CENTRAL EMBEDDINGS IN SEMI-SIMPLE RINGS

Shimshon A. Amitsur
CENTRAL EMBEDDINGS IN SEMI-SIMPLE RINGS

S. A. AMITSUR

A ring \( S \) is a central extension of a subring \( R \) if \( S = RC \) and \( C \) is the centralizer of \( R \) in \( S \), i.e., \( C = \{ s \in S; sr = rs \} \) for every \( r \in R \). We shall also say that \( R \) is centrally embedded in \( S \).

We have shown that if a ring \( R \) is centrally embedded in a simple artinian ring then \( R \) is a prime \( \text{Ore} \) ring and its quotient ring \( Q \) is the minimal central extension of \( R \) which is a simple artinian ring; furthermore, the centralizer of \( R \) can be characterized. In the present note we extend these results and show that rings which can be centrally embedded in semi-simple artinian rings are semi-prime \( \text{Ore} \) rings with a finite number of minimal primes and their rings of quotients are the minimal central extension of this type.

2. The Ring \( Q_0(R) \). We recall some definitions and results of [1].

Let \( R \) be an associative ring (not necessarily with a unit) and let \( L_0(R) \) be the set of all (two-sided) ideals \( A \) of \( R \) with the property:

(A) \( \forall x \in R, Ax = 0 \Rightarrow x = 0 \).

The set \( L_0(R) \) is a filter. That is: closed under finite intersection and inclusion. We shall also assume henceforth that \( R \in L_0(R) \) i.e. \( Rx = 0 \Rightarrow x = 0 \).

Consider every \( A \in L_0(R) \) as left \( R \)-module and define the ring \( Q_0(R) = \lim \text{Hom}_R(A, R) \), where \( A \) ranges over all \( A \in L_0(R) \). A more detailed description of \( Q_0(R) \) is as follows: Let \( U = \bigcup \text{Hom}_R(A, R), A \in L_0(R) \), and in \( U \) we define an equivalence relation, addition and multiplication as follows:

For \( \alpha: A \to R, \beta: B \to R \) and \( A, B \in L_0(R) \) we put:

(i) \( \alpha + \beta: A \cap B \to R \) defined by \( x(\alpha + \beta) = x\alpha + x\beta \) for \( x \in A \cap B \).

(ii) \( \alpha\beta: BA \to R \) by: \( (\Sigma ba)\alpha\beta = \Sigma [b(a\alpha)]\beta \) for \( b \in B, a \in A \).

(iii) \( \alpha \equiv \beta \) if there exists \( C \subseteq A \cap B, C \in L_0(R) \) for which \( c\alpha = c\beta \) for every \( c \in C \).

The ring \( Q_0(R) \) is the ring of equivalence classes of \( U \) with respect to preceding definitions. Furthermore, \( R \) is canonically mapped into \( Q_0(R) \) by identifying \( R \) with the right multiplications on \( R \).

The center \( \Gamma = \Gamma(R) \) of \( Q_0(R) \) can be characterized as the set of all \( \bar{\gamma} \in Q_0(R) \) which have a representative \( \gamma \in \text{Hom}(A, R) \) such that \( \gamma \) is in
fact a bi-R-module homomorphism of \( A \) into \( R \), i.e. it satisfies \((ax)\gamma = (a\gamma)x\) and \((xa)\gamma = x(ay)\) for \( a \in A, x \in R \). Also \( \bar{\gamma} \in \Gamma \) if and only if it commutes with the element of \( R \).

From the results of [1] we quote the following:

If \( R \) is semi-simple artinian, then \( R \) is both a right and left \( \text{\Ore} \) ring and its quotient ring is \( Q_0(R) = R\Gamma \). [1, Theorem 6].

If \( S = RC \) is a simple artinian central extension of \( R \) then \( \Gamma \subseteq C \). \( S = R\Gamma \otimes_C C \) and \( R\Gamma = Q_0(R) \) is also simple artinian [1, Theorem 18].

The ring \( R\Gamma \) is semi-simple artinian if and only if the number of minimal primes \( P \) of \( R \) is finite, and for each \( P, (R/P)\Gamma(R/P) \) is simple artinian. [1, Corollary 13].

It follows also from the proofs of [1, Theorem 10] that the number of simple components of \( R \) equals the number of minimal primes of \( R \).

3. The main result. Let \( S = S_1 \oplus \cdots \oplus S_m \) a direct sum of a finite number of simple rings \( S_i \) with units \( e_i \), and \( 1 = e_1 + e_2 + \cdots + e_n \). The ring \( S \) will be said an extension of minimal length of a subring \( R \) if for every \( i \) there exist \( 0 \neq r \in R \) such that \( re_i = 0 \) for all \( j \neq i \), or equivalently \( r(1 - e_i) = 0 \). This means that for no subring \( S(1 - e_i) = S_1 \oplus \cdots \oplus S_{i-1} \oplus S_{i+1} \oplus \cdots \oplus S_m \) the subring \( R(1 - e_i) \) is isomorphic with \( R \).

**Lemma 1.** Let \( S = RC \) be a central extension of \( R \), and let \( S = S_1 \oplus \cdots \oplus S_m \) be a direct sum of simple rings \( S_i \) with units \( e_i \). Then:

1. For every central idempotent \( e \), \( Se \) is a central extension of \( Re \); and it is also a direct sum of simple rings with a unit.
2. There exists a direct summand \( Se \) of \( S \) such that \( R \cong Re \), and \( Se \) is a central extension of \( R \) of minimal length.

**Proof.** A central idempotent \( e \) of \( S \) is of the form \( e = e_{i_1} + \cdots + e_{i_r} \), and hence \( Se = S_{i_1} \oplus S_{i_2} \oplus \cdots \oplus S_{i_r} \). Furthermore \( S = RC \) yields that \( Se = (RC)e = (Re)(Ce) \) and the elements of \( Ce \) commute with the elements of \( Ce \), which readily implies that \( Se \) is a central extension of \( Re \).

To prove the second part, we consider the set of all central idempotents \( e \) of \( S \) with the property: "\( re = 0, r \in R \Rightarrow r = 0 \)." Clearly for such \( e \), \( R \cong Re \) by corresponding: \( r \rightarrow re \). The set of these idempotents is not empty since the unit 1 has this property. Each of the central idempotent \( e \) has the form \( e = e_{i_1} + \cdots + e_{i_r}, i_1 < i_2 < \cdots < i_r \). So choose \( e \) of this set with minimal \( \rho \). Then \( Se \) is a central extension of \( Re \) of minimal length, since the minimality of \( \rho \) implies that for any \( 1 \leq \lambda \leq \rho \), there exists \( r \neq 0 \) such that \( r(e - e_{i_\lambda}) = 0 \).

The preceding lemma shows that if a ring \( R \) has a central extension
S of the type described above, then replacing S by a direct summand we get a central extension of minimal length of a ring isomorphic with R. We can, therefore, restrict ourselves to the study of central extension of minimal length. Our results is the following.

**Theorem A.** Let $S = RC$ be a central extension of $R$ of minimal length then $R$ is semi-prime and we can embed $\Gamma \subseteq C$. Furthermore, $R \Gamma$ is also a central extension of $R$ of the same type with the same number of components.

**Theorem B.** Let $S = RC$ be a semi-simple artinian ring and a central extension of $R$ of minimal length then $R = Q_0(R)$ is also semi-simple artinian and $S = R \Gamma \otimes_R C$.

In view of the results quoted from [1] we deduce that:

**Corollary C.** If $R$ has a central extension which is a semi-simple artinian ring, then $R$ is a semi-prime (right and left) $\mathcal{O}$re ring with a finite number of minimal primes. Its ring of quotient is $Q_0(R)$ and it is a minimal semi-simple artinian central extension of $R$.

4. **Proofs.** Before proceeding with the proof we need a few lemmas.

**Lemma 2.** Let $S = RC$ be a central extension of $R$ of minimal length, then an ideal $A$ in $R$ belongs to $L_0(R)$ if and only if $AC = S$.

Indeed, let $S = S_1 \oplus \cdots \oplus S_n$, $S_i$ simple with a unit $\epsilon_i$. If $AC = S$ and $Ax = 0$ for some $x \in R$, then $Sx = (AC)x = (Ax)C = 0$ but $S$ has a unit and so $x = 0$, i.e. $A \in L_0(R)$. Conversely, it suffices to show that $AC \cap S_i \neq 0$, since then $AC \cap S_i$ is a nonzero ideal in the simple ring implies that $S_i = AC \cap S_i$. This in turn yields that $AC \supseteq S_i$ and, therefore $AC \supseteq S_1 \oplus \cdots \oplus S_n = S$. To prove that $AC \cap S_i \neq 0$, we note that if $AC \cap S_i = 0$ then $A \epsilon_i \subseteq AS_i \subseteq ARC \cap S_i \subseteq AC \cap S_i = 0$. Let $P = \{r, re_i = 0\}$ and $Q = \{r \in R, r(1-\epsilon_i) = 0\}$. Since $S$ is of minimal length it follows that $P \cap Q = 0$. $Q \neq 0$ and $P \supseteq A$. Thus $AQ \subseteq P \cap Q = 0$ which contradicts the assumption that $A \in L_0(R)$ (i.e., $A$ satisfies (A) of §2).

We can follow now the proofs of [1] Lemma 14 and show:

**Lemma 3.** If $S$ is as above then there is an embedding of $\Gamma$ into the center of $S$ which contains $C$.

**Proof.** Let $\alpha: A \to R$, $A \in L_0(R)$ be a representative of an element $\bar{\alpha} \in \Gamma$. First we show that there is a unique element $c_\alpha \in C$
depending on \( \tilde{\alpha} \) (and not on the representative \( \alpha \)) such that \( a\alpha = ac_\alpha \) for every \( a \in A \). Next we prove that the correspondence: \( \tilde{\alpha} \to \delta_\alpha \) is the required embedding. The proof follows the proof of [I] Lemma 14.

Since \( A \in L_0(R) \), it follows by Lemma 2 that \( AC = S \) and hence \( 1 = \Sigma a_i c_i \) for some \( a_i \in A \) and \( c_i \in C \). Set \( c_\alpha = \Sigma (a_\alpha) c_i \). Since \( \tilde{\alpha} \in \Gamma \), \( \alpha \) is a bi-\( R \) hence for every \( a \in A \):

\[
\alpha \alpha = (\alpha \alpha) 1 = \Sigma (a\alpha) a c_i = \Sigma (a\alpha) \alpha c_i = a \Sigma (a\alpha) c_i = ac_\alpha.
\]

To prove that \( c_\alpha \in C \), we observe that for every \( a \in A \) and \( x \in R \):

\[
(ax)c_\alpha = (ax)\alpha = a(\alpha x) = ac_\alpha x.
\]

Hence, \( a(xc_\alpha - c_\alpha x) = 0 \). Consequently, \( S(xc_\alpha - c_\alpha x) = (CA) (xc_\alpha - c_\alpha x) = 0 \) and since \( 1 \in S \) it follows that \( xc_\alpha - c_\alpha x = 0 \) for every \( x \in R \), i.e. \( c_\alpha \in C \).

The element \( c_\alpha \) which belongs to \( C \), actually commutes also with the elements of \( R \) and hence belongs to the center of \( S \). Indeed, let \( c \in C \) and \( a \in A \) then since \( C \) centralizes \( A \) we have \( \alpha \alpha x = c(\alpha \alpha) \) as \( a\alpha \in R \). Also \( a\alpha = ac_\alpha = c_\alpha a \) and, therefore:

\[
c_\alpha(ac) = (ac_\alpha)c = (a\alpha)c = c(a\alpha) = (ca)c_\alpha = (ac)c_\alpha.
\]

That is, \( c_\alpha \) commutes with all the elements of \( AC = S \) and this means that \( c_\alpha \) is in the center of \( S \).

Next we show that \( c_\alpha \) depends only on \( \tilde{\alpha} \in F \): let \( \beta : B \to R \) be another representative of \( \tilde{\alpha} \) then \( \alpha = \beta \) on some \( D \subseteq A \cap B \) which belongs to \( L_0(R) \). Hence for \( d \in D \):

\[
dc_\alpha = d\alpha = d\beta = dc_\beta.
\]

which implies that \( D(c_\alpha - c_\beta) = 0 \) and therefore \( S(c_\alpha - c_\beta) = (CD) (c_\alpha - c_\beta) = 0 \) which yields \( c_\alpha - c_\beta = 0 \).

Finally \( c_\alpha + \beta = c_\alpha + c_\beta \), \( c_{\alpha \beta} = c_\alpha c_\beta \) since for some ideals in \( L_0(R) \) we have the following relations for their elements:

\[
x(c_\alpha + \beta) = x(\alpha + \beta) = x\alpha + x\beta = xc_\alpha + xc_\beta = x(c_\alpha + c_\beta)
\]

\[
y(c_\alpha c_\beta) = y(\alpha \beta) = (y\alpha)\beta = (y\alpha)c_\beta = y(c_\alpha c_\beta)
\]

and as in preceding proofs this implies that \( c_{\alpha + \beta} = c_\alpha + c_\beta \) and \( c_{\alpha \beta} = c_\alpha c_\beta \).

We, henceforth, identify \( \Gamma \) with its image in \( C \) and thus we may assume that \( \Gamma \subseteq C \).

**Lemma 4.** Let \( S = RC = S_1 \oplus \cdots \oplus S_n \), \( S \), simple with unit \( \varepsilon_i \), be a central extension of \( R \) of minimal type, then \( \varepsilon_i \in \Gamma \).

For let \( P = \{ r \in R, r\varepsilon_i = 0 \} \) and \( Q = \{ r \in R, r(1 - \varepsilon_i) = 0 \} \). Since \( S \) of minimal length, \( P \neq 0 \), \( Q \neq 0 \) and \( P \cap Q = 0 \). We first assert that \( P + Q \in L_0(R) \) and, indeed, \( (QC)\varepsilon_i = (Q\varepsilon_i)C = QC = QRC = QS = \)
$Q \neq 0$ and so $QC \subseteq S$, but $QC$ is an ideal in $S$ and therefore, also in $S$, which yields $QC = S$, since $S$ is simple. A similar proof which uses the fact that $P \epsilon_i \neq 0$ for $j \neq i$ shows that $(PC)\epsilon_i = S_j$. Hence

$$(P + Q)C = \Sigma (P + Q)C_k = \Sigma S_k = S$$

and thus $P + Q \in L_0(R)$ by Lemma 1. Consider now the map $\epsilon: P + Q \to Q$ given by $(p + q)\epsilon = q$. Clearly, this is a bi-$R$-homomorphism, hence $\alpha \in \Gamma$ and so there exists $c_\epsilon \in C$ such that $(p + q)c_\epsilon = q$. Consequently, $(p + q)c_\epsilon = q = q\epsilon_i = (p + q)\epsilon_i$. By the uniqueness of $c_\epsilon$ it follows that $c_\epsilon = \epsilon_i$

We are now in position to prove the main theorems.

$R$ is semi-prime, for if $A^2 = 0$ then $(AC)^2 = 0$ in $S$, but $S$ is semi-prime and so $AC = 0$ which implies that $A = 0$.

Let $S = RC = S_1 \oplus \cdots \oplus S_n$ be a central extension of $R$ of minimal length, with $\epsilon_i$ the units of $S_i$. Put $P = \{r \in R, re_\epsilon = 0\}$, and consider $R$ as a subring of $Q_0(R)$. Then we readily have, since $\epsilon_i \in \Gamma \subseteq Q_0(R)$ that $P = R \cap Q_0(R) (1 - \epsilon_i)$. Furthermore, $P$ is a prime ideal: indeed let $AB \subseteq P$ with $A, B$ ideals in $R$ containing $P$, then since $B \subseteq P, B\epsilon_i \neq 0$ and, therefore, $(BC)\epsilon_i$ is a nonzero ideal in $S_i$ which implies that $BC\epsilon_i = S_i$. Thus:

$$0 = (CP)\epsilon_i \supseteq (CAB)\epsilon_i = A(CB)\epsilon_i = AS_i.$$

This yields that $A\epsilon_i = 0$ and so $A \subseteq P$. We can now apply [1] Theorem 8, which in our case means that $Q_0(R/P) \equiv Q_0(R)\epsilon_i$ and $\Gamma(R/P) \equiv \Gamma(R)\epsilon_i = \Gamma\epsilon_i$.

Denote, $R_1 = Re_\epsilon$ (which isomorphic with $R/P$) and $c_1 = c\epsilon_i$ then $R_1\epsilon_i = R_1C_1 = S_1$ which shows that $R_1$ is a prime ring with a central extension which is a simple ring $S_1$ with a unit. It follows, therefore, by [1] Theorem 18 that $R_1\Gamma(R_1)$ is simple with a unit. Now $\Gamma(R_1) = \Gamma(R/P) = \Gamma\epsilon_i$ by the preceding result. So $R_1(\Gamma\epsilon_i)$ is simple with a unit and note also that $R_1\Gamma\epsilon_i = (R\Gamma)\epsilon_i$. The same follows for all the other idempotents $\epsilon_i$ and so we get that $R\Gamma = R\Gamma\epsilon_i + R\Gamma\epsilon_2 + \cdots + R\Gamma\epsilon_n$ is a direct sum of simple rings with units, which completes the proof of Theorem A.

The proof of Theorem B follows the same lines by applying the second part of [1] Theorem 18 which was quoted in the present note (§2). Namely, if $S$ is semi-simple artinian then each summand $S_i$ is simple artinian and hence, by [1] Theorem 18 $R_1\Gamma_1 = (R\epsilon_i) (\Gamma\epsilon_i) = (R\Gamma)\epsilon_i$ is simple artinian. Furthermore, we also have $R_1\Gamma_1 = Q_0(R/P) = Q_0(R)\epsilon_i$ by (iii) of [1] Theorem B. Thus, $Q_0(R) = \Sigma Q_0(R)\epsilon_i = \Sigma R_1\Gamma_1 = R\Gamma$. 

Finally, \((RC)\varepsilon_i = R_i \Gamma_i \otimes \Gamma_i \varepsilon_i\) for every \(i\), from which it follows that:

\[
RC = \sum R \Gamma_i \otimes \Gamma_i \varepsilon_i = R \Gamma \otimes C
\]

since \(\Gamma = \sum \Gamma \varepsilon_i\) and the elements \(\varepsilon_i\) belong to the center of \(S = RC\). The last isomorphism is given by the mappings \(r \alpha \otimes \varepsilon \rightarrow \sum (r \alpha) \varepsilon_i \otimes \varepsilon_i; r \alpha_i \otimes \varepsilon_i \rightarrow r \alpha_i \otimes \varepsilon_i\).

Corollary C follows now immediately by Theorem 6 and Corollary 13 of [1].

We finish with an immediate corollary of the fact that \(\Gamma \subseteq \text{Cents } S\), and \(\text{Cent } S \subseteq C\):

**Corollary D.** If \(RC\) is a central embedding of \(R\) in a direct sum of simple rings of minimal length, then so is \(R(\text{Cent } C)\).

**References**


Received December 28, 1973.

The Hebrew University of Jerusalem
Shimshon A. Amitsur, Central embeddings in semi-simple rings .............. 1
David Marion Arnold and Charles Estep Murley, Abelian groups, A, such
that \( \text{Hom}(A, - - -) \) preserves direct sums of copies of A .............. 7
Martin Bartelt, An integral representation for strictly continuous linear
operators .................................................................................. 21
Richard G. Burton, Fractional elements in multiplicative lattices ............ 35
James Alan Cochran, Growth estimates for the singular values of
square-integrable kernels .................................................................. 51
C. Martin Edwards and Peter John Stacey, On group algebras of central
group extensions ............................................................................ 59
Peter Fletcher and Pei Liu, Topologies compatible with homeomorphism
groups ......................................................................................... 77
George Gasper, Jr., Products of terminating \( _3F_2(1) \) series ................. 87
Leon Gerber, The orthocentric simplex as an extreme simplex ............... 97
Burrell Washington Helton, A product integral solution of a Riccati
equation ....................................................................................... 113
Melvyn W. Jeter, On the extremal elements of the convex cone of
superadditive \( n \)-homogeneous functions .................................... 131
R. H. Johnson, Simple separable graphs ............................................ 143
Margaret Humm Kleinfeld, More on a generalization of commutative and
alternative rings ............................................................................ 159
A. Y. W. Lau, The boundary of a semilattice on an \( n \)-cell .................. 171
Robert F. Lax, The local rigidity of the moduli scheme for curves .......... 175
Glenn Richard Luecke, A note on quasidiagonal and quasitriangular
operators ....................................................................................... 179
Paul Milnes, On the extension of continuous and almost periodic
functions ....................................................................................... 187
Hidegoro Nakano and Kazumi Nakano, Connector theory ..................... 195
James Michael Osterburg, Completely outer Galois theory of perfect
ing rings ...................................................................................... 215
Lavon Barry Page, Compact Hankel operators and the F. and M. Riesz
theorem ......................................................................................... 221
Joseph E. Quinn, Intermediate Riesz spaces ....................................... 225
Shlomo Vinner, Model-completeness in a first order language with a
generalized quantifier .................................................................. 265
Jorge Viola-Prioli, On absolutely torsion-free rings .............................. 275
Philip William Walker, A note on differential equations with all solutions of
integrable-square ......................................................................... 285
Stephen Jeffrey Willson, Equivariant maps between representation
spheres ......................................................................................... 291