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Let B denote the algebra of bounded analytic functions on the open unit disc D in the complex plane. Let (B, τ) denote B endowed with the topology τ, where τ is chosen from κ, β or σ, respectively, the topology of uniform convergence on compact subsets of D, the strict topology and the topology of uniform convergence on D. This note obtains an integral representation of the form $Tf(z) = \int_\Gamma f(w)K(z,w)dw$ where $\Gamma = \{z : |z| = 1\}$ for the linear operators which are continuous from (B, κ) into (B, σ). This representation is then used to study the convergence of operators in the full algebra of all continuous linear operators from (B, β) into (B, β).

1. Introduction. Let $M(D)$ denote the set of bounded complex valued Borel measures on D. R. C. Buck **[5]** showed that L is a continuous linear functional on $(C(D), \beta)$ if and only if $Lf = \int_D f d\mu$, $\forall f \in C(D)$ for some $\mu \in M(D)$. L. A. Rubel and A. L. Shields **[7]** showed that for any $\mu \in M(D)$ there exists a function h in $L^1(\Gamma)$ such that $\int_D f d\mu = \int_\Gamma f(x)h(x)dx$, $\forall f \in B$ and conversely, that any $h \in L^1(\Gamma)$ determines a measure $\mu \in M(D)$ for which this equality holds. Thus the continuous linear functionals on (B, β) can be represented as integration over Γ with respect to functions in $L^1(\Gamma)$.

Letting both τ_1 and τ_2 be one of the topologies κ, β or σ, let $[\tau_1 : \tau_2]$ denote the algebra of all continuous linear operators from (B, τ_1) into (B, τ_2).

In Theorem 1 it is shown that any linear operator T in $[\beta : \beta]$ can be represented in the form

$$Tf(z) = \int_\Gamma f(w)K(z,w)dw, \qquad \forall f \in B.$$

However, a necessary and sufficient condition on $K(z, w)$ that such a T be in $[\beta : \beta]$ is not known.

The algebra $[\kappa : \sigma]$ is a dense subalgebra of $[\beta : \beta]$ in the compact open topology. In Theorem 3 it is shown that a linear operator T is in $[\kappa : \sigma]$ if and only if $Tf(z) = \int_\Gamma f(w)\ K(z,w)dw$ where the kernel

$K(z, w)$ satisfies certain fixed conditions. One can then associate with every linear operator in $[\beta : \beta]$ an explicit kernel $K(z, w)$. In §4, the convergence of linear operators in $[\beta : \beta]$ is characterized by using the convergence of the sequence of associated kernels. In the last section this convergence criterion is applied to the special type of operators in $[\beta : \beta]$ called multipliers.

2. Definitions. The topology σ, of uniform convergence on D, is defined by the norm

$$\|f\| = \sup\{|f(z)| : |z| < 1\}.$$

The topology κ, of uniform convergence on compact subsets of D, can be defined by the family of semi-norms

$$\|f\|_r = \sup\{|f(z)| : |z| < r\}$$

where $0 < r < 1$. The strict topology β was introduced by R. C. Buck in **[3]** as a topology on the set of bounded continuous functions on a space. It is defined by the family of semi-norms

$$|f|_\phi = \|f\phi\|, \ \phi \in C_0[D],$$

the continuous functions on D which vanish at infinity. The strict topology was first employed to study B in **[4]**. For properties of β and its relation to κ and σ see **[3]**, **[4]**, **[5]** and **[7]**. In particular, a sequence of functions $\{f_n\}$ in B converges strictly to zero if and only if it is uniformly bounded and converges κ (or pointwise) to zero. Also the β bounded subsets of B are precisely the σ bounded subsets.

In **[2]** two appropriate topologies were employed to study $[\beta : \beta]$. From $[\sigma : \sigma]$, the subalgebra $[\beta : \beta]$ inherits the usual operator norm topology where

$$\|T\| = \sup\{\|Tf\| : \|f\| \leqq 1, \ f \in B\}.$$

The second topology is that of uniform convergence on bounded subsets of B which in fact is equivalent on $[\beta : \beta]$ to the compact open topology.

Definition. A net of operators $\{T_\alpha\}$ in $[\beta : \beta]$ converges uniformly on bounded subsets (witten u.b.) to T if and only if given any β open set G in B with $0 \in G$ and any β bounded set S in B, there exists an α' such that if $\alpha > \alpha'$, then $(T_\alpha - T)(S) \subseteq G$.

For properties of $[\beta:\beta]$ in these two topologies see **[2]**. In particular the u.b. bounded subsets are precisely the norm bounded subsets and $[\beta:\beta]$ is a u.b. (and hence norm) closed subalgebra of $[\sigma:\sigma]$.

In should be observed **[1]** that the continuity classes $[\kappa:\sigma]$, $[\beta:\sigma]$, $[\beta:\beta]$ and $[\sigma:\sigma]$ are in fact algebras, they are related by the proper inclusions $[\kappa:\sigma]\subset[\beta:\sigma]\subset[\beta:\beta]\subset[\sigma:\sigma]$, and $[\kappa:\sigma]$ is dense in $[\beta:\beta]$ in the u.b. topology, but in the norm topology $[\kappa:\sigma]$ is dense in only $[\beta:\sigma]$. In the study of the u.b. denseness of $[\kappa:\sigma]$ in $[\beta:\beta]$, the operators T_r play a significant role. Given an operator T in $[\beta:\beta]$, the operator $[T]_r$ (sometimes written T_r) is defined by $T_rf(z)=T(f_r)(z)$ where $f_r(z)=f(rz)$, $f\in B$, and $0<r<1$. An operator T_r is in $[\kappa:\sigma]$ and it is known **[2]** that $\{T_r\}$ converges u.b. to T as $r\uparrow 1$.

Finally, a result of P. Hessler (see **[1]** or **[6]**) shows that a linear operator T is in $[\beta:\tau]$ if and only if whenever a sequence $\{f_n\}$ in B converges strictly to zero, it follows that $\{Tf_n\}$ converges τ to zero, where τ is κ,β or σ.

3. An integral representation. Let z be a fixed point in D. Then given a linear operator T in $[\beta:\beta]$, the linear functional L defined on B by $Lf=Tf(z)$ is a continuous linear functional on (B,β). Therefore, $Tf(z)=Lf=\int_\Gamma f(w)K_z(w)dw$ for some function $K_z(w)$ in $L^1(\Gamma)$. It is difficult to determine the relationship between the various functions K_z that is necessary and sufficient to ensure that $Tf(z)=\int_\Gamma f(w)K(z,w)dw$ will represent an operator in $[\beta:\beta]$. The following gives a necessary condition and a different sufficient condition.

THEOREM 1. *For any linear operator T in $[\beta:\beta]$,*

$$Tf(z)=\int_\Gamma f(w)K(z,w)dw, \qquad \forall f\in B$$

where $K(z',w)=K_{z'}$ is in $L^1(\Gamma)$ for each z' in D and the L^1 norms of all the functions $K_{z'}$ are uniformly bounded.

If $K(z,w)$ satisfies the above necessary conditions and $K(z,w)$ is analytic in D for each fixed w in Γ and bounded on $D\times\Gamma$, then any T so defined is in $[\beta:\beta]$.

Proof. Let T be in $[\beta:\beta]$. Then, as before, let $L_z(f)=Tf(z)$ for z fixed in D. Since $L_z(f)=\int_\Gamma f(w)K_z(w)dw$, we have $\|L_z\|=\|K_z\|_{L^1}$

and $|L_z(f)| = |Tf(z)| \leqq \|T\| \, \|f\|$, where $\|K_z\|_{L^1}$ denotes the usual L^1 norm of K_z on Γ. Hence $\|K_z\|_{L^1} = \|L_z\| \leqq \|T\|$ for each z in D.

For the converse, define $Tf(z) = \int_\Gamma f(w)\,K_z(w)dw = \int_\Gamma f(w)\,K(z,w)dw$. Then $Tf(z)$ is continuous in D since

$$Tf(z_1) - Tf(z) = \int_\Gamma f(w)[K(z_1, w) - K(z, w)]dw$$

$$= \int_\Gamma f(w)(z_1 - z)(2\pi i)^{-1} \int_\gamma K(s, w)[(s - z_1)(s - z)]^{-1}\, ds\, dw$$

where γ is a circle in D with center z_1 and containing z in its interior and hence

$$|Tf(z_1) - Tf(z)| \leqq \|f\| \, |z_1 - z| \sup |(s - z_1)(s - z)|^{-1} \|K\|_R$$

where $\|K\|_R$ is the sup of $|K(z,w)|$ taken over $R = D \times \Gamma$. Thus $|Tf(z_1) - Tf(z)|$ tends to zero as z approaches z_1. Then for any triangle Δ in D, $\int_\Delta Tf(z) = \int_\Delta \int_\Gamma f(w)\,K(z,w)dw = \int_\Gamma \int_\Delta f(w)\,K(z,w)dw = 0$. By Morera's theorem, $Tf(z)$ is analytic in D.

Now $Tf(z)$ is a bounded function since

$$|Tf(z)| \leqq \int_\Gamma |f(w)\,K(z,w)|\, d\,|w| \leqq 2\pi \|K_z\|_{L^1} \|f\| \leqq M \|f\|$$

for all z in D. If $\{f_n\}$ converges strictly to zero, then $Tf_n(z) = L_z(f_n)$ converges to zero. Hence $\{Tf_n\}$ converges pointwise to zero and is uniformly bounded, which implies $\{Tf_n\}$ converges strictly to zero.

Note that additional conditions are imposed on $K(z,w)$ in the converse only to ensure that $Tf(z)$ is analytic. Any $K(z,w)$ which satisfies the necessary conditions and makes Tf analytic will yield a T in $[\beta : \beta]$. It is certainly not necessary that $K(z,w)$ be analytic in z because for any function h in $L^1(\Gamma)$, $Tf(z) = \int_\Gamma f(w)h(w)dw = \int_\Gamma f(w)\,K(z,w)dw$ is strictly continuous and $K(z,w) = h(w)$ need only be defined a.e..

We consider now the case when C is some rectifiable curve inside D and $Tf(z) = \int_C f(w)\,K(z,w)dw$ with $K(z,w)$ in $L^1(C)$ for any z. As in the previous theorem, if T is in $[\beta : \beta]$, then the functions $K_z(w)$ are uniformly bounded in $L^1(C)$ norm. Hence T is in $[\kappa : \sigma]$, because if $\{f_n\}$ converges κ to zero, then $\{f_n\}$ converges uniformly to zero on C and $|Tf_n(z)| \leqq$ (length of C) $\|f_n\|_C \|K_z\|_{L^1(C)}$.

Now we obtain a representation formula for the operators in $[\kappa : \sigma]$. Given an operator T in $[\kappa : \sigma]$, there is an M and an $r < 1$ such that $\|Tf\| \leqq M\|f\|_r$ for all f in B. Letting $f(z) = z^k$, we obtain $\|T(z^k)\| \leqq M\|z^k\|_r = M(r)^k$. Hence $\|T(z^k)\|^{1/k} \leqq rM^{1/k}$ and lim sup $\|T(z^k)\|^{1/k} \leqq r$.

THEOREM 2. *If T is in $[\kappa : \sigma]$, then there exists a function $K(z, w)$ analytic for $|z| < 1$ and $\infty > |w| > r_0$ for some $r_0 < 1$ and such that if $1 > r_1 > r_0$, then there exists an M such that $|K(z, w)| \leqq M$ for all $|z| < 1$, $|w| \geqq r_1$ and such that*

$$Tf(z) = \int_{|w|=r_1} f(w) \quad K(z, w)dw, \quad \forall f \in B.$$

Conversely, using this representation formula, any such $K(z, w)$ yields an operator T in $[\kappa : \sigma]$.

Proof. Explicitly the analyticity condition on the function $K(z, w)$ is that for w fixed with $|w| > r_0$, $K(z, w)$ is an analytic function of z for z in D, and for z fixed in $D, K(z, w)$ is an analytic function of w in $\{w : |w| > r_0\}$.

Now let $K(z, w)$ satisfy the conditions of the theorem and put $Tf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w)dw$. Then $Tf(z)$ is analytic for $|z| <$ 1 just as in Theorem 1. Since $|Tf(z)| \leqq Mr_1 \cdot \|f\|_{|w|=r_1}$, it follows that Tf is in B. Also T is in $[\kappa : \sigma]$ since $\{f_n\}$ converging κ to zero implies $\{f_n\}$ converges to zero uniformly on $|w| = r_1$.

Now assume that T is in $[\kappa : \sigma]$ and let $K(z, w) = \Sigma_{k=0}^{\infty}(u_k(z)/w^{k+1})$ where $T(z^k) = u_k$. For fixed z in D, $\limsup |u_k(z)|^{1/k} \leqq \limsup \|u_k\|^{1/k} = r_0$ for some real number $r_0 < 1$. Hence $\sup_{z\in D} \limsup |u_k(z)|^{1/k} \leqq r_0$. Hence $K(z, w)$ is analytic for $|w| > r_0$ for any fixed z in D. Let r_1 be such that $1 > r_1 > r_0$. For large $k, \|u_k\| \leqq (r_0 + \epsilon)^k$ with $r_0 + \epsilon < r_1 < 1$ and hence for $|w| = r_1$, $|z| < 1$, $|K(z, w)| \leqq \Sigma_{k=0}^{\infty}((r_0 + \epsilon)^k / |w|^{k+1}) = 1/r_1 \Sigma_{k=0}^{\infty}((r_0 + \epsilon)/r_1)^k < \infty$. Now $K(z, w)$ is analytic in D for fixed $|w_0|$ with $|w_0| > r_0$ because $\Sigma_{k=0}^{n}(u_k(z)/w_0^{k+1})$ converges uniformly in D to $K(z, w_0)$.

Now put $Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w)dw$. Since $K(z, w)$ satisfies the conditions of the sufficiency part of the theorem, S is in

$[\kappa : \sigma]$. If $f(z) = z^n$, then

$$Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} w^n \sum_{k=0}^{\infty} (u_k(z)/w^{k+1}) dw = u_n(z) = T(z^n).$$

Hence $S = T$ because they are both in $[\beta : \beta]$ and they agree on the polynomials, a β dense subset of B.

Now that there is a representation for T in $[\kappa : \sigma]$ on a curve inside the disk, the curve can be pushed to the boundary.

THEOREM 3. *A linear operator T is in $[\kappa : \sigma]$ if and only if*

$$Tf(z) = \int_{\Gamma} f(w) K(z, w) dw, \qquad \forall f \in B$$

where $K(z, w)$ is analytic for $|w| > r_0$, $|z| < 1$ for some $r_0 < 1$ and if $1 > r_1 > r_0$, then there exists an M such that $|K(z, w)| \leqq M$ for $|z| < 1$ and $|w| \geqq r_1$.

Proof. Let $K(z, w)$ be given and put $Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w) dw$. Then by Theorem 2, S is in $[\kappa : \beta]$. Since $K(z, w)$ is analytic for $|w| > r_0$, it can be represented as $\sum_{k=0}^{\infty} g_k(z)/w^{k+1}$ where $g_k(z_0)$, $k = 0, 1, \cdots$ is the sequence of coefficients in the series expansion of $K(z_0, w)$. Now $K(z, w)$ is bounded on $D \times \Gamma$ and analytic for $|z| < 1$ for any fixed w_0 with $|w_0| = 1$. Hence $K(z, w)$ satisfies the sufficiency conditions of Theorem 1. Let $Tf(z) = \int_{\Gamma} f(w) K(z, w) dw$. Then by Theorem 1, T is in $[\beta : \beta]$. But

$$\begin{aligned} S(z^n) &= \sum_{k=0}^{\infty} g_k(z)(2\pi i)^{-1} \int_{|w|=r_1} w^n / w^{k+1} dw \\ &= g_n(z) \end{aligned}$$

and

$$\begin{aligned} T(z^n) &= (2\pi i)^{-1} \int_{\Gamma} w^n \sum_{k=0}^{\infty} (g_k(z)/w^{k+1}) dw \\ &= \sum_{k=0}^{\infty} g_k(z)(2\pi i)^{-1} \int_{\Gamma} w^n / w^{k+1} dw \\ &= g_n(z). \end{aligned}$$

Since S and T agree on the polynomials and both are in $[\beta : \beta]$, they are equal. Hence T is in $[\kappa : \sigma]$.

Let T be in $[\kappa : \sigma]$ and put $K(z,w) = \Sigma_{k=0}^{\infty}(u_k(z)/w^{k+1})$ where $T(z^k) = u_k(z)$. Then by Theorem 2, $K(z,w)$ satisfies the conditions of Theorem 3 and hence of Theorem 1. Let $Sf(z) = (2\pi i)^{-1} \int_\Gamma f(w)K(z,w)dw$. As above it follows that $S = T$.

Recall that if T is an operator in $[\beta : \beta]$, then the operators T_r for $0 < r < 1$ are in $[\kappa : \sigma]$ and $\{T_r\}$ converges uniformly on bounded subsets to T. This gives a limit representation for an operator in $[\beta : \beta]$. Are there any non-limiting representations of any operators in $[\beta : \beta]$ other than those in $[\kappa : \sigma]$?

COROLLARY. *Let T be in $[\beta : \beta]$. Then*

$$Tf(z) = \lim_{r \uparrow 1} \int_{|w|=(1/2)(1+1/r)} f_r(w)K(z,w)dw, \quad \forall f \in B$$

where $K(z,w) = (2\pi i)^{-1}\Sigma_{k=0}^{\infty}(T(z^k)/w^{k+1})dw$.

Proof. Since T_r is in $[\kappa : \sigma]$,

$$\begin{aligned} T_rf(z) &= (2\pi i)^{-1} \int_{|w|=(1+r)/2} f(w) \sum_{k=0}^{\infty} (T_r(z^k)/w^{k+1})dw \\ &= (2\pi i)^{-1} \int_{|w|=(1+r)/2} f(w) \sum_{k=0}^{\infty} (T(z^k)r^k/w^{k+1})dw \\ &= (2\pi i)^{-1} \int_{|t|=(1+1/r)/2} f_r(t) \sum_{k=0}^{\infty} (T(z^k)/t^{k+1})dt, \end{aligned}$$

by letting $w = rt$.

Now we use the integral representation for operators in $[\kappa : \sigma]$ to show that $[\kappa : \sigma] = \{T_r : T \in [\beta : \beta]\}$. This characterization of $[\kappa : \sigma]$ is a useful tool in the study of $[\kappa : \sigma]$ (see **[2]**).

THEOREM 4. $[\kappa : \sigma] = \{T_r : T \in [\beta : \beta], 0 < r < 1\}$.

Proof. We have to show that if T is in $[\kappa : \sigma]$, then there exists an operator S in $[\kappa : \sigma]$ and an $s < 1$ such that $T = S_s$. Since T is in $[\kappa : \sigma]$, $Tf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w)K(z,w)dw$ where $K(z,w) = \Sigma_{k=0}^{\infty} (T(z^k)/w^{k+1})$ is analytic for $|w| > r_0$ and $1 > r_1 > r_0$. Let $K_1(z,w) =$

$s\,\Sigma_{k=0}^{\infty}(T(z^k)/(sw)^{k+1})$ where $s<1$ and $r_0<r_0/s<1$. Define S by $Sf(z)=(2\pi i)^{-1}\int_{|w|=s_1} f(w)K_1(z,w)dw$ where $1>s_1>r_0/s$. Then S is in $[\kappa:\sigma]$ since $K_1(z,w)$ is analytic for $|z|<1$, $|w|>r_0/s$ and for $|z|<1$, $|w|=s_1$, $|K_1(z,w)|\leqq s\,|\Sigma_{k=0}^{\infty}(T(z^k)/(sw)^{k+1})|=s\,|K(z,sw)|<M$ since $r_0<s_1s=s\,|w|$. Let $f(z)=z^n$. Then $S_sf(z)=Sf_s(z)=(2\pi i)^{-1}\int_{|w|=s_1}(sw)^n s\,\Sigma_{k=0}^{\infty}(T(z^k)/(sw)^{k+1})dw=T(f)$ and hence $T=S_s$.

4. Convergence in $[\beta:\beta]$. In the Corollary to Theorem 3 of the last section it was shown that to any operator T in $[\beta:\beta]$ there corresponds a kernel $K(z,w)$ by which T is determined. Two operators T_1 and T_2 in $[\beta:\beta]$ should be close (e.g. $\|T_1-T_2\|$ small) if the corresponding kernels K_1 and K_2 are close (e.g. $\|K_1-K_2\|_R$ small for some region R).

However in relating $\|K_1-K_2\|_R$ to $\|T_1-T_2\|$ it seems that a suitable region R can not be determined. For example if $T_1=0$ and $T_2=I$, the zero and identity operators respectively, then the kernel $K_2(z,w)$ corresponding to I is $\Sigma_{k=0}^{\infty}(z^k/w^{k+1})$ and

$$\|K_1-K_2\|_R=\sup\left\{\left|\sum_{k=0}^{\infty}(z^k/w^{k+1})\right|:|z|<1,\ |w|>1\right\}=\infty,$$

where $R=\{(z,w):|z|<1,\ |w|>1\}$. On any region properly contained in R, uniform convergence of a sequence of functions $\{K_n\}$ is related to u.b. and not norm convergence of the corresponding operators $\{T_n\}$. One might be able to use $\|K_1-K_2\|_R$ where $R=\{(z,w)=|z|<1,|w|>1\}$ if one considered only operators bounded away from I in norm.

Obviously if $\{T_n\}$ and T are in $[\beta:\beta]$ and the sequence of corresponding kernels $\{K_n\}$ converges to K uniformly on $\{(z,w):|z|<1,\ |w|>1\}$, then $\{T_n\}$ converges to T in norm.

We will characterize the u.b. sequential convergence of operators in $[\beta:\beta]$ in terms of the corresponding kernels. Although the u.b. topology in $[\beta:\beta]$ is determined by the convergence of nets, the u.b. topology restricted to a norm (equivalently u.b.) bounded subset of $[\beta:\beta]$ is determined by sequential convergence [2].

The first step is to describe the u.b. convergence of a sequence of operators in $[\beta:\beta]$ in terms of their associated operators in $[\kappa:\sigma]$.

Let C denote the algebra of functions in B which are uniformly continuous on D. Recall that $[T_n]_rf=T_n(f_r)$ and observe that $T_r=TI_r$ where I is the identity operator.

THEOREM 5. *Let $\{T_n\}$, $n=1,2,\cdots$, and T be linear operators in $[\beta:\beta]$. Then $\{T_n\}$ converges uniformly on bounded subsets to T if and* *only if* $[T_n]_r$ *converges uniformly on bounded subsets to* T_r *for every* $0 < r < 1$ *and there exists an* M *such that* $\|T_n\| \leqq M$, $n = 1, 2, \cdots$.

Proof. Let $\{T_n\}$ converge u.b. to T. Then $T_n f$ converges strictly to Tf for every fixed f in C. Hence for fixed f in $C, \{T_n f\}$ is uniformly bounded in norm, because strictly convergent sequences are bounded. By the uniform boundedness principle, the set $\{\|T_n\|\}$ is uniformly bounded, where $\|T_n\| = \sup\{\|T_n f\| : f \in C, \|f\| \leqq 1\}$. It follows **[2]** that this is the norm of T_n as an operator on all of B.

Now fix $0 < r < 1$ and let S be a bounded set and G an open set in (B, β). Then $([T_n]_r - T_r)(S) = (T_n - T)(I_r)(S) = (T_n - T)S_r \subseteq G$ for $n > N$ for some N, because $S_r = \{f_r : f \in S\}$ is a bounded set and T_n converges u.b. to T.

For the converse let $G = \{g : |g|_\psi < 3\epsilon\}$ be an open set and S a bounded set in (B, β). Let $G_1 = \{g : |g|_\psi < \epsilon\}$. For f in S,

$$\begin{aligned} |([T_n]_r - T_n)f|_\psi &= \|\psi([T_n]_r - T_n)f\| \\ &= \|\psi T_n(I_r - I)f\| \\ &\leqq M\|\psi(I_r - I)f\| \\ &= M|(I_r - I)f|_\psi \\ &< \epsilon \end{aligned}$$

for $r \geqq r_0$ for some $r_0 < 1$ because I_r converges u.b. to I. Hence for $r \geqq r_0$, $([T_n]_r - T_n)S \subseteq G_1$.

Since T_r converges u.b. to T, there is an r_1 such that $1 > r \geqq r_1$ implies $(T - T_r)S \subseteq G_1$. Fix t larger than r_0 and r_1 and let N be such that $n > N$ implies $(T_t - [T_n]_t)S \subseteq G_1$. Then for $n > N$,

$$\begin{aligned} (T - T_n)S &= (T - T_t)S + (T_t - [T_n]_t)S + ([T_n]_t - T_n)S \\ &\subseteq G_1 + G_1 + G_1 \\ &\subseteq G. \end{aligned}$$

LEMMA 6. *Let* $\{T_n\}$, $n = 1, 2, \cdots$, *and* T *be in* $[\beta : \beta]$. *Then* $[T_n]_r$ *converges uniformly on bounded subsets to* T_r *for every* $0 < r < 1$ *if and only if the corresponding kernels* $K_n(z, w)$ *converge to* $K(z, w)$ *uniformly on compact subsets of* $D \times \{w : |w| > 1\}$ *and given* $\rho > 1$, *there exists an* M_ρ *such that* $|K_n(z, w)| \leqq M_\rho$ *for all* n *and* $|z| < 1$, $|w| \geqq \rho > 1$.

Proof. Let $[T_n]_r$ converge u.b. to T_r. Fix $r < 1$ and $s < 1$. Then it will be shown that $\{K_n(z, w)\}$ converges to $K(z, w)$ uniformly on $R = \{(z, w) : |z| \leqq s$ and $|w| \geqq \rho > 1/r\}$.

The operators $[T_n]_r$ and T_r are maps from C into B. As in the previous Theorem it follows that $\|[T_n]_r\|$ is uniformly bounded for $n = 1, 2, \cdots$.

Given $\epsilon > 0$ and s let ψ in $C_0[D]$ be 1 on $\{z : |z| < s\}$. There is an N such that for $n > N$ and for all f in the bounded set $\{f : \|f\| \leqq 1\}$, $\|([T_n]_r - T_r)f\|_s \leqq |([T_n]_r - T_r)f|_\psi < \epsilon$ since $[T_n]_r$ converges u.b. to T_r. Thus for $\|f\| \leqq 1$,

$$\begin{aligned}\|([T_n]_r - T_r)f\| &\leqq \|\psi([T_n]_r - T_r)f\| \\ &= |([T_n]_r - T_r)f|_\psi \\ &< \epsilon.\end{aligned}$$

For $j = 0, 1, \cdots$, let $f_j(w) = w^j$, $u_j = T(f_j)$ and $u_{j,n} = T_n(f_j)$. Then $[T_n]_r f_j(z) - T_r f_j(z) = r^j u_{j,n}(z) - r^j u_j(z)$. Hence $\|r^j[u_{j,n}(z) - u_j(z)]\|_s < \epsilon$ for $n > N$ for all $j = 0, 1, \cdots$. For $j = 1, 2, \cdots, J$, we have $\|u_{j,n}(z) - u_j(z)\|_s < \epsilon / r^J$ for $n > N$ since $r^j \geqq r^J$.

Now $K_n(z, w) - K(z, w) = \sum_{k=0}^\infty (u_{k,n}(z) - u_k(z)) r^k / w^{k+1} r^k$. For $n > N$, $\|r^k[u_{k,n}(z) - u_k(z)]\|_s < \epsilon < 1$. Since $r\rho > 1$ there is a J so large that $\sum_{k=J+1}^\infty (1/r\rho)^k < \epsilon/2$. Then

$$\|K_n(z, w) - K(z, w)\|_R \leqq \left\| \sum_{k=0}^{J} (u_{k,n}(z) - u_k(z))/w^{k+1} \right\|_R + \epsilon/2$$

since $1/|wr| \leqq 1/\rho r$. Also for $n > N$, $\|u_{k,n}(z) - u_k(z)\|_s < \epsilon/2$ for $k = 1, 2, \cdots, J$. Therefore $\|K_n - K\|_R < \epsilon$.

It remains to be shown that given $\rho > 1$, there exists a constant M_ρ such that $|K_n(z, w)| \leqq M_\rho$ for all $|z| < 1$ and $|w| > \rho > 1$. Given $\rho > 1$, fix $0 < r < 1$ with $r\rho > 1$. Since $[T_n]_r$ converges u.b. to T_r it follows from Theorem 6 that there exists an M with $\|[T_n]_r\| \leqq M$ for all $n = 1, 2, \cdots$. Now

$$\begin{aligned}K_n(z, w) &= \sum_{k=0}^\infty T_n(z^k)/w^{k+1} \\ &= r \sum_{k=0}^\infty r^k T_n(z^k)/r^{k+1} w^{k+1} \\ &= r \sum_{k=0}^\infty [T_n]_r(z^k)/(wr)^{k+1}\end{aligned}$$

and therefore for $|z| < 1$ and $|w| \geqq \rho$,

$$\begin{aligned}|K_n(z, w)| &\leqq rM \sum_{k=0}^\infty 1/|wr|^{k+1} \\ &\leqq Mr \sum_{k=0}^\infty (1/(\rho r))^{k+1}\end{aligned}$$

where the last expression is M_ρ.

For the converse, fix $0 < r < 1$ and let $\gamma = (1 + 1/r)/2$. Now

$$\|([T_n]_r - T_r)f\|_s = \left\| \int_{|w|=\gamma} f_r(w)(K_n(z,w) - K(z,w))dw \right\|_s$$
$$\leqq \|K_n(z,w) - K(z,w)\|_R \, \|f\|$$

where $R = \{(z,w) : |z| < s, \ |w| = (1 + 1/r)/2\}$. This last expression tends to zero as $n \to \infty$. Hence if S is a bounded set, $([T_n]_r - T_r)S$ converges κ to zero.

Let f in B satisfy $\|f\| \leqq 1$. Then

$$\|[T_n]_r f(z)\| = \left\| \int_{|w|=\gamma} f_r(w) K_n(z,w) dw \right\|$$
$$\leqq \|K_n(z,w)\|_R \, \|f\|$$
$$\leqq M$$

by assumption on the kernels K_n where $R = \{(z,w) : |z| < 1, |w| = (1 + 1/r)/2\}$. Hence $\|[T_n]_r\| \leqq M$ for all n.

Let S be a bounded set in (B, β) and let $G = \{g : |g|_\psi < \epsilon, \psi \not\equiv 0\}$ be an open set in (B, β). We have $\|([T_n]_r - T_r)\| \leqq 2M$. Let r' be such that for $|z| > r'$, $|\psi(z)| < \epsilon/2M$. Then

$$\epsilon > \|([T_n]_r - T_r)f\psi\|_{r'<|z|<1}.$$

For $|z| \leqq r'$, choose N such that $n > N$ implies

$$\|([T_n]_r - T_r)f\|_{r'} < \epsilon \|\psi\|^{-1} \text{ for all } f \text{ in } S.$$

Then $\|([T_n]_r - T_r)f\psi\|_{r'} < \epsilon$ and $([T_n]_r - T_r)f \in G$ for $n > N$ and all f in S.

Theorem 5 and Lemma 6 taken together characterize u.b. convergence on bounded sets in $[\beta : \beta]$ in terms of the kernel functions $K(z,w)$.

THEOREM 7. *Let* $\{T_n\}$, $n = 1, 2, \cdots$ *and* T *be in* $[\beta : \beta]$. *Then* $\{T_n\}$ *converges* u.b. *to* T *if and only if the corresponding kernels* $\{K_n(z,w)\}$ *converge uniformly on compact subsets of* $D \times \{w : |w| > 1\}$ *to* $K(z,w)$ *and for any* $\rho > 1$, *there exists a number* M_ρ *such that* $|K_n(z,w)| \leqq M_\rho$ *for* $|z| < 1$ *and* $|w| \geqq \rho > 1$.

COROLLARY. *Let* S *be a norm* (equivalently u.b.) *bounded subset of* $[\beta : \beta]$. *Let* $\{T_n\}$, $n = 1, 2, \cdots$, *and* T *be in* S *with corresponding* *kernels* $\{K_n(z,w)\}$ *and* $K(z,w)$. *Then* $\{T_n\}$ *converges u.b. to* T *if and only if* $\{K_n(z,w)\}$ *converges uniformly on compact subsets of* $D \times \{w : |w| > 1\}$ *to* $K(z,w)$.

Proof. The condition that $\{K_n(z,w)\}$ converges κ to $K(z,w)$ on $D \times \{w : |w| > 1\}$ is necessary by the above theorem. Let T in S imply $\|T\| \leqq M$. Then it follows that $|K_n(z,w)| \leqq M_\rho$ for $|z| < 1$ and $|w| > \rho$ and the condition is also sufficient.

On the locally compact Hausdorff space $D \times \{w : |w| > 1\}$, a sequence $\{K_n(z,w)\}$ converges strictly to a function $K(z,w)$, **[5]**, if and only if $\{K_n(z,w)\}$ converges uniformly on compact subsets of $D \times \{w : |w| > 1\}$ to $K(z,w)$ and $|K_n(z,w)|$ is uniformly bounded on $D \times \{w : |w| > 1\}$. The next corollary follows immediately from the previous theorem, but it is not known if the converse holds. See Theorem 8 for a similar result.

COROLLARY. *Let* $\{T_n\}$, $n = 1, 2, \cdots$ *and* T *be in* $[\beta : \beta]$ *with corresponding kernels* $\{K_n(z,w)\}$ *and* $K(z,w)$. *If* $\{K_n(z,w)\}$ *converges strictly to* $K(z,w)$ *on* $D \times \{w : |w| > 1\}$, *then* $\{T_n\}$ *converges u.b. to* T.

5. Convergence of multipliers. The characterization of u.b. convergence in the last section is applied to the multiplier operators.

DEFINITION. A multiplier on B is a linear operator T such that there exists a sequence $\{c_n\}$ with the property that $T(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n c_n z^n$ for every function $\sum_{n=0}^{\infty} a_n z^n$ in B. It is known **[1]** that an operator T is a multiplier from B into B if and only if the sequence $\{c_n\}$ is one side of the sequence of Fourier-Stieltjes coefficients of a bounded complex valued regular Borel measure μ on Γ and also $\|\mu\| = \|T\|$. Also if T is a multiplier from B into B, then T is in $[\kappa : \kappa]$, a subalgebra of $[\beta : \beta]$. Let $\hat{\mu}(k)$ denote the kth Fourier-Stieltjes coefficient of the measure μ.

Clearly, if $\{T_n\}$, $n = 1, 2, \cdots$ and T are multipliers in $[\beta : \beta]$, and $\{T_n\}$ converges in norm to T then $\lim_{n \to \infty} \hat{\mu}_n(k) = \hat{\mu}(k)$ uniformly in k, where μ_n and μ are the measures associated with T_n and T respectively. In other words, the sequence of functions $\{\hat{\mu}_n\}$ defined on P, the nonnegative integers, converges uniformly to $\hat{\mu}$ on P. One expects then that for u.b. convergence the functions $\{\hat{\mu}_n\}$ will converge strictly to $\hat{\mu}$ on P. On the locally compact Hausdorff space P, a sequence of functions $\{\hat{\mu}_n\}$ converges strictly to a function $\hat{\mu}$ if and only if $\{\hat{\mu}_n\}$ is uniformly bounded and $\{\hat{\mu}_n\}$ converges uniformly on compact subsets to $\hat{\mu}$ **[5]**, i.e., pointwise on P.

THEOREM 8. *Let* $\{T_n\}$, $n = 0, 1, \cdots$, *and* T *be multipliers from* B *into* B *with associated measures* $\{\mu_n\}$ *and* μ. *Then* $\{T_n\}$ *converges u.b. to* T *if and only if* $\{\hat{\mu}_n\}$ *converges strictly to* $\hat{\mu}$.

Proof. For necessity we must show that there exists an M such that $|\hat{\mu}_n(k)| \leqq M$ for all $n, k = 0, 1, \cdots$, and $\lim_{n\to\infty} \hat{\mu}_n(k) = \hat{\mu}(k)$, $k = 0, 1, \cdots$. Since $\{T_n\}$ converges u.b. to T there is an M such that $\|T_n\| \leqq M$, $n = 0, 1, \cdots$. Since $\|T_n\| = \|\mu_n\|$, $|\hat{\mu}_n(k)| \leqq \|\mu_n\| \leqq M$. Let $\hat{\mu}_n(k) = c_{n,k}$. Now since $\{T_n(z^k)\}$ converges strictly to $T(z^k)$, we have $\{c_{n,k}z^k\}$ converges strictly to $c_k z^k$ as $n \to \infty$. Hence $\lim_{n\to\infty} c_{n,k} = c_k$.

For the sufficiency part of the proof, let $|z| \leqq s < 1$ and $|w| \geqq \rho > 1$. Then

$$|K_n(z, w) - K(z, w)| = \left| \sum_{k=0}^{\infty} (c_{n,k} - c_k) z^k / w^{k+1} \right|$$

$$\leqq \sum_{k=0}^{\infty} |c_{n,k} - c_k| (s/\rho)^k.$$

Let k' be such that $\sum_{k=k'}^{\infty} (s/\rho)^k < \epsilon/4M$ and let N be so large that $n > N$ implies $|c_{n,k} - c_k| < \epsilon\rho/2(s-\rho)$ for $k = 0, 1, \cdots, k'$. Then for $|z| < s$ and $|w| \geqq \rho$, $|K_n(z, w) - K(z, w)| < \epsilon$. Also $|\sum_{k=0}^{\infty} (c_{n,k} z^k / w^{k+1})| \leqq M\rho(\rho - 1)^{-1}$ for all $|z| < 1$ and $|w| \geqq \rho$.

The multipliers from B into B which are in the algebra $[\beta : \sigma]$ correspond to the absolutely continuous measures on Γ **[1]**. Let ϕ_n in $L^1(\Gamma)$ correspond to the multiplier T_n.

COROLLARY. *Let* $\{T_n\}$, $n = 1, 2, \cdots$ *and* T *be multipliers in* $[\beta : \sigma]$. *Then* $\{T_n\}$ *converges uniformly on bounded subsets to* T *if and only if* $\|\phi_n\|_{L^1} \leqq M$ *and* $\lim_{n\to\infty} \hat{\phi}_n(k) = \hat{\phi}(k)$.

COROLLARY. *Let* $\{T_n\}$, $n = 1, 2, \cdots$ *and* T *be multipliers in* $[\beta : \sigma]$. *If* $\{\phi_n\}$ *converges to* ϕ *in* L^1, *then* $\{T_n\}$ *converges u.b. to* T.

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