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**AN INTEGRAL REPRESENTATION FOR STRICTLY
CONTINUOUS LINEAR OPERATORS**

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Let B denote the algebra of bounded analytic functions on the open unit disc D in the complex plane. Let (B, τ) denote B endowed with the topology τ , where τ is chosen from κ, β or σ , respectively, the topology of uniform convergence on compact subsets of D , the strict topology and the topology of uniform convergence on D . This note obtains an integral representation of the form $Tf(z) = \int_{\Gamma} f(w) K(z, w) dw$ where $\Gamma = \{z : |z| = 1\}$ for the linear operators which are continuous from (B, κ) into (B, σ) . This representation is then used to study the convergence of operators in the full algebra of all continuous linear operators from (B, β) into (B, β) .

1. Introduction. Let $M(D)$ denote the set of bounded complex valued Borel measures on D . R. C. Buck [5] showed that L is a continuous linear functional on $(C(D), \beta)$ if and only if $Lf = \int_D f d\mu$, $\forall f \in C(D)$ for some $\mu \in M(D)$. L. A. Rubel and A. L. Shields [7] showed that for any $\mu \in M(D)$ there exists a function h in $L^1(\Gamma)$ such that $\int_D f d\mu = \int_{\Gamma} f(x)h(x)dx$, $\forall f \in B$ and conversely, that any $h \in L^1(\Gamma)$ determines a measure $\mu \in M(D)$ for which this equality holds. Thus the continuous linear functionals on (B, β) can be represented as integration over Γ with respect to functions in $L^1(\Gamma)$.

Letting both τ_1 and τ_2 be one of the topologies κ, β or σ , let $[\tau_1 : \tau_2]$ denote the algebra of all continuous linear operators from (B, τ_1) into (B, τ_2) .

In Theorem 1 it is shown that any linear operator T in $[\beta : \beta]$ can be represented in the form

$$Tf(z) = \int_{\Gamma} f(w)K(z, w)dw, \quad \forall f \in B.$$

However, a necessary and sufficient condition on $K(z, w)$ that such a T be in $[\beta : \beta]$ is not known.

The algebra $[\kappa : \sigma]$ is a dense subalgebra of $[\beta : \beta]$ in the compact open topology. In Theorem 3 it is shown that a linear operator T is in $[\kappa : \sigma]$ if and only if $Tf(z) = \int_{\Gamma} f(w) K(z, w)dw$ where the kernel

$K(z, w)$ satisfies certain fixed conditions. One can then associate with every linear operator in $[\beta : \beta]$ an explicit kernel $K(z, w)$. In §4, the convergence of linear operators in $[\beta : \beta]$ is characterized by using the convergence of the sequence of associated kernels. In the last section this convergence criterion is applied to the special type of operators in $[\beta : \beta]$ called multipliers.

2. Definitions. The topology σ , of uniform convergence on D , is defined by the norm

$$\|f\| = \sup\{|f(z)| : |z| < 1\}.$$

The topology κ , of uniform convergence on compact subsets of D , can be defined by the family of semi-norms

$$\|f\|_r = \sup\{|f(z)| : |z| < r\}$$

where $0 < r < 1$. The strict topology β was introduced by R. C. Buck in [3] as a topology on the set of bounded continuous functions on a space. It is defined by the family of semi-norms

$$|f|_\phi = \|f\phi\|, \phi \in C_0[D],$$

the continuous functions on D which vanish at infinity. The strict topology was first employed to study B in [4]. For properties of β and its relation to κ and σ see [3], [4], [5] and [7]. In particular, a sequence of functions $\{f_n\}$ in B converges strictly to zero if and only if it is uniformly bounded and converges κ (or pointwise) to zero. Also the β bounded subsets of B are precisely the σ bounded subsets.

In [2] two appropriate topologies were employed to study $[\beta : \beta]$. From $[\sigma : \sigma]$, the subalgebra $[\beta : \beta]$ inherits the usual operator norm topology where

$$\|T\| = \sup\{\|Tf\| : \|f\| \leq 1, f \in B\}.$$

The second topology is that of uniform convergence on bounded subsets of B which in fact is equivalent on $[\beta : \beta]$ to the compact open topology.

DEFINITION. A net of operators $\{T_\alpha\}$ in $[\beta : \beta]$ converges uniformly on bounded subsets (written u.b.) to T if and only if given any β open set G in B with $0 \in G$ and any β bounded set S in B , there exists an α' such that if $\alpha > \alpha'$, then $(T_\alpha - T)(S) \subseteq G$.

For properties of $[\beta : \beta]$ in these two topologies see [2]. In particular the u.b. bounded subsets are precisely the norm bounded subsets and $[\beta : \beta]$ is a u.b. (and hence norm) closed subalgebra of $[\sigma : \sigma]$.

It should be observed [1] that the continuity classes $[\kappa : \sigma]$, $[\beta : \sigma]$, $[\beta : \beta]$ and $[\sigma : \sigma]$ are in fact algebras, they are related by the proper inclusions $[\kappa : \sigma] \subset [\beta : \sigma] \subset [\beta : \beta] \subset [\sigma : \sigma]$, and $[\kappa : \sigma]$ is dense in $[\beta : \beta]$ in the u.b. topology, but in the norm topology $[\kappa : \sigma]$ is dense in only $[\beta : \sigma]$. In the study of the u.b. denseness of $[\kappa : \sigma]$ in $[\beta : \beta]$, the operators T_r play a significant role. Given an operator T in $[\beta : \beta]$, the operator $[T]_r$ (sometimes written T_r) is defined by $T_r f(z) = T(f_r)(z)$ where $f_r(z) = f(rz)$, $f \in B$, and $0 < r < 1$. An operator T_r is in $[\kappa : \sigma]$ and it is known [2] that $\{T_r\}$ converges u.b. to T as $r \uparrow 1$.

Finally, a result of P. Hessler (see [1] or [6]) shows that a linear operator T is in $[\beta : \tau]$ if and only if whenever a sequence $\{f_n\}$ in B converges strictly to zero, it follows that $\{Tf_n\}$ converges τ to zero, where τ is κ , β or σ .

3. An integral representation. Let z be a fixed point in D . Then given a linear operator T in $[\beta : \beta]$, the linear functional L defined on B by $Lf = Tf(z)$ is a continuous linear functional on (B, β) . Therefore, $Tf(z) = Lf = \int_{\Gamma} f(w)K_z(w)dw$ for some function $K_z(w)$ in $L^1(\Gamma)$. It is difficult to determine the relationship between the various functions K_z that is necessary and sufficient to ensure that $Tf(z) = \int_{\Gamma} f(w)K(z, w)dw$ will represent an operator in $[\beta : \beta]$. The following gives a necessary condition and a different sufficient condition.

THEOREM 1. *For any linear operator T in $[\beta : \beta]$,*

$$Tf(z) = \int_{\Gamma} f(w)K(z, w)dw, \quad \forall f \in B$$

where $K(z', w) = K_{z'}$ is in $L^1(\Gamma)$ for each z' in D and the L^1 norms of all the functions $K_{z'}$ are uniformly bounded.

If $K(z, w)$ satisfies the above necessary conditions and $K(z, w)$ is analytic in D for each fixed w in Γ and bounded on $D \times \Gamma$, then any T so defined is in $[\beta : \beta]$.

Proof. Let T be in $[\beta : \beta]$. Then, as before, let $L_z(f) = Tf(z)$ for z fixed in D . Since $L_z(f) = \int_{\Gamma} f(w)K_z(w)dw$, we have $\|L_z\| = \|K_z\|_{L^1}$

and $|L_z(f)| = |Tf(z)| \leq \|T\| \|f\|$, where $\|K_z\|_{L^1}$ denotes the usual L^1 norm of K_z on Γ . Hence $\|K_z\|_{L^1} = \|L_z\| \leq \|T\|$ for each z in D .

For the converse, define $Tf(z) = \int_{\Gamma} f(w) K_z(w) dw = \int_{\Gamma} f(w) K(z, w) dw$. Then $Tf(z)$ is continuous in D since

$$\begin{aligned} Tf(z_1) - Tf(z) &= \int_{\Gamma} f(w)[K(z_1, w) - K(z, w)]dw \\ &= \int_{\Gamma} f(w)(z_1 - z)(2\pi i)^{-1} \int_{\gamma} K(s, w)[(s - z_1)(s - z)]^{-1} ds dw \end{aligned}$$

where γ is a circle in D with center z_1 and containing z in its interior and hence

$$|Tf(z_1) - Tf(z)| \leq \|f\| |z_1 - z| \sup |(s - z_1)(s - z)|^{-1} \|K\|_R$$

where $\|K\|_R$ is the sup of $|K(z, w)|$ taken over $R = D \times \Gamma$. Thus $|Tf(z_1) - Tf(z)|$ tends to zero as z approaches z_1 . Then for any triangle

$$\Delta \quad \text{in} \quad D, \quad \int_{\Delta} Tf(z) = \int_{\Delta} \int_{\Gamma} f(w) K(z, w) dw = \int_{\Gamma} \int_{\Delta} f(w) K(z, w) dw = 0.$$

By Morera's theorem, $Tf(z)$ is analytic in D .

Now $Tf(z)$ is a bounded function since

$$|Tf(z)| \leq \int_{\Gamma} |f(w) K(z, w)| d|w| \leq 2\pi \|K_z\|_{L^1} \|f\| \leq M \|f\|$$

for all z in D . If $\{f_n\}$ converges strictly to zero, then $Tf_n(z) = L_z(f_n)$ converges to zero. Hence $\{Tf_n\}$ converges pointwise to zero and is uniformly bounded, which implies $\{Tf_n\}$ converges strictly to zero.

Note that additional conditions are imposed on $K(z, w)$ in the converse only to ensure that $Tf(z)$ is analytic. Any $K(z, w)$ which satisfies the necessary conditions and makes Tf analytic will yield a T in $[\beta : \beta]$. It is certainly not necessary that $K(z, w)$ be analytic in z because for any function h in $L^1(\Gamma)$, $Tf(z) = \int_{\Gamma} f(w)h(w)dw = \int_{\Gamma} f(w) K(z, w)dw$ is strictly continuous and $K(z, w) = h(w)$ need only be defined a.e..

We consider now the case when C is some rectifiable curve inside D and $Tf(z) = \int_C f(w) K(z, w)dw$ with $K(z, w)$ in $L^1(C)$ for any z . As

in the previous theorem, if T is in $[\beta : \beta]$, then the functions $K_z(w)$ are uniformly bounded in $L^1(C)$ norm. Hence T is in $[\kappa : \sigma]$, because if $\{f_n\}$ converges κ to zero, then $\{f_n\}$ converges uniformly to zero on C and $|Tf_n(z)| \leq (\text{length of } C) \|f_n\|_C \|K_z\|_{L^1(C)}$.

Now we obtain a representation formula for the operators in $[\kappa : \sigma]$. Given an operator T in $[\kappa : \sigma]$, there is an M and an $r < 1$ such that $\|Tf\| \leq M \|f\|_r$ for all f in B . Letting $f(z) = z^k$, we obtain $\|T(z^k)\| \leq M \|z^k\|_r = M(r)^k$. Hence $\|T(z^k)\|^{1/k} \leq rM^{1/k}$ and $\limsup \|T(z^k)\|^{1/k} \leq r$.

THEOREM 2. *If T is in $[\kappa : \sigma]$, then there exists a function $K(z, w)$ analytic for $|z| < 1$ and $\infty > |w| > r_0$ for some $r_0 < 1$ and such that if $1 > r_1 > r_0$, then there exists an M such that $|K(z, w)| \leq M$ for all $|z| < 1$, $|w| \geq r_1$ and such that*

$$Tf(z) = \int_{|w|=r_1} f(w) K(z, w)dw, \quad \forall f \in B.$$

Conversely, using this representation formula, any such $K(z, w)$ yields an operator T in $[\kappa : \sigma]$.

Proof. Explicitly the analyticity condition on the function $K(z, w)$ is that for w fixed with $|w| > r_0$, $K(z, w)$ is an analytic function of z for z in D , and for z fixed in D , $K(z, w)$ is an analytic function of w in $\{w : |w| > r_0\}$.

Now let $K(z, w)$ satisfy the conditions of the theorem and put $Tf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w)dw$. Then $Tf(z)$ is analytic for $|z| < 1$ just as in Theorem 1. Since $|Tf(z)| \leq Mr_1 \cdot \|f\|_{|w|=r_1}$, it follows that Tf is in B . Also T is in $[\kappa : \sigma]$ since $\{f_n\}$ converging κ to zero implies $\{f_n\}$ converges to zero uniformly on $|w| = r_1$.

Now assume that T is in $[\kappa : \sigma]$ and let $K(z, w) = \sum_{k=0}^{\infty} (u_k(z)/w^{k+1})$ where $T(z^k) = u_k$. For fixed z in D , $\limsup |u_k(z)|^{1/k} \leq \limsup \|u_k\|^{1/k} = r_0$ for some real number $r_0 < 1$. Hence $\sup_{z \in D} \limsup |u_k(z)|^{1/k} \leq r_0$. Hence $K(z, w)$ is analytic for $|w| > r_0$ for any fixed z in D . Let r_1 be such that $1 > r_1 > r_0$. For large k , $\|u_k\| \leq (r_0 + \epsilon)^k$ with $r_0 + \epsilon < r_1 < 1$ and hence for $|w| = r_1$, $|z| < 1$, $|K(z, w)| \leq \sum_{k=0}^{\infty} ((r_0 + \epsilon)^k / |w|^{k+1}) = 1/r_1 \sum_{k=0}^{\infty} ((r_0 + \epsilon)/r_1)^k < \infty$. Now $K(z, w)$ is analytic in D for fixed $|w_0|$ with $|w_0| > r_0$ because $\sum_{k=0}^{\infty} (u_k(z)/w_0^{k+1})$ converges uniformly in D to $K(z, w_0)$.

Now put $Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w)dw$. Since $K(z, w)$ satisfies the conditions of the sufficiency part of the theorem, S is in

$[\kappa : \sigma]$. If $f(z) = z^n$, then

$$Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} w^n \sum_{k=0}^{\infty} (u_k(z)/w^{k+1}) dw = u_n(z) = T(z^n).$$

Hence $S = T$ because they are both in $[\beta : \beta]$ and they agree on the polynomials, a β dense subset of B .

Now that there is a representation for T in $[\kappa : \sigma]$ on a curve inside the disk, the curve can be pushed to the boundary.

THEOREM 3. *A linear operator T is in $[\kappa : \sigma]$ if and only if*

$$Tf(z) = \int_{\Gamma} f(w)K(z, w)dw, \quad \forall f \in B$$

where $K(z, w)$ is analytic for $|w| > r_0$, $|z| < 1$ for some $r_0 < 1$ and if $1 > r_1 > r_0$, then there exists an M such that $|K(z, w)| \leq M$ for $|z| < 1$ and $|w| \geq r_1$.

Proof. Let $K(z, w)$ be given and put $Sf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w)K(z, w)dw$. Then by Theorem 2, S is in $[\kappa : \beta]$. Since $K(z, w)$ is analytic for $|w| > r_0$, it can be represented as $\sum_{k=0}^{\infty} g_k(z)/w^{k+1}$ where $g_k(z)$, $k = 0, 1, \dots$ is the sequence of coefficients in the series expansion of $K(z, w)$. Now $K(z, w)$ is bounded on $D \times \Gamma$ and analytic for $|z| < 1$ for any fixed w_0 with $|w_0| = 1$. Hence $K(z, w)$ satisfies the sufficiency conditions of Theorem 1. Let $Tf(z) = \int_{\Gamma} f(w)K(z, w)dw$. Then by Theorem 1, T is in $[\beta : \beta]$. But

$$\begin{aligned} S(z^n) &= \sum_{k=0}^{\infty} g_k(z)(2\pi i)^{-1} \int_{|w|=r_1} w^n/w^{k+1} dw \\ &= g_n(z) \end{aligned}$$

and

$$\begin{aligned} T(z^n) &= (2\pi i)^{-1} \int_{\Gamma} w^n \sum_{k=0}^{\infty} (g_k(z)/w^{k+1})dw \\ &= \sum_{k=0}^{\infty} g_k(z)(2\pi i)^{-1} \int_{\Gamma} w^n/w^{k+1} dw \\ &= g_n(z). \end{aligned}$$

Since S and T agree on the polynomials and both are in $[\beta : \beta]$, they are equal. Hence T is in $[\kappa : \sigma]$.

Let T be in $[\kappa : \sigma]$ and put $K(z, w) = \sum_{k=0}^{\infty} (u_k(z)/w^{k+1})$ where $T(z^k) = u_k(z)$. Then by Theorem 2, $K(z, w)$ satisfies the conditions of Theorem 3 and hence of Theorem 1. Let $Sf(z) = (2\pi i)^{-1} \int_{\Gamma} f(w) K(z, w) dw$. As above it follows that $S = T$.

Recall that if T is an operator in $[\beta : \beta]$, then the operators T_r for $0 < r < 1$ are in $[\kappa : \sigma]$ and $\{T_r\}$ converges uniformly on bounded subsets to T . This gives a limit representation for an operator in $[\beta : \beta]$. Are there any non-limiting representations of any operators in $[\beta : \beta]$ other than those in $[\kappa : \sigma]$?

COROLLARY. *Let T be in $[\beta : \beta]$. Then*

$$Tf(z) = \lim_{r \uparrow 1} \int_{|w|=(1/2)(1+1/r)} f_r(w) K(z, w) dw, \quad \forall f \in B$$

where $K(z, w) = (2\pi i)^{-1} \sum_{k=0}^{\infty} (T(z^k)/w^{k+1}) dw$.

Proof. Since T_r is in $[\kappa : \sigma]$,

$$\begin{aligned} T_r f(z) &= (2\pi i)^{-1} \int_{|w|=(1+r)/2} f(w) \sum_{k=0}^{\infty} (T_r(z^k)/w^{k+1}) dw \\ &= (2\pi i)^{-1} \int_{|w|=(1+r)/2} f(w) \sum_{k=0}^{\infty} (T(z^k) r^k / w^{k+1}) dw \\ &= (2\pi i)^{-1} \int_{|t|=(1+1/r)/2} f_r(t) \sum_{k=0}^{\infty} (T(z^k)/t^{k+1}) dt, \end{aligned}$$

by letting $w = rt$.

Now we use the integral representation for operators in $[\kappa : \sigma]$ to show that $[\kappa : \sigma] = \{T_r : T \in [\beta : \beta]\}$. This characterization of $[\kappa : \sigma]$ is a useful tool in the study of $[\kappa : \sigma]$ (see [2]).

THEOREM 4. $[\kappa : \sigma] = \{T_r : T \in [\beta : \beta], 0 < r < 1\}$.

Proof. We have to show that if T is in $[\kappa : \sigma]$, then there exists an operator S in $[\kappa : \sigma]$ and an $s < 1$ such that $T = S_s$. Since T is in $[\kappa : \sigma]$, $Tf(z) = (2\pi i)^{-1} \int_{|w|=r_1} f(w) K(z, w) dw$ where $K(z, w) = \sum_{k=0}^{\infty} (T(z^k)/w^{k+1})$ is analytic for $|w| > r_0$ and $1 > r_1 > r_0$. Let $K_1(z, w) =$

$s \sum_{k=0}^{\infty} (T(z^k))/(sw)^{k+1}$ where $s < 1$ and $r_0 < r_0/s < 1$. Define S by $Sf(z) = (2\pi i)^{-1} \int_{|w|=s_1} f(w)K_1(z, w)dw$ where $1 > s_1 > r_0/s$. Then S is in $[\kappa : \sigma]$ since $K_1(z, w)$ is analytic for $|z| < 1$, $|w| > r_0/s$ and for $|z| < 1$, $|w| = s_1$, $|K_1(z, w)| \leq s |\sum_{k=0}^{\infty} (T(z^k))/(sw)^{k+1}| = s |K(z, sw)| < M$ since $r_0 < s_1, s = s_1 |w|$. Let $f(z) = z^n$. Then $S_s f(z) = S f_s(z) = (2\pi i)^{-1} \int_{|w|=s_1} (sw)^n s \sum_{k=0}^{\infty} (T(z^k))/(sw)^{k+1} dw = T(f)$ and hence $T = S_s$.

4. Convergence in $[\beta : \beta]$. In the Corollary to Theorem 3 of the last section it was shown that to any operator T in $[\beta : \beta]$ there corresponds a kernel $K(z, w)$ by which T is determined. Two operators T_1 and T_2 in $[\beta : \beta]$ should be close (e.g. $\|T_1 - T_2\|$ small) if the corresponding kernels K_1 and K_2 are close (e.g. $\|K_1 - K_2\|_R$ small for some region R).

However in relating $\|K_1 - K_2\|_R$ to $\|T_1 - T_2\|$ it seems that a suitable region R can not be determined. For example if $T_1 = 0$ and $T_2 = I$, the zero and identity operators respectively, then the kernel $K_2(z, w)$ corresponding to I is $\sum_{k=0}^{\infty} (z^k/w^{k+1})$ and

$$\|K_1 - K_2\|_R = \sup \left\{ \left| \sum_{k=0}^{\infty} (z^k/w^{k+1}) \right| : |z| < 1, |w| > 1 \right\} = \infty,$$

where $R = \{(z, w) : |z| < 1, |w| > 1\}$. On any region properly contained in R , uniform convergence of a sequence of functions $\{K_n\}$ is related to u.b. and not norm convergence of the corresponding operators $\{T_n\}$. One might be able to use $\|K_1 - K_2\|_R$ where $R = \{(z, w) = |z| < 1, |w| > 1\}$ if one considered only operators bounded away from I in norm.

Obviously if $\{T_n\}$ and T are in $[\beta : \beta]$ and the sequence of corresponding kernels $\{K_n\}$ converges to K uniformly on $\{(z, w) : |z| < 1, |w| > 1\}$, then $\{T_n\}$ converges to T in norm.

We will characterize the u.b. sequential convergence of operators in $[\beta : \beta]$ in terms of the corresponding kernels. Although the u.b. topology in $[\beta : \beta]$ is determined by the convergence of nets, the u.b. topology restricted to a norm (equivalently u.b.) bounded subset of $[\beta : \beta]$ is determined by sequential convergence [2].

The first step is to describe the u.b. convergence of a sequence of operators in $[\beta : \beta]$ in terms of their associated operators in $[\kappa : \sigma]$.

Let C denote the algebra of functions in B which are uniformly continuous on D . Recall that $[T_n]f = T_n(f)$ and observe that $T_r = TI$, where I is the identity operator.

THEOREM 5. *Let $\{T_n\}$, $n = 1, 2, \dots$, and T be linear operators in $[\beta : \beta]$. Then $\{T_n\}$ converges uniformly on bounded subsets to T if and*

only if $\{T_n\}_r$ converges uniformly on bounded subsets to T_r for every $0 < r < 1$ and there exists an M such that $\|T_n\| \leq M$, $n = 1, 2, \dots$.

Proof. Let $\{T_n\}$ converge u.b. to T . Then $T_n f$ converges strictly to Tf for every fixed f in C . Hence for fixed f in C , $\{T_n f\}$ is uniformly bounded in norm, because strictly convergent sequences are bounded. By the uniform boundedness principle, the set $\{\|T_n\|\}$ is uniformly bounded, where $\|T_n\| = \sup\{\|T_n f\| : f \in C, \|f\| \leq 1\}$. It follows [2] that this is the norm of T_n as an operator on all of B .

Now fix $0 < r < 1$ and let S be a bounded set and G an open set in (B, β) . Then $(\{T_n\}_r - T_r)(S) = (T_n - T)(I_r)(S) = (T_n - T)S_r \subseteq G$ for $n > N$ for some N , because $S_r = \{f_r : f \in S\}$ is a bounded set and T_n converges u.b. to T .

For the converse let $G = \{g : |g|_\psi < 3\epsilon\}$ be an open set and S a bounded set in (B, β) . Let $G_1 = \{g : |g|_\psi < \epsilon\}$. For f in S ,

$$\begin{aligned} |(\{T_n\}_r - T_n)f|_\psi &= \|\psi(\{T_n\}_r - T_n)f\| \\ &= \|\psi T_n(I_r - I)f\| \\ &\leq M \|\psi(I_r - I)f\| \\ &= M |(I_r - I)f|_\psi \\ &< \epsilon \end{aligned}$$

for $r \geq r_0$ for some $r_0 < 1$ because I_r converges u.b. to I . Hence for $r \geq r_0$, $(\{T_n\}_r - T_n)S \subseteq G_1$.

Since T_r converges u.b. to T , there is an r_1 such that $1 > r \geq r_1$ implies $(T - T_r)S \subseteq G_1$. Fix t larger than r_0 and r_1 and let N be such that $n > N$ implies $(T_t - \{T_n\}_t)S \subseteq G_1$. Then for $n > N$,

$$\begin{aligned} (T - T_n)S &= (T - T_t)S + (T_t - \{T_n\}_t)S + (\{T_n\}_t - T_n)S \\ &\subseteq G_1 + G_1 + G_1 \\ &\subseteq G. \end{aligned}$$

LEMMA 6. Let $\{T_n\}$, $n = 1, 2, \dots$, and T be in $[\beta : \beta]$. Then $\{T_n\}_r$ converges uniformly on bounded subsets to T_r for every $0 < r < 1$ if and only if the corresponding kernels $K_n(z, w)$ converge to $K(z, w)$ uniformly on compact subsets of $D \times \{w : |w| > 1\}$ and given $\rho > 1$, there exists an M_ρ such that $|K_n(z, w)| \leq M_\rho$ for all n and $|z| < 1$, $|w| \geq \rho > 1$.

Proof. Let $\{T_n\}_r$ converge u.b. to T_r . Fix $r < 1$ and $s < 1$. Then it will be shown that $\{K_n(z, w)\}$ converges to $K(z, w)$ uniformly on $R = \{(z, w) : |z| \leq s \text{ and } |w| \geq \rho > 1/r\}$.

The operators $[T_n]_r$ and T_r are maps from C into B . As in the previous Theorem it follows that $\|[T_n]_r\|$ is uniformly bounded for $n = 1, 2, \dots$.

Given $\epsilon > 0$ and s let ψ in $C_0[D]$ be 1 on $\{z : |z| < s\}$. There is an N such that for $n > N$ and for all f in the bounded set $\{f : \|f\| \leq 1\}$, $\|([T_n]_r - T_r)f\|_s \leq \|([T_n]_r - T_r)f\|_\psi < \epsilon$ since $[T_n]_r$ converges u.b. to T_r . Thus for $\|f\| \leq 1$,

$$\begin{aligned} \|([T_n]_r - T_r)f\| &\leq \|\psi([T_n]_r - T_r)f\| \\ &= \|([T_n]_r - T_r)f\|_\psi \\ &< \epsilon. \end{aligned}$$

For $j = 0, 1, \dots$, let $f_j(w) = w^j$, $u_j = T(f_j)$ and $u_{j,n} = T_n(f_j)$. Then $[T_n]_r f_j(z) - T f_j(z) = r^j u_{j,n}(z) - r^j u_j(z)$. Hence $\|r^j [u_{j,n}(z) - u_j(z)]\|_s < \epsilon$ for $n > N$ for all $j = 0, 1, \dots$. For $j = 1, 2, \dots, J$, we have $\|u_{j,n}(z) - u_j(z)\|_s < \epsilon/r^j$ for $n > N$ since $r^j \geq r^J$.

Now $K_n(z, w) - K(z, w) = \sum_{k=0}^{\infty} (u_{k,n}(z) - u_k(z)) r^k / w^{k+1} r^k$. For $n > N$, $\|r^k [u_{k,n}(z) - u_k(z)]\|_s < \epsilon < 1$. Since $r\rho > 1$ there is a J so large that $\sum_{k=J+1}^{\infty} (1/r\rho)^k < \epsilon/2$. Then

$$\|K_n(z, w) - K(z, w)\|_R \leq \left\| \sum_{k=0}^J (u_{k,n}(z) - u_k(z)) / w^{k+1} \right\|_R + \epsilon/2$$

since $1/|wr| \leq 1/pr$. Also for $n > N$, $\|u_{k,n}(z) - u_k(z)\|_s < \epsilon/2$ for $k = 1, 2, \dots, J$. Therefore $\|K_n - K\|_R < \epsilon$.

It remains to be shown that given $\rho > 1$, there exists a constant M_ρ such that $|K_n(z, w)| \leq M_\rho$ for all $|z| < 1$ and $|w| > \rho > 1$. Given $\rho > 1$, fix $0 < r < 1$ with $r\rho > 1$. Since $[T_n]_r$ converges u.b. to T_r it follows from Theorem 6 that there exists an M with $\|[T_n]_r\| \leq M$ for all $n = 1, 2, \dots$. Now

$$\begin{aligned} K_n(z, w) &= \sum_{k=0}^{\infty} T_n(z^k) / w^{k+1} \\ &= r \sum_{k=0}^{\infty} r^k T_n(z^k) / r^{k+1} w^{k+1} \\ &= r \sum_{k=0}^{\infty} [T_n]_r(z^k) / (wr)^{k+1} \end{aligned}$$

and therefore for $|z| < 1$ and $|w| \geq \rho$,

$$\begin{aligned} |K_n(z, w)| &\leq rM \sum_{k=0}^{\infty} 1/|wr|^{k+1} \\ &\leq Mr \sum_{k=0}^{\infty} (1/(\rho r))^{k+1} \end{aligned}$$

where the last expression is M_ρ .

For the converse, fix $0 < r < 1$ and let $\gamma = (1 + 1/r)/2$. Now

$$\begin{aligned} \|([T_n]_r - T)f\| &= \left\| \int_{|w|=\gamma} f_r(w)(K_n(z, w) - K(z, w))dw \right\|, \\ &\leq \|K_n(z, w) - K(z, w)\|_R \|f\| \end{aligned}$$

where $R = \{(z, w) : |z| < s, |w| = (1 + 1/r)/2\}$. This last expression tends to zero as $n \rightarrow \infty$. Hence if S is a bounded set, $([T_n]_r - T)S$ converges κ to zero.

Let f in B satisfy $\|f\| \leq 1$. Then

$$\begin{aligned} \|[T_n]_r f(z)\| &= \left\| \int_{|w|=\gamma} f_r(w)K_n(z, w)dw \right\| \\ &\leq \|K_n(z, w)\|_R \|f\| \\ &\leq M \end{aligned}$$

by assumption on the kernels K_n where $R = \{(z, w) : |z| < 1, |w| = (1 + 1/r)/2\}$. Hence $\|[T_n]_r\| \leq M$ for all n .

Let S be a bounded set in (B, β) and let $G = \{g : |g|_\psi < \epsilon, \psi \neq 0\}$ be an open set in (B, β) . We have $\|([T_n]_r - T)\| \leq 2M$. Let r' be such that for $|z| > r', |\psi(z)| < \epsilon/2M$. Then

$$\epsilon > \|([T_n]_r - T)f\psi\|_{r' < |z| < 1}.$$

For $|z| \leq r'$, choose N such that $n > N$ implies

$$\|([T_n]_r - T)f\|_{r'} < \epsilon \|\psi\|^{-1} \text{ for all } f \text{ in } S.$$

Then $\|([T_n]_r - T)f\psi\|_{r'} < \epsilon$ and $([T_n]_r - T)f \in G$ for $n > N$ and all f in S .

Theorem 5 and Lemma 6 taken together characterize u.b. convergence on bounded sets in $[\beta : \beta]$ in terms of the kernel functions $K(z, w)$.

THEOREM 7. *Let $\{T_n\}, n = 1, 2, \dots$ and T be in $[\beta : \beta]$. Then $\{T_n\}$ converges u.b. to T if and only if the corresponding kernels $\{K_n(z, w)\}$ converge uniformly on compact subsets of $D \times \{w : |w| > 1\}$ to $K(z, w)$ and for any $\rho > 1$, there exists a number M_ρ such that $|K_n(z, w)| \leq M_\rho$ for $|z| < 1$ and $|w| \geq \rho > 1$.*

COROLLARY. *Let S be a norm (equivalently u.b.) bounded subset of $[\beta : \beta]$. Let $\{T_n\}, n = 1, 2, \dots$, and T be in S with corresponding*

kernels $\{K_n(z, w)\}$ and $K(z, w)$. Then $\{T_n\}$ converges u.b. to T if and only if $\{K_n(z, w)\}$ converges uniformly on compact subsets of $D \times \{w : |w| > 1\}$ to $K(z, w)$.

Proof. The condition that $\{K_n(z, w)\}$ converges κ to $K(z, w)$ on $D \times \{w : |w| > 1\}$ is necessary by the above theorem. Let T in S imply $\|T\| \leq M$. Then it follows that $|K_n(z, w)| \leq M_\rho$ for $|z| < 1$ and $|w| > \rho$ and the condition is also sufficient.

On the locally compact Hausdorff space $D \times \{w : |w| > 1\}$, a sequence $\{K_n(z, w)\}$ converges strictly to a function $K(z, w)$, [5], if and only if $\{K_n(z, w)\}$ converges uniformly on compact subsets of $D \times \{w : |w| > 1\}$ to $K(z, w)$ and $|K_n(z, w)|$ is uniformly bounded on $D \times \{w : |w| > 1\}$. The next corollary follows immediately from the previous theorem, but it is not known if the converse holds. See Theorem 8 for a similar result.

COROLLARY. Let $\{T_n\}$, $n = 1, 2, \dots$ and T be in $[\beta : \beta]$ with corresponding kernels $\{K_n(z, w)\}$ and $K(z, w)$. If $\{K_n(z, w)\}$ converges strictly to $K(z, w)$ on $D \times \{w : |w| > 1\}$, then $\{T_n\}$ converges u.b. to T .

5. Convergence of multipliers. The characterization of u.b. convergence in the last section is applied to the multiplier operators.

DEFINITION. A multiplier on B is a linear operator T such that there exists a sequence $\{c_n\}$ with the property that $T(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n c_n z^n$ for every function $\sum_{n=0}^{\infty} a_n z^n$ in B . It is known [1] that an operator T is a multiplier from B into B if and only if the sequence $\{c_n\}$ is one side of the sequence of Fourier-Stieltjes coefficients of a bounded complex valued regular Borel measure μ on Γ and also $\|\mu\| = \|T\|$. Also if T is a multiplier from B into B , then T is in $[\kappa : \kappa]$, a subalgebra of $[\beta : \beta]$. Let $\hat{\mu}(k)$ denote the k th Fourier-Stieltjes coefficient of the measure μ .

Clearly, if $\{T_n\}$, $n = 1, 2, \dots$ and T are multipliers in $[\beta : \beta]$, and $\{T_n\}$ converges in norm to T then $\lim_{n \rightarrow \infty} \hat{\mu}_n(k) = \hat{\mu}(k)$ uniformly in k , where μ_n and μ are the measures associated with T_n and T respectively. In other words, the sequence of functions $\{\hat{\mu}_n\}$ defined on P , the nonnegative integers, converges uniformly to $\hat{\mu}$ on P . One expects then that for u.b. convergence the functions $\{\hat{\mu}_n\}$ will converge strictly to $\hat{\mu}$ on P . On the locally compact Hausdorff space P , a sequence of functions $\{\hat{\mu}_n\}$ converges strictly to a function $\hat{\mu}$ if and only if $\{\hat{\mu}_n\}$ is uniformly bounded and $\{\hat{\mu}_n\}$ converges uniformly on compact subsets to $\hat{\mu}$ [5], i.e., pointwise on P .

THEOREM 8. *Let $\{T_n\}$, $n = 0, 1, \dots$, and T be multipliers from B into B with associated measures $\{\mu_n\}$ and μ . Then $\{T_n\}$ converges u.b. to T if and only if $\{\hat{\mu}_n\}$ converges strictly to $\hat{\mu}$.*

Proof. For necessity we must show that there exists an M such that $|\hat{\mu}_n(k)| \leq M$ for all $n, k = 0, 1, \dots$, and $\lim_{n \rightarrow \infty} \hat{\mu}_n(k) = \hat{\mu}(k)$, $k = 0, 1, \dots$. Since $\{T_n\}$ converges u.b. to T there is an M such that $\|T_n\| \leq M$, $n = 0, 1, \dots$. Since $\|T_n\| = \|\mu_n\|$, $|\hat{\mu}_n(k)| \leq \|\mu_n\| \leq M$. Let $\hat{\mu}_n(k) = c_{n,k}$. Now since $\{T_n(z^k)\}$ converges strictly to $T(z^k)$, we have $\{c_{n,k}z^k\}$ converges strictly to $c_k z^k$ as $n \rightarrow \infty$. Hence $\lim_{n \rightarrow \infty} c_{n,k} = c_k$.

For the sufficiency part of the proof, let $|z| \leq s < 1$ and $|w| \geq \rho > 1$. Then

$$\begin{aligned} |K_n(z, w) - K(z, w)| &= \left| \sum_{k=0}^{\infty} (c_{n,k} - c_k) z^k / w^{k+1} \right| \\ &\leq \sum_{k=0}^{\infty} |c_{n,k} - c_k| (s/\rho)^k. \end{aligned}$$

Let k' be such that $\sum_{k=k'}^{\infty} (s/\rho)^k < \epsilon/4M$ and let N be so large that $n > N$ implies $|c_{n,k} - c_k| < \epsilon\rho/2(s - \rho)$ for $k = 0, 1, \dots, k'$. Then for $|z| < s$ and $|w| \geq \rho$, $|K_n(z, w) - K(z, w)| < \epsilon$. Also $|\sum_{k=0}^{\infty} (c_{n,k}z^k/w^{k+1})| \leq M\rho(\rho - 1)^{-1}$ for all $|z| < 1$ and $|w| \geq \rho$.

The multipliers from B into B which are in the algebra $[\beta : \sigma]$ correspond to the absolutely continuous measures on Γ [1]. Let ϕ_n in $L^1(\Gamma)$ correspond to the multiplier T_n .

COROLLARY. *Let $\{T_n\}$, $n = 1, 2, \dots$ and T be multipliers in $[\beta : \sigma]$. Then $\{T_n\}$ converges uniformly on bounded subsets to T if and only if $\|\phi_n\|_{L^1} \leq M$ and $\lim_{n \rightarrow \infty} \hat{\phi}_n(k) = \hat{\phi}(k)$.*

COROLLARY. *Let $\{T_n\}$, $n = 1, 2, \dots$ and T be multipliers in $[\beta : \sigma]$. If $\{\phi_n\}$ converges to ϕ in L^1 , then $\{T_n\}$ converges u.b. to T .*

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