}^i \cdot \tilde{p}_i = p_0 \cdot FUzR.
\]

Here \((\tilde{P}, \tilde{d}^i)\) is the kernel pair of \( \pi \), and \( QS \) preserves finite limits. Now the unique map \( u: P \to \tilde{P} \) such that \( \tilde{d}^i \cdot u = d^i \) is surjective, so there is \( t: S\tilde{U}\tilde{P} \to SUP \) splitting it. Using this map and adjointness we produce \( t': FUQS\tilde{P} \to (QS)^iP \) such that \( (QS)^u \cdot t' = \eta QS\tilde{P} \cdot \varepsilon QS\tilde{P} \).

Define \( \tilde{p}_i: (FU)^iR \to (QS)^iP \) by \( \tilde{p}_i = t' \cdot FU\tilde{p}_i \) and then

\[
p_i = \mu P \cdot \tilde{p}_i \cdot \tilde{d}^i GR
\]

where \( \mu = \) multiplication for \( QS \), \( \tilde{d}^i = \) comultiplication for \( FU \). One computes that \( QS\tilde{u} \cdot p_i = \tilde{p}_i \), from which it follows that there is a unique \( p_2: (FU)^2R \to QSB \) such that \( d^i \cdot p_2 = p_i \cdot \varepsilon^i \), \( 0 \leq i \leq 2 \) where \( \varepsilon^i = (FU)^i\varepsilon(FU)^{2-i}R \). By the naturality of \( \varepsilon \), \( T\tilde{d} \cdot p_2 \cdot \sum_{i=0}^{2} (-1)^i \varepsilon^i = 0 \).

On the other hand,
so there is a unique $\tilde{q}_i: \text{FUR} \to (QS)^i \tilde{P}$ such that $(QS)^i \tilde{d}^i \cdot \tilde{q}_i = \eta^i E \cdot p_0$, $i = 0, 1$, where $\eta^i E$ is defined as was $\epsilon^i$ above. Let as before $t''$: \text{FU}(QS)^i \tilde{P} \to (QS)^i P$ be such that $(QS)^i u \cdot t'' = \eta( QS)^i \tilde{P} \cdot \epsilon( QS)^i \tilde{P}$. Define $\tilde{q}_i = t'' \cdot \text{FU} \tilde{q}_i$ and $q_i = \mu QS\tilde{P} \cdot \tilde{q}_i \cdot \delta R$. Then $(QS)^i u \cdot q_i = \tilde{q}_i$, and $q_i$ induces $q_i: \text{FUR} \to (QS)^i B$ such that $(QS)^i d^i = \eta^i P \cdot q_i$, $0 \leq i \leq 2$. The induced derivation $(QS)^i \delta \cdot q_i$ has the property that $\sum_{i=0}^2 (\epsilon^i)^i \cdot T^i \delta \cdot q_2 = 0$.

Finally, for $i = 0, 1$ consider $(QS)^i \tilde{d}^i \cdot \eta^i \cdot \tilde{p}_i; \text{FU}^i R \to (QS)^i E$. One computes that $(QS)^i \pi \cdot (QS)^i \tilde{d}^i \cdot \eta QS \tilde{P} \cdot \tilde{p}_i = (QS)^i \pi \cdot (QS)^i \tilde{d}^i \cdot QS \tilde{P} \cdot \tilde{p}_i$, and concludes that there exists $\tilde{v}: (FU)^i R \to (QS)^i \tilde{P}$ such that $(QS)^i \tilde{d}^i \cdot \tilde{v} = (QS)^i \tilde{d}^i \cdot \eta^i \cdot \tilde{p}_i$ for $i = 0, 1$. As before, the fact that $u: P \to \tilde{P}$ is surjective allows us to define $v: (FU)^i R \to (QS)^i P$ such that $(QS)^i u \cdot v = \tilde{v}$. Let $r_i: (FU)^i R \to (QS)^i B$ be the unique map such that $(QS)^i d^i \cdot r_i = \eta QS \tilde{P} \cdot p_i$, $(QS)^i d^i \cdot r_i = v$, $(QS)^i d^i \cdot r_i = q_i \cdot \text{FU} \epsilon R$ (it is easy to see that such $r_i$ exists, because $(QS)^i B$ is the kernel triple of $(QS)^i d^0$ and $(QS)^i d^3$). Similarly let $(QS)^i d^0 \cdot r_2 = q_1 \cdot \epsilon F U R$, $(QS)^i d^2 \cdot r_2 = v$, and $(QS)^i d^2 \cdot r_2 = QS \eta P \cdot p_1$. Now we have:

$$( (QS)^i \delta \cdot r_2 - (QS)^i \delta \cdot r_1 ) \cdot \sum_{i=0}^2 (\epsilon^i)^i$$

$$= (QS)^i (P - s^0 \cdot d^0) \cdot (q_1 \cdot \epsilon^0 - v + QS \eta P \cdot p_1) \cdot \sum_{i=0}^2 (\epsilon^i)^i$$

$$- (QS)^i (P - s^0 \cdot d^0) \cdot (\eta QS \tilde{P} \cdot p_1 - v + q_1 \cdot \epsilon^i) \cdot \sum_{i=0}^2 (\epsilon^i)^i$$

$$= - (QS)^i (P - s^0 \cdot d^0) \cdot \left( \sum_{j=0}^1 (\epsilon^i)^j \right) \cdot p_1 \cdot \left( \sum_{i=0}^2 (\epsilon^i)^i \right)$$

$$= \left( \sum_{j=0}^1 (\epsilon^j)^j \right) \cdot (QS)^i \delta \cdot p_2,$$

and similarly

$$( \sum_{j=0}^2 (\epsilon^j)^j ) \cdot ( (QS)^i \delta \cdot r_2 - (QS)^i \delta \cdot r_1 ) = (QS)^i \delta \cdot q_2 \cdot \sum_{i=0}^2 (\epsilon^i)^i .$$

Hence $(QS \delta \cdot p_2, (QS)^i \delta \cdot r_2 - (QS)^i \delta \cdot r_1, (QS)^i \delta \cdot q_2)$ is a cocycle in the bicohomology double complex; we will denote its cohomology class by $[p]$ and call $[p]$ the obstruction of $p$. We say $p$ is unabstructed if $[p] = 0$. This terminology is justified by the next two results.

**Proposition 2.1.** The cohomology class of $(QS \delta \cdot p_2, (QS)^i \delta \cdot r_2 - (QS)^i \delta \cdot r_1, (QS)^i \delta \cdot q_2)$ is independent of the choices of $s$: $\text{SUM} \to \text{SUE}$
and \( t: \text{SUP} \to \text{SUP} \).

**Proof.** Once we have \( p, q, \) and \( v \) the maps \( p_2, q_2, r_1, \) and \( r_2 \) are uniquely determined. So suppose \( \sigma_0, \sigma_1, \tau_1, \rho_1, \rho_2 \) are different choices of \( p_0, p_1, q_1, r_1, r_2 \) and construct simplicial homotopies as in [1]. Specifically let \( Q S d_0 \cdot \tilde{h}^0 = p_0, Q S d_1 \cdot \tilde{h}^0 = \sigma_0, Tu \cdot \tilde{h}^0 = \tilde{h}^0 \), and

\[
Q S d_0 \cdot v' = Q S d_0 \cdot p, Q S d_1 \cdot v' = Q S d_1 \cdot \sigma_1.
\]

Considering the maps \( p, v', \) and \( h^0 \cdot e^1 \) from \((FU)^{d}R \) to \( Q SP \) we see that there exists \( h^*:(FU)^{d}R \to Q SB \) such that \( Q S d_0 \cdot h^0 = p, Q S d_1 \cdot h^0 = v', \) and \( Q S d_2 \cdot h^0 = h^0 \cdot e^1 \). Similarly there exists \( h^1:(FU)^{d}R \to Q SB \) such that \( Q S d_0 \cdot h^1 = h^0 \cdot e^0, Q S d_1 \cdot h^1 = v', \) and \( Q S d_2 \cdot h^1 = \sigma_1 \). From these relations it is easy to compute that \( (Q S \partial \cdot h_0 - Q S \partial \cdot h_1) \cdot \sum_{i=0}^{2} (-1)^i e^i = Q S \partial \cdot p_2 - Q S \partial \cdot \sigma_2 \).

Now let \( w: F U R \to (Q S)^{d}P \) be such that \((Q S)^{d}w = (Q S)^{d} \cdot q_1 \) and \((Q S)^{d}w = (Q S)^{d} \cdot \tau_1 \), where \( \tau_1 \) “lifts” \( \sigma_0 \). As above let \( k^0, k^1: F U R \to (Q S)^{d}B \) be determined by the conditions

\[
(Q S)^{d} \cdot k^0 = q_1, (Q S)^{d} \cdot k^0 = w, (Q S)^{d} \cdot k^1 = Q S \eta P \cdot h^0,
\]

\[
(Q S)^{d} \cdot k^0 = \varepsilon \eta Q S P \cdot h^0, (Q S)^{d} \cdot k^1 = w,
\]

and \((Q S)^{d} \cdot k^1 = \tau_1 \). Again one finds that \( (Q S)^{d} \cdot \varepsilon \cdot \sum_{i=0}^{2} (-1)^i e^i - ((Q S)^{d} \cdot \partial \cdot k^0 - (Q S)^{d} \cdot \partial \cdot k^1) = (Q S)^{d} \cdot \partial \cdot q_2 - (Q S)^{d} \cdot \partial \cdot \tau_2 \). Finally,

\[
(Q S)^{d} \cdot \partial \cdot k^0 - (Q S)^{d} \cdot \partial \cdot k^1 \cdot \sum_{i=0}^{1} (-1)^i e^i - \left( \sum_{j=0}^{1} (-1)^j \varepsilon^j \right) \cdot (Q S \partial \cdot h^0 - Q S \partial \cdot h^1) = (Q S)^{d} \cdot \partial \cdot p_1 - (Q S)^{d} \cdot \partial \cdot p_2 - (Q S)^{d} \cdot \partial \cdot r_1 + (Q S)^{d} \cdot \partial \cdot r_2.
\]

Hence the cohomology class of \((Q S \partial \cdot p_2, (Q S)^{d} \cdot \partial \cdot r_2 - (Q S)^{d} \cdot \partial \cdot r_1, (Q S)^{d} \cdot q_1)\) agrees with that of \((Q S \partial \cdot p_2, (Q S)^{d} \partial \cdot p_2 - (Q S)^{d} \partial \cdot r_1, (Q S)^{d} \partial \cdot \tau_2)\), as was to be shown.

**Theorem 2.2.** A surjection \( p: R \to M \) arises from an extension if and only if \( p \) is unobstructed.

**Proof.** Suppose \( p \) arises from an extension \( 0 \to A \to C \xrightarrow{\theta} R \to 0 \) and let \( K \) be the kernel pair of \( \theta \). Then we have a commutative diagram:

![Diagram](attachment:image.png)

Hence the cohomology class of \((Q S \partial \cdot p_2, (Q S)^{d} \partial \cdot r_2 - (Q S)^{d} \partial \cdot r_1, (Q S)^{d} \partial \cdot q_1)\) agrees with that of \((Q S \partial \cdot p_2, (Q S)^{d} \partial \cdot p_2 - (Q S)^{d} \partial \cdot r_1, (Q S)^{d} \partial \cdot \tau_2)\), as was to be shown.
Moreover we can find $\sigma_0: \text{FUR} \to \text{QSC}$ such that $\text{QS} \circ \sigma_0 = \eta R \cdot \varepsilon R$. If we let $\sigma_i: (\text{FU})^i R \to \text{QSK}$ be such that $\text{QSe} \cdot \sigma_i = \sigma_0 \cdot \varepsilon^i$ and $\tau_i: \text{FUR} \to (\text{QS})^i K$ such that $(\text{QS})^i \varepsilon^i \cdot \tau_i = \gamma^i \cdot \sigma_0$ for $i = 0, 1$ then $\text{QSV}_s \cdot \sigma_0$ serves as $p_0$, $\text{QSV}_s \cdot \sigma_1$ as $p_1$, and $(\text{QS})^i \nu_i \cdot \tau_1$ as $q_i$. By 2.1 we can assume that things have been so arranged. But then using the fact that $(\text{QS})^i \varepsilon^0$, $(\text{QS})^i \varepsilon^i$ is a kernel pair for each $j \geq 0$, one can show that

$$\text{QSe} \cdot \sigma_i \cdot \sum_{j=0}^i (-1)^j \varepsilon^j = 0,$$

$$(\text{QS})^i (K - t^0 \cdot \varepsilon^0) \cdot \left[ (\sum_{j=0}^i (-1)^j \gamma^j) \cdot \sigma_1 - \tau_1 \cdot (\sum_{j=0}^i (-1)^j \varepsilon^j) \right] = 0,$$

and

$$(\text{QS})^i (K - t^0 \cdot \varepsilon^0) \cdot \left( \sum_{j=0}^i (-1)^j \gamma^j \right) \cdot \tau_1 = 0.$$

From this it follows that $\text{QSe} \cdot p_2 = 0$, $(\text{QS})^2 \partial \cdot r_2 - (\text{QS})^2 \partial \cdot r_1 = 0$, and $(\text{QS})^3 \partial \cdot q_3 = 0$. Thus $[p] = 0$.

Conversely, suppose $[p] = 0$. Then there exist $\tau: (\text{FU})^i R \to \text{QSC}(Z A)$, $\rho: \text{FUR} \to (\text{QS})^i Z A$ with $\varepsilon = \text{QSe} \cdot p_2$, $\gamma \cdot \rho = (\text{QS})^2 \partial \cdot q_3$, and $\rho \cdot \varepsilon - \gamma \cdot \tau = (\text{QS})^2 \partial \cdot r_2 - (\text{QS})^2 \partial \cdot r_1$. Here we abbreviate $\sum_{j=0}^i (-1)^j \varepsilon^j = \varepsilon$ and similarly for $\gamma$. Now $p_1 = p_1 - \tau$, $q_1 = q_1 - \rho$ serve as new $p_1$, $q_1$, and also give $\bar{p}_2$, $\bar{q}_2$, $\bar{r}_1$, $\bar{r}_2$. We have

$$\text{QSe} \cdot (P - s^0 \cdot d^0) \cdot \text{QSe} \cdot \bar{p}_2 = \text{QSe} \cdot (P - s^0 \cdot d^0) \cdot p_1 \cdot \varepsilon$$

$$= \text{QSe} \cdot (P - s^0 \cdot d^0) \cdot p_1 \cdot \varepsilon - \text{QSe} \cdot (P - s^0 \cdot d^0) \cdot \tau \cdot \varepsilon$$

$$= \text{QSe} \cdot (P - s^0 \cdot d^0) \cdot p_1 \cdot \varepsilon$$

$$- \tau \cdot \varepsilon + \text{QSe}^0 \cdot \text{QSe}^0 \cdot \tau \cdot \varepsilon$$

$$= \text{QSe} \cdot p_2 - \tau \cdot \varepsilon$$

$$= 0$$

because the kernel of $\text{QSe}^0$ is $\text{QSe}(Z(A, P))$ which contains $\text{QSe}(Z A)$. Similar computations yield $(\text{QS})^i (P - s^0 \cdot d^0) \cdot (\text{QS})^i \partial \cdot \bar{q}_2 = 0$ and

$$(\text{QS})^i (P - s^0 \cdot d^0) \cdot (\text{QS})^i \partial \cdot \bar{r}_2 - (\text{QS})^i (P - s^0 \cdot d^0) \cdot (\text{QS})^i \partial \cdot \bar{r}_1 = 0.$$
subscripts adjoined. For example, \( \theta \cdot s = M \) means \( \theta_x \cdot s_x = M_x \) for each \( x \) in \( X \). Our assumption that \( (QS)^s \delta \cdot q_2 \) e.t.c. are all zero translates into the following three equations in \( (\text{Sets}^{X})_{\mathbb{G}} \):

\[
\begin{align*}
(\text{i}) & \quad p_i \cdot u_1 \cdot T \xi_1 + p_i \cdot u_i \cdot \mu R + p_i \cdot u_i \cdot T u_1 = 0 \\
(\text{ii}) & \quad G^p p_i \cdot G u_1 \cdot q_i - G^p p_i \cdot \delta'(A \times E) \cdot q_i + G^p p_i \cdot G u_1 \cdot \xi_2 = 0 \\
(\text{iii}) & \quad G p_i \cdot q_i \cdot \xi_1 - G p_i \cdot G u_1 \cdot \lambda P \cdot T q_i = G p_i \cdot G u_1 \cdot \lambda R \cdot T \xi_2 - G p_i \cdot u_i \cdot T u_1.
\end{align*}
\]

Here \( \lambda : TG \to GT \) is the distributive law (see [5]), and \( p_1 \) (or \( p_2 \)) is the first (or second) projection from the appropriate product. Since our presentation has now begun to differ significantly from that of Barr [1], we will provide more detail than earlier in the paper. Let \( C = A \times R \), and define \( \zeta_1 : TC \to C, \zeta_2 : C \to GC \) by the conditions \( p_i \cdot \xi_1 = p_i \cdot u_1 \cdot T(p_i \times s \cdot p \cdot p_2) + p_i \cdot u_i \cdot T(p_2) \cdot \xi_1 = \xi_1 \cdot T(p_2) \cdot G p_i \cdot \xi_2 = \lambda \cdot p_i + G p_i \cdot q_i \cdot p_2, \) \( G p_2 \cdot \xi_2 = \xi_2 \cdot p_2 \). We claim that \((C, \zeta_1, \zeta_2)\) is in \( (\text{Sets}^{X})_{\mathbb{G}} \). Besides the \"cocycle identities\" listed above, the only fact we need is that

\[ \nu_i : T(A \times E) \to A \times E \]

has the following property: For each \( g : X \to A \) and \( f : X \to A \times E \) we have

(iv) \( p_i \cdot u_1 \cdot T([g + p_i \cdot f] \times d^i \cdot f) = p_i \cdot u_1 \cdot T(g \times d^i \cdot f) + p_i \cdot u_i \cdot T f \). Since this amounts to a combinatorial identity, we relegate its proof to the Appendix. Using (i) and (iv) we can prove that \( \zeta_1 \) is associative:

\[
\begin{align*}
p_i \cdot \zeta_1 \cdot T \xi_1 & = [p_i \cdot u_1 \cdot T(p_i \times s \cdot p \cdot p_2) + p_i \cdot u_i \cdot T(p_2) \cdot T \xi_1 \]
& = p_i \cdot u_1 \cdot T([p_i \cdot u_1 \cdot T(p_i \times s \cdot p \cdot p_2) + p_i \cdot u_i \cdot T(p_2)] \times s \cdot p \cdot \zeta_1) \cdot T p_2 \\
& \quad + p_i \cdot u_i \cdot T \xi_1 \cdot T^2 p_2 \\
& = p_i \cdot u_1 \cdot T([p_i \cdot u_1 \cdot T(p_i \times s \cdot p \cdot p_2)] \times \gamma \cdot T s \cdot T p \cdot T p_2) \\
& \quad + p_i \cdot u_i \cdot T u_i \cdot T^2 p_2 + p_i \cdot u_i \cdot T \xi_1 \cdot T^2 p_2 \\
& = p_i \cdot u_i \cdot T(\nu_1 \cdot T(p_i \times s \cdot p \cdot p_2)) + p_i \cdot u_i \cdot \mu R \cdot T^2 p_2 \\
& = p_i \cdot u_1 \cdot \mu(A \times E) \cdot T^2(p_i \times s \cdot p \cdot p_2) + p_i \cdot u_i \cdot \mu R \cdot T^2 p_2 \\
& = p_i \cdot \zeta_1 \cdot \mu(A \times R);
\end{align*}
\]

the fact that \( p_2 \cdot \zeta_1 \cdot T \xi_1 = p_2 \cdot \zeta_1 \cdot \mu(A \times R) \) is an easy computation. Notice that in the above computation we have taken

\[ g = p_i \cdot u_1 \cdot T(p_i \times s \cdot p \cdot p_2) \]

and \( f = u_i \cdot T p_2 \) in (iv). Before proving that \( \zeta_1 \) is unitary, we show that \( u_i \) is \"normalized\":

\[
\begin{align*}
0 = (p_i \cdot u_1 \cdot T \xi_1 - p_i \cdot u_i \cdot \mu R + p_i \cdot u_i \cdot T u_i) \cdot \eta TR \\
& = p_i \cdot u_1 \cdot \eta R \cdot \xi_1 - p_i \cdot u_i + p_i \cdot u_i \cdot \eta R \cdot u_i \\
& = p_i \cdot u_1 \cdot \eta R \cdot \xi_1.
\end{align*}
\]

But composing this equation with \( \eta R \) gives \( p_i \cdot u_i \cdot \eta R = 0 \), and from
this it follows that $\zeta_1$ is unitary:

$$
\zeta_1 \cdot \eta(A \times R) = [p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) \cdot \eta(A \times R) \\
+ p_1 \cdot u_1 \cdot Tp_2 \cdot \eta(A \times R)] \times \xi_1 \cdot Tp_2 \cdot \eta(A \times R) \\
= [p_1 \cdot (p_1 \times s \cdot p \cdot p_2) \cdot \eta(A \times R) \\
+ p_1 \cdot u_1 \cdot \eta R \cdot T^s p_2] \times \xi_1 \cdot \eta R \cdot T^s p_2 \\
= p_1 \times p_2.
$$

The computations which show that $\zeta_2$ is counitary and coassociative use only (ii) above, and will be omitted. The "compatibility" of $\zeta_1$ and $\zeta_2$ uses (iii) and (iv) above, and proceeds as follows:

$$
Gp_1 \cdot G\zeta_1 \cdot \lambda(A \times R) \cdot T_{\zeta_2} \\
= G(p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) + p_1 \cdot u_1 \cdot Tp_2) \cdot \lambda(A \times R) \cdot T_{\zeta_2} \\
+ Gp_1 \cdot G\nu_1 \cdot GT(p_1 \times s \cdot p \cdot p_2) \cdot \lambda(A \times R) \cdot T_{\zeta_2} \\
= Gp_1 \cdot G\nu_1 \cdot (A \times E) \cdot TG(p_1 \times s \cdot p \cdot p_2) \cdot T_{\zeta_2} \\
+ Gp_1 \cdot G\nu_1 \cdot \lambda R \cdot Tp_2 \cdot T_{\zeta_2} \\
= Gp_1 \cdot G\nu_1 \cdot \lambda(A \times E) \cdot T([\alpha_2 \cdot p_1 + Gp_1 \cdot q_1 \cdot p_2] \times Gs \cdot Gp \cdot \xi_2 \cdot p_3) \\
+ Gp_1 \cdot G\nu_1 \cdot \lambda R \cdot T(p_{1} \cdot \eta p_2) \\
= Gp_1 \cdot G\nu_1 \cdot \lambda(A \times E) \cdot T(\alpha_2 \cdot p_1 \times \gamma_2 \cdot s \cdot p \cdot p_2) \\
+ Gp_1 \cdot G\nu_1 \cdot \lambda(A \times E) \cdot Tq_1 \cdot Tp_2 + Gp_1 \cdot G\nu_1 \cdot \lambda R \cdot T_{\zeta_2} \cdot Tp_2 \\
= Gp_1 \cdot G\nu_1 \cdot \lambda(A \times E) \cdot T\nu_2 \cdot T(p_1 \times s \cdot p \cdot p_2) \\
+ Gp_1 \cdot \nu_2 \cdot \xi_1 \cdot Tp_2 + Gp_1 \cdot \nu_2 \cdot u_1 \cdot Tp_2 \\
= Gp_1 \cdot \nu_2 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) + \alpha_2 \cdot p_1 \cdot u_1 \cdot Tp_2 + Gp_1 \cdot q_1 \cdot \xi_1 \cdot Tp_2 \\
= \alpha_2 \cdot p_1 \cdot \zeta_1 + Gp_1 \cdot q_1 \cdot p_2 \cdot \zeta_1 \\
= Gp_1 \cdot \zeta_2 \cdot \zeta_1;
$$

here, again, that $Gp_2 \cdot G\zeta_2 \cdot \lambda(A \times R) \cdot T_{\zeta_2} = Gp_2 \cdot \zeta_2 \cdot \zeta_1$ is obvious. Notice that we have not used (iv) as it stands, but rather the analog of (iv) for $GP = G(A \times E)$. We have taken $g = \alpha_2 \cdot p_1$ and $f = q_1 \cdot p_2$. At any rate, $(C, \zeta_1, \zeta_2)$ is in $(Sets^{[X]})_o^i$ and the first injection, second projection give us an exact sequence $0 \rightarrow A \xrightarrow{i} A \times R = C \xrightarrow{p_2} R \rightarrow 0$ in $(Sets^{[X]})_o^i$. Define $h: C \rightarrow E$ by $h = \omega \cdot p_1 + s \cdot p \cdot p_2$. Clearly $\pi \cdot h = p_1 \cdot p_2$ and $h \cdot i = \omega$, so that if $h$ is a morphism in $(Sets^{[X]})_o^i$ then we will have produced an extension from which $p$ arises, and the proof will be complete. But we have:

$$
h \cdot \zeta_1 = \omega \cdot (p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) + p_1 \cdot u_1 \cdot Tp_2) + s \cdot p \cdot \xi_1 \cdot Tp_2 \\
= \omega \cdot p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) \\
+ \gamma_1 \cdot Ts \cdot Tp \cdot Tp_2 - s \cdot p \cdot \xi_1 \cdot Tp_2 + s \cdot p \cdot \xi_1 \cdot Tp_2 \\
= \omega \cdot p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) + p_1 \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2) \\
= d^p \cdot \nu_1 \cdot T(p_1 \times s \cdot p \cdot p_2)
$$
\[ \gamma_1 \cdot Td^g \cdot T(p_1 \times s \cdot p_d) = \gamma_1 \cdot T(\omega \cdot p_1 + s \cdot p_d) = \gamma_1 \cdot T \theta , \]

and

\[ Gh \cdot \zeta_2 = G(\alpha_2 p_1 + Gp_1 q_1 p_2) + Gs \cdot Gp \cdot \xi_2 p_2 = \gamma_2 \cdot \omega \cdot p_1 + \gamma_2 \cdot s \cdot p_2 - Gs \cdot Gp \cdot \xi_2 p_2 + Gs \cdot Gp \cdot \xi_2 p_2 = \gamma_2 \cdot h . \]

III. The Action of \( H^1 \).

**Theorem 3.1.** Let \( p : R \to M \) be unobstructed, and let \( \Sigma \) denote the equivalence classes of extensions of \( R \) by \( A \) which induce \( p \). Then the group \( H^1(R, ZA) \) acts on \( \Sigma \) as a principal homogeneous representation.

**Proof.** It is shown in [5] that \( H^1(R, ZA) \) is in one-one correspondence with the set of equivalence classes of singular extensions of \( R \) by \( ZA \). Once this is known, Barr's proof of this proposition [1] translates almost verbatim into a proof for sheaves.

**Appendix.** In this appendix we give a proof of equation (iv) above (§Ⅱ), and compare Barr's constructions [1] to our own. To dispose of equation (iv), recall that given a commutative algebra \( A \), its structure map \( \alpha : TA \to A \) takes a polynomial in elements of \( A \) to the "value" of the polynomial. That is, \( \alpha \) remembers that \( A \) is an algebra and uses the algebra operations in \( A \) to compute the polynomial. Now multiplication in \( P = A \times E \) is defined by \((a_1, x_1)(a_2, x_2) = (a_1 a_2 + x_1 a_2 + a_1 x_2, x_1 x_2) \) where \( a_1 a_2 \) and \( a_1 x_2 \) denote the value of \( x \) on \( a \).

**Proposition A.1.** Given \( a_i \in A \), \( x_i \in E \) for \( 1 \leq i \leq n \) we have

\[
\prod_{i=1}^{n} (a_i, x_i) = \left( \sum f(1), \cdots, f(n), x_1 \cdots x_n \right)
\]

where the sum is taken over all functions \( f : n = \{1, 2, \cdots, n\} \to \{a, x\} \) such that \( f \) is not identically equal to \( x \).

**Proof.** By induction on \( n \). We have

\[
\prod_{i=1}^{n} (a_i, x_i) = \left( \sum f(1), \cdots, f(n - 1), x_1 \cdots x_{n-1} \right)(a_n, x_n)
\]

\[
= \left( \sum f(1), \cdots, f(n - 1) a_n + \sum f(1) a_n + f(n - 1) x_n \right) + x_1 \cdots x_n a_n, x_1 \cdots x_n
\]

\[
= \left( \sum f(1), \cdots, f(n), x_1 \cdots x_n \right)
\]
where the indexing sets for the sums are clear.

**Proposition A.2.** Given \( a_i, b_i \in A, x_i \in E \) for \( 1 \leq i \leq n \) we have that \( \prod_{i=1}^{n} (a_i + b_i, x_i) \) and \( \prod_{i=1}^{n} (b_i, \omega a_i + x_i) + \prod_{i=1}^{n} (a_i, x_i) \) have the same first coordinates.

**Proof.** Induction on \( n \) and Proposition A.1.

\[
\prod_{i=1}^{n} (a_i + b_i, x_i)
= (\Sigma g(1), \cdots g(n - 1), h(1), \cdots h(n - 1), x_1 \cdots x_n) (a_n + b_n, x_n)
\]

where \( g \)'s run through the set of functions from \( n \rightarrow \{ b, \omega a + x \} \) which are not identically \( \omega a + x \) and the \( h \)'s through all \( n - 1 \rightarrow \{ a, x \} \) which are not identically \( x \). Hence we get as first coordinate

\[
\Sigma g(1), \cdots g(n - 1), a_n + \Sigma g(1), \cdots g(n - 1), b_n
+ \Sigma h(1), \cdots h(n - 1), a_n + \Sigma h(1), \cdots h(n - 1), b_n
+ \Sigma g(1), \cdots g(n - 1), x_n + \Sigma h(1), \cdots h(n - 1), x_n
+ x_1 \cdots x_{n-1} a_n + x_1 \cdots x_{n-1} b_n.
\]

The third, sixth, and seventh terms of this sum give us

\[
\Sigma h(1), \cdots h(n),
\]

Since \( \Sigma h(1), \cdots h(n - 1), b_n = \prod_{i=1}^{n-1} (\omega a_i + x_i) b_n - x_i \cdots x_{n-1} b_n \) the remaining terms give us \( \Sigma g(1), \cdots g(n) \). This completes the proof.

Taking into account the remarks preceding Proposition A.1, equation (iv) follows immediately from A.2.

In [1] Barr constructs the extension which realizes an unobstructed \( p \) as a certain coequalizer. In the notation of our §II, his diagram on page 365 would look like:

\[
\begin{array}{ccc}
0 & \rightarrow & (A, \alpha_i) \\
\downarrow & & \downarrow A \\
(T^2 R, \mu TR) & \xrightarrow{O \times \mu R} & (A \times TR, -) \\
\downarrow & & \downarrow (p_1 + p_1 \cdot u_1 \cdot p_2) \times \xi_1 \cdot p_2 \\
(T^2 R) & \xrightarrow{\mu R} & (TR, \mu R) \\
\downarrow & & \downarrow \xi_1 \\
0 & \rightarrow & (R, \xi_1)
\end{array}
\]
He uses the coequalizer \((p_1 + p_1 \cdot u_1 \cdot p_2) \times \xi_1 \cdot p_2\) to define the algebra which gives the extension, and then must make some rather tedious computations to verify that all requirements are met. One knows that if an extension exists, then its underlying set will have to be \(A \times \mathbb{R}\); the only question is how the algebra structure on \(A \times \mathbb{R}\) is “twisted”. By passing to the equivalent category of \(T\)-algebras it becomes clear exactly how the cocycle should be used to produce this twisted structure. All of this was first noticed by Beck in the case of singular extensions [2]. At any rate, the “globalization” of Barr’s results seems to require that we pass to \((\text{Sets}^{|x|})^T\).

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