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An abstract study of the theory of fractional ideals of a commutative ring is begun. In particular, the definition of principal element in a multiplicative lattice L is used to define a lattice of fractional elements, L^*, associated with L. As one application of this definition a theory of Dedekind lattices is developed. This construction also allows the development of an abstract theory of integral closure for a Noether lattice. This theory will be presented in a further paper.

By a multiplicative lattice we mean a complete lattice L together with a commutative, associative multiplication on L such that (i) $a(b \cup c) = ab \cup ac$ and (ii) $ab \leqq a \cap b$ for all a, b, c in L. We further assume that L has a greatest element e such that $ea = a$ for all a in L and a least element 0. We denote the meet and join of two elements a, b in L by $a \cup b$ and $a \cap b$, respectively, and we use $\leqq$ to denote the order relation on L. A lattice with a multiplication satisfying condition (i) above is a lattice ordered semi-group.

An element m in L is join principal if $(a \cup bm): m = a: m \cup b$ for all a, b in L, meet principal if $(a \cap b: m)m = am \cap b$ for all a, b in L, and principal if it is both join and meet principal. This definition of principal element was given by Dilworth in **[1]**.

The author wishes to express appreciation to Dr. William M. Cunnea.

1. Definition and basic properties of L^*. Let L be a multiplicative lattice and consider the set of all ordered pairs of the form (p, q), where $p, q \in L$ and q is a principal nonzero divisor of L. We define a relation, denoted by "$\sim$", on this set as follows:

$$(p, q) \sim (p', q') \quad \text{iff} \quad pq' = qp'.$$

LEMMA 1.1. *"$\sim$" is an equivalence relation on the set of ordered pairs defined above.*

Proof. It is clear that the relation is reflexive and symmetric. To show transitivity, assume $(p, q) \sim (p', q')$ and $(p', q') \sim (p'', q'')$. Then $pq'q'' = qp'q''$ and since $p'q'' = q'p''$ this can be rewritten $pq''q' = qp''q'$. Therefore,

$$pq'' = pq''q' \colon q' = qp''q' \colon q' = qp'',$$

where the first and last equalities follow from the fact that q' is a principal nonzero divisor in L.

Let L^* denote the set of equivalence classes defined by the above equivalence relation. We denote the equivalence class containing (p,q) by $\langle p,q\rangle$. If $\langle p,q\rangle$ and $\langle r,s\rangle$ are elements of L^* we define $\langle p,q\rangle \leqq \langle r,s\rangle$ iff $ps \leqq qr$.

LEMMA 1.2. *The relation "$\leqq$" is a partial order on L^*.*

Proof. To show that "$\leqq$" is well defined, assume that $(p,q) \sim (p',q')$ and $(r,s) \sim (r',s')$. Then $pq' = qp'$ and $rs' = sr'$. Now, suppose $ps \leqq qr$. Then

$$(p's')qs = s'q'ps \leqq s'q'qr = (r'q')qs.$$

Therefore, since qs is a principal nonzero divisor in L,

$$p's' = [(p's')(qs)] \colon (qs) \leqq [(r'q')(qs)] \colon (qs) = r'q'$$

and "$\leqq$" is well defined.

It is clear the relation is reflexive and antisymmetric. To show transitivity, suppose $\langle p,q\rangle \leqq \langle r,s\rangle$ and $\langle r,s\rangle \leqq \langle r',s'\rangle$. Then $pss' \leqq qrs' \leqq qsr'$. Thus,

$$ps' = ps's \colon s \leqq qr's \colon s = qr'.$$

THEOREM 1.1. *The set L^* together with the relation $\leqq$ is a lattice with least upper bound and greatest lower bound given by the following equations*:

(1) $\langle p,q\rangle \cup \langle p',q'\rangle = \langle pq' \cup qp', qq'\rangle$

(2) $\langle p,q\rangle \cap \langle p',q'\rangle = \langle pq' \cap qp', qq'\rangle$.

Proof. Let $\langle p,q\rangle$ and $\langle p',q'\rangle$ be any two elements of L^*. Then

$$pqq' \leqq pqq' \cup qqp' = q(pq' \cup qp').$$

Therefore, $\langle p,q\rangle \leqq \langle pq' \cup qp', qq'\rangle$. Similarly, $\langle p',q'\rangle \leqq \langle pq' \cup qp', qq'\rangle$.

Thus, $\langle pq' \cup qp', qq'\rangle$ is an upper bound for $\langle p,q\rangle$ and $\langle p',q'\rangle$. Moreover, if $\langle p,q\rangle \leqq \langle r,s\rangle$ and $\langle p',q'\rangle \leqq \langle r,s\rangle$, then $ps \leqq qr$ and $p's \leqq q'r$. Therefore

$$(pq' \cup qp')s = pq's \cup qp's \leqq qq'r \cup qq'r = qq'r.$$

Thus, $\langle pq' \cup qp', qq' \rangle \leqq \langle r, s \rangle$ and $\langle pq' \cup qp', qq' \rangle$ is the least upper bound for $\langle p, q \rangle$ and $\langle p', q' \rangle$.

Since q is a principal nonzero divisor,

$$(pq' \cap qp')q = (pq' \cap (qp'q) \colon q)q = pq'q \cap qp'q \leqq qq'p.$$

Thus, $\langle pq' \cap qp', qq' \rangle \leqq \langle p, q \rangle$ and a similar argument shows that $\langle pq' \cap qp', qq' \rangle \leqq \langle p', q' \rangle$.

If $\langle r, s \rangle \leqq \langle p, q \rangle$ and $\langle r, s \rangle \leqq \langle p', q' \rangle$, then $rq \leqq sp$ and $rq' \leqq sp'$. Therefore, since s is a principal nonzero divisor,

$$s(pq' \cap qp') = spq' \cap sqp' \leqq rqq' \cap rqq' = rqq'.$$

Thus, $\langle pq' \cap qp', qq' \rangle$ is the greatest lower bound of $\langle p, q \rangle$ and $\langle p', q' \rangle$.

DEFINITION 1.1. The lattice L^* will be called the lattice of fractional elements of L.

We now define a multiplication on L^* as follows: If $\langle p, q \rangle$ and $\langle r, s \rangle$ are elements of L^*, then

$$\langle p, q \rangle \langle r, s \rangle = \langle pr, qs \rangle.$$

It is easy to see that this multiplication is well defined.

PROPOSITION 1.1. *With the above multiplication, L^* is a commutative, associative lattice ordered semigroup. The element $\langle e, e \rangle$ is a multiplicative identity.*

Proof. For arbitrary elements $\langle a, b \rangle$, $\langle c, d \rangle$, and $\langle f, g \rangle$ in L^* we have

$$\begin{aligned} \langle a, b \rangle (\langle c, d \rangle \cup \langle f, g \rangle) &= \langle a, b \rangle \langle cg \cup df, dg \rangle = \langle acg \cup adf, bdg \rangle \\ &= \langle b(acg \cup adf), b(bdg) \rangle = \langle ac, bd \rangle \cup \langle af, bg \rangle \\ &= \langle a, b \rangle \langle c, d \rangle \cup \langle a, b \rangle \langle f, g \rangle, \end{aligned}$$

where we have used the fact that

$$(b(acg \cup adf), b(bdg)) \sim (acg \cup adf, bdg).$$

Commutativity and associativity for multiplication are obvious as is the fact that $\langle e, e\rangle$ is a multiplicative identity.

We remark that L^* is not a multiplicative lattice since it does not satisfy the condition

$$\langle p, q\rangle\langle p', q'\rangle \leqq \langle p, q\rangle \cap \langle p', q'\rangle.$$

The original lattice, L, can be embedded in the lattice L^* as follows: Let $\bar{L}$ be the sublattice of L^* consisting of all elements of the form $\langle p, e\rangle$, where $p \in L$ and e is the largest element of L. Then $\bar{L}$ is a residuated multiplicative lattice. In fact,

$$\langle p, e\rangle : \langle q, e\rangle = \langle p : q, e\rangle.$$

The mapping $\phi: L \to \bar{L}$ defined by $\phi(p) = \langle p, e\rangle$ for all p in L is then a lattice isomorphism of the residuated multiplicative lattice L onto the residuated multiplicative lattice $\bar{L}$.

PROPOSITION 1.2. $\bar{L} = \{\langle p, q\rangle \in L^* | \langle p, q\rangle \leqq \langle e, e\rangle\}$. *If* $\langle p, q\rangle \in \bar{L}$, *then* $\langle p, q\rangle = \langle p : q, e\rangle$.

Proof. Clearly $\langle p, e\rangle \leqq \langle e, e\rangle$ for all p in L. If $\langle p, q\rangle \leqq \langle e, e\rangle$, then $p \leqq q$. Therefore, since q is principal, $(p : q)q = p \cap q = p$. Thus, $\langle p, q\rangle = \langle q(p : q), q\rangle = \langle p : q, e\rangle$.

Let $a \in L$ and suppose that $\{a_i | i \in I\}$ is a subset of L. Then $a(\cup_{i \in I} a_i) = \cup_{i \in I} aa_i$. This result can be found in **[6]**.

THEOREM 1.2. *Let* $p' \in L$ *such that there exists a principal nonzero divisor* $A \in L$ *with* $a \leqq p'$. *If* q' *is any principal nonzero divisor in* L, *the residual* $\langle p, q\rangle : \langle p', q'\rangle$ *exists for all elements* $\langle p, q\rangle$ *in* L^*.

Proof. For an arbitrary element $\langle p, q\rangle \in L^*$, define

$$A = \{\langle r, s\rangle | \langle r, s\rangle \in L^* \quad \text{and} \quad \langle r, s\rangle\langle p', q'\rangle \leqq \langle p, q\rangle\}.$$

A is nonempty since $\langle 0, e\rangle \in A$. We will show that there exists a greatest element, $\langle c, d\rangle$, in the set A. It is clear that if such an element exists then $\langle c, d\rangle = \langle p, q\rangle : \langle p', q'\rangle$.

We first show there exists a principal nonzero divisor d in L such that

(i) $\langle d, e\rangle\langle r, s\rangle \leqq \langle e, e\rangle$ for all $\langle r, s\rangle \in A$.

Let a be a principal nonzero divisor such that $a \leqq p'$. Then $\langle a, e\rangle \leqq \langle p', q'\rangle$ since $aq' \leqq p'q' \leqq p'$. Therefore,

$$\langle r, s\rangle\langle a, e\rangle \leqq \langle r, s\rangle\langle p', q'\rangle \leqq \langle p, q\rangle$$

for all $\langle r, s\rangle \in A$. Hence

$$\langle aq, e\rangle\langle r, s\rangle = \langle a, e\rangle\langle q, e\rangle\langle r, s\rangle \leqq \langle q, e\rangle\langle p, q\rangle$$
$$= \langle qp, q\rangle = \langle p, e\rangle \leqq \langle e, e\rangle.$$

Therefore, if we set $d = aq$, (i) is satisfied.

With d defined as in the preceding paragraph, let $c = \bigcup\{dr\colon s \mid \langle r, s\rangle \in A\}$. This element exists since L is a complete lattice. With c and d defined as above, $\langle c, d\rangle$ is the greatest element of A. To show this, let $\langle r, s\rangle \in A$. Then $rp'q \leqq sq'p$. Since $\langle dr, s\rangle \leqq \langle e, e\rangle$, $dr \leqq s$. Combining this with the fact that s is principal gives

$$(dr\colon s)p'qs = (dr \cap s)p'q = drp'q \leqq dq'ps$$

for all $\langle r, s\rangle \in A$. Therefore,

$$(dr\colon s)p'q = [(dr\colon s)p'qs]\colon s \leqq (dq'ps)\colon s = dq'p$$

for all $\langle r, s\rangle \in A$. Thus,

$$\bigcup_{\langle r, s\rangle \in A} ((dr\colon s)p'q) \leqq dq'p$$

and so

$$cp'q = \left(\bigcup_{\langle r, s\rangle \in A} dr\colon s\right) p'q = \bigcup_{\langle r, s\rangle \in A} ((dr\colon s)p'q) \leqq dq'p.$$

Therefore, $\langle c, d\rangle\langle p', q'\rangle \leqq \langle p, q\rangle$ and $\langle c, d\rangle$ is an element of A. If $\langle r, s\rangle$ is an arbitrary element of A then, since $dr\colon s \leqq c$,

$$rd = s(rd\colon s) \leqq sc.$$

Thus, $\langle r, s\rangle \leqq \langle c, d\rangle$ so $\langle c, d\rangle$ is the greatest element of A.

We now investigate the existence of a multiplicative inverse for elements of the lattice of fractional elements. If $\langle p, q\rangle$ is an invertible element of L^*, $\langle p, q\rangle^{-1}$ will denote the multiplicative inverse of $\langle p, q\rangle$ in L^*. This inverse is unique if it exists.

PROPOSITION 1.3. *A nonzero element $p \in L$ is invertible in L^* if and only if there exists an element $q \in L$ such that pq is a principal nonzero divisor.*

Proof. If $\langle p,e\rangle\langle x,y\rangle=\langle e,e\rangle$, then $\langle px,y\rangle=\langle e,e\rangle$, i.e., $px=y$ with y principal. If there exists $q\in L$ such that $pq=y$ is a principal nonzero divisor, then $\langle q,y\rangle$ is the inverse of $\langle p,e\rangle$ in L^*.

COROLLARY. *Every principal nonzero divisor in L is invertible in* L^*.

PROPOSITION 1.4. *Let* $\langle p,q\rangle\in L^*$ *with p a nonzero divisor. If* $\langle p,q\rangle$ *is invertible in* L^*, *then* $\langle p,q\rangle^{-1}=\langle e,e\rangle:\langle p,q\rangle$.

Proof. Since $\langle p,q\rangle$ is invertible, there exists $\langle x,y\rangle\in L^*$ such that $px=qy$. Thus, px is a principal nonzero divisor and $px\leqq p$. Therefore, by Theorem 1.2, $\langle e,e\rangle:\langle p,q\rangle$ exists.

Clearly, $\langle p,q\rangle^{-1}\leqq\langle e,e\rangle:\langle p,q\rangle$. Moreover,

$$[\langle e,e\rangle:\langle p,q\rangle]\,\langle p,q\rangle\leqq\langle e,e\rangle.$$

Therefore,

$$[\langle e,e\rangle:\langle p,q\rangle]\,\langle p,q\rangle\langle p,q\rangle^{-1}\leqq\langle p,q\rangle^{-1}\langle e,e\rangle=\langle p,q\rangle^{-1}.$$

Thus, $\langle e,e\rangle:\langle p,q\rangle\leqq\langle p,q\rangle^{-1}$.

The multiplicative lattice, L, is an M-lattice if and only if it satisfies the following condition:

(M) If a and b are elements of L with $a\leqq b$, there exists an element $c\in L$ such that $a=bc$.

We list here two important properties of such lattices:

(1) L is an M-lattice if and only if every element of L is meet principal.

(2) An M-lattice is distributive.

For proofs of these properties as well as a more complete discussion of M-lattices, see **[3]** and **[7]**.

PROPOSITION 1.5. *If the nonzero elements of* L^* *form a group then L is an M-lattice.*

Proof. Let a and b be elements of L with $a\leqq b$. Then there exists $\langle x,y\rangle\in L^*$ such that

(i) $\langle b,e\rangle\langle x,y\rangle=\langle a,e\rangle$.

Thus, $bx=ay$ with y a principal nonzero divisor in L. Since $a\leqq b$, $a=a\cap b$ and so

$$bx=ay=(a\cap b)y=ay\cap by=bx\cap by.$$

Thus, $bx \leqq by$ and therefore $x \leqq y$. Thus, by Proposition 1.2, $\langle x, y\rangle = \langle x: y, e\rangle$. Therefore, (i) may be rewritten

$$\langle b, e\rangle\langle x: y, e\rangle = \langle a, e\rangle$$

or, $b(x: y) = a$.

THEOREM 1.3. *The nonzero elements of L^* from a group if and only if every nonzero element of L is a principal nonzero divisor.*

Proof. If the nonzero elements of L^* from a group then L is an M-lattice by the previous proposition so that every element of L is meet principal. To show every element is join principal, let $a, b \in L$, $b \neq 0$. Then $(ab: b)b \leqq ab$ which implies $ab: b \leqq a$ since b has an inverse in L^*. Since clearly $a \leqq ab: b$, we have

(i) $ab: b = a$

for all $a, b \in L$, $b \neq 0$.

Let a, b, c be elements of L with $c \neq 0$. Then

$$((a: c) \cup b)c = (a: c)c \cup bc = (a \cap c) \cup bc$$

since c is meet principal. Since L is distributive,

$$(a \cap c) \cup bc = (a \cup bc) \cap (c \cup bc) = (a \cup bc) \cap c.$$

Thus,

(ii) $((a: c) \cup b)c = (a \cup bc) \cap c.$

Using equations (i) and (ii) gives

$$(a: c) \cup b = [((a: c) \cup b)c]: c = [(a \cup bc) \cap c]: c$$
$$= (a \cup bc): c.$$

Thus, every nonzero element of L is a principal nonzero divisor.

Conversely, if every nonzero element of L is a principal nonzero divisor and if $\langle p, q\rangle \in L^*$, $\langle p, q\rangle \neq \langle 0, e\rangle$, then $\langle q, p\rangle \in L^*$. Thus

$$\langle p, q\rangle\langle q, p\rangle = \langle e, e\rangle$$

so $\langle p, q\rangle$ is invertible in L^*.

PROPOSITION 1.6. *If every nonzero element of L is invertible in L^* then the nonzero elements of L^* from a group.*

Proof. Let $\langle p, q\rangle \in L^*$, $p \neq 0$. Since p is invertible in L^*, there exists $\langle x, y\rangle \in L^*$ such that $px = y$. Then $\langle xq, y\rangle$ is the multiplicative inverse for $\langle p, q\rangle$ in L^*.

PROPOSITION 1.7. *Suppose L satisfies the following conditions:*

(1) *Every element of L contains a principal element.*

(2) *L contains no zero divisors.*

Then L is an M-lattice if and only if every nonzero element of L is a principal nonzero divisor.

Proof. If every element is principal, L is clearly an M-lattice. Suppose L is an M-lattice and let $p \in L$, $p \neq 0$. Let $q \leqq p$ be a principal element of L. Then $q = pr$ for some r in L. Thus, p is invertible in L^* by Proposition 1.3. The Proposition then follows from Proposition 1.6 and Theorem 1.3.

EXAMPLE. Let $L(R)$ be the lattice of ideals of a commutative ring with identity R. Let $L(Q(R))$ denote the lattice of fractional ideals of R. If $A \in L(Q(R))$, then $A = \frac{1}{d}B$, where B is an ideal of R. The mapping $\phi: L(Q(R)) \to L^*$ defined by $\phi\left(\frac{1}{d}B\right) = \langle B, (d)\rangle$ is an isomorphism of $L(Q(R))$ onto L^*. Thus, in this case, the lattice of fractional elements defined above is isomorphic to the lattice of fractional ideals of R.

2. Dedekind lattices. Throughout this section we will assume that L is a multiplicative lattice that satisfies the following conditions:

(A) L is modular.

(B) Every element of L is a join of principal elements.

(C) If p is a principal element of L and $p \leqq \cup_{\iota \in I} q_\iota$ where each q_ι is principal, then there exists a finite subset I' of I such that $p \leqq \cup_{\iota \in I'} q_\iota$.

(D) L contains no zero divisors.

L^* will denote the lattice of fractional elements of L.

If $L(R)$ is the lattice of ideals of a commutative ring with identity R, then $L(R)$ satisfies (A) and (B). Since every principal element of $L(R)$ is a finitely generated ideal of R ([**3**], p. 655), $L(R)$ also satisfies (C). We also remark that a Noether lattice satisfies (A) through (C). A further discussion of (B) and (C) can be found in [**6**].

DEFINITION 2.1. A Dedekind lattice is a multiplicative lattice satisfying (A) through (D) above in which every element can be written as a finite product of prime elements.

LEMMA 2.1. *Let $\{p_i | i = 1,\cdots,n\}$ be a set of elements of L. If $\Pi_{i=1}^n p_i$ is invertible in L^*, then each p_i is invertible in L^*.*

Proof. By Proposition 1.3, $\Pi_{i=1}^n p_i$ is invertible if and only if there exists $x, y \in L$ with y principal such that $x\Pi_{i=1}^n p_i = y$. Then, for $j = 1,\cdots,n$,

$$p_j\left(x\prod_{i\neq j} p_i\right) = y$$

so p_j is invertible by Proposition 1.3.

LEMMA 2.2. *For products of invertible prime elements of L, the factorization into prime elements is unique.*

Proof. Suppose $a = \Pi_{i=1}^n p_i = \Pi_{j=1}^m q_j$ where p_i and q_j are prime in L and a is invertible in L^*. Further, assume p_1 is minimal among the set $\{p_i | i = 1,\cdots, n\}$. Then $\Pi_{j=1}^m q_j \leqq p_1$ so there exists q_j such that $q_j \leqq p_1$. Without loss of generality we may assume $j = 1$ so that $q_1 \leqq p_1$. Now, $\Pi_{i=1}^n p_i \leqq q_1$. Thus, there exists an integer s such that $p_s \leqq q_1$. Then $p_s \leqq q_1 \leqq p_1$ which implies $q_1 = p_1$ since p_1 was assumed to be minimal among the p_i. By Lemma 2.1, p_1 is invertible in L^*. Therefore,

$$\prod_{i=2}^n p_i = p_1^{-1}p_1 \prod_{i=2}^n p_i = p_1^{-1}p_1 \prod_{j=2}^m q_j = \prod_{j=2}^m q_j.$$

Clearly, $\Pi_{i=2}^n p_i = \Pi_{j=2}^m q_j$ is invertible in L^*, so the above argument can be repeated.

PROPOSITION 2.1. *If $p \in L$ is invertible in L^*, then p can be written as a finite join of principal elements.*

Proof. If $p \in L$ is invertible in L^* there exists $\langle r, s\rangle \in L^*$ such that $\langle p, e\rangle\langle r, s\rangle = \langle e, e\rangle$. By condition (B) on the lattice L, we can write

$$p = \bigcup_{i\in I} p_i \quad \text{and} \quad r = \bigcup_{j\in J} r_j$$

where p_i and r_j are principal for all $i \in I$ and all $j \in J$. Therefore,

$$\langle e, e\rangle = \langle p, e\rangle\langle r, s\rangle = \left\langle \bigcup_{i\in I} p_i, e\right\rangle\left\langle \bigcup_{j\in J} r_j, e\right\rangle$$

$$= \left\langle \bigcup_{i,j} (p_i r_j), s\right\rangle.$$

Thus, $s = \cup_{i,j} (p_i r_j)$. Since s is principal, by condition (C), s can be written as a join of finitely many of the elements $p_i r_j$. Thus,

$$s = \bigcup_{k=1}^{n} p_k r_k$$

where, for all k, $p_k \leqq p$ and $r_k \leqq r$ and p_k, r_k are principal. Therefore $\langle e, e\rangle = \cup_{k=1}^{n} (\langle p_k, e\rangle\langle r_k, s\rangle)$ and so,

$$\langle p, e\rangle = \langle p, e\rangle\langle e, e\rangle = \bigcup_{k=1}^{n} (\langle p, e\rangle\langle p_k, e\rangle\langle r_k, s\rangle)$$

$$\leqq \bigcup_{k=1}^{n} (\langle p, e\rangle\langle r, s\rangle\langle p_k, e\rangle) = \bigcup_{k=1}^{n} \langle e, e\rangle\langle p_k, e\rangle$$

$$= \bigcup_{k=1}^{n} \langle p_k, e\rangle.$$

Since $p_k \leqq p$ for all k,

$$p = \bigcup_{k=1}^{n} p_k.$$

PROPOSITION 2.2. *If $p \in L$ is invertible in L^*, then $qp : p = q$ for all $q \in L$.*

Proof. Clearly $q \leqq qp : p$. Moreover, $(qp : p)p \leqq qp$ and so, since p is invertible,

$$qp : p = (qp : p)pp^{-1} \leqq qpp^{-1} = q.$$

THEOREM 2.1. *In a Dedekind lattice every proper, nonzero prime element is maximal in L and invertible in L^*.*

Proof. We first show that every invertible prime of L is maximal. Because of condition (B) it will suffice to show that if $q \in L$ is principal and $q \not\leqq p$, then $p \cup q = e$. Thus, assume $q \in L$ is principal and $q \not\leqq p$ and consider the elements $p \cup q$ and $p \cup q^2$. Since L is a Dedekind lattice

(i) $p \cup q = \prod_{i=1}^{r} p_i$

(ii) $p \cup q^2 = \prod_{j=1}^{s} q_j$

where p_i and q_j are prime. Clearly, $p \cup q$, $p \cup q^2$ as well as the elements p_i and q_j belong to the factor lattice L/p. We will denote elements of L/p by a/p, b/p, etc.

Since p is prime in L, L/p has no zero divisors and since q and q^2 are principal in L, $(p \cup q)/p$ and $(p \cup q^2)/p$ are principal in L/p.

Let $(L/p)^*$ denote the lattice of fractional elements of L/p. Since $(p \cup q)/p$ and $(p \cup q^2)/p$ are principal nonzero divisors in L/p, they are invertible in $(L/p)^*$ by the Corollary to Proposition 1.3. The elements p_i/p and q_j/p are prime in L/p since they are prime in L. Thus (i) and (ii) give $(p \cup q)/p$ and $(p \cup q^2)/p$ as a product of primes of L/p.

Since $\Pi_{i=1}^{r}(p_i/p) = (p \cup q)/p$ is invertible in $(L/p)^*$, each p_i/p is invertible in $(L/p)^*$ by Lemma 2.1. Similarly, each q_j/p is invertible in $(L/p)^*$.

We now note that $p \cup (p \cup q)^2 = p \cup q^2$. Therefore, in L/p

$$\prod_{i=1}^{r} (p_i/p)^2 = (p \cup q)^2/p = (p \cup q^2)/p = \prod_{j=1}^{s} (q_j/p).$$

Thus, since each p_i/p and q_j/p is invertible in $(L/p)^*$, by Lemma 2.2 the q_j/p must be the p_i/p each repeated twice. Specifically, in L/p we have $s = 2r$ and after a possible renumbering of the q_j, $q_{2i}/p = q_{2i-1}/p = p_i/p$. Therefore, since $p_i \geqq p$ for all i and $q_j \geqq p$ for all j,

$$q_{2i} = q_{2i-1} = p_i$$

in the lattice L. Therefore, in the lattice L,

$$\text{(iii) } p \leqq p \cup q^2 = \prod_{j=1}^{s} q_j = \prod_{i=1}^{r} p_i^2 = (p \cup q)^2 = p^2 \cup q(p \cup q)$$

$$\leqq p^2 \cup q.$$

Since p is prime and $q \not\leqq p$, $rq \leqq p$ implies that $r \leqq p$. Therefore, $p : q \leqq p$ and so $p \cap q = (p:q)q \leqq pq$, where the first equality follows from the fact that q is principal. Since L is a multiplicative lattice, $pq \leqq p \cap q$ and therefore

(iv) $pq = p \cap q$

By assumption, p is invertible. Therefore, by Proposition 2.2 and (iv),

(v) $q = qp : p = (q \cap p) : p = q : p$.

We now establish the following equation:

(vi) $(p^2 \cup q): p = p^2: p \cup q: p$.

By Proposition 2.2, $(p^2: p)p = p^2$ and by (iv) and (v), $(q: p)p = qp = q \cap p$. Therefore

$$(p^2: p) \cup (q: p) = ((p^2: p) \cup (q: p))p: p = ((p^2: p)p \cup (q: p)p): p$$
$$= (p^2 \cup (p \cap q)): p = ((p^2 \cup q) \cap p): p = (p^2 \cup q): p$$

where we have again used Proposition 2.2 as well as the fact that L is modular.

By equation (iii), $p \leqq p^2 \cup q$. Therefore, using (vi) and Proposition 2.2 gives

$$e = p: p \leqq (p^2 \cup q): p = (p^2: p) \cup (q: p) = p \cup (q: p) = p \cup q.$$

Therefore, $p \cup q = e$ and every invertible prime is maximal. We now show that every prime is invertible. Let p be prime and let q be a principal element with $q \leqq p$. Then $q = \Pi_{i=1}^{n} p_i$ where each p_i is prime. Since q is principal it is invertible in L^*. Therefore p_i is invertible in L^* for all i. Thus, each p_i is maximal in L by the first part of the proof. But this implies that $p_i = p$ for some i and so p is invertible.

COROLLARY 2.1. *In a Dedekind lattice, the factorization of an element into a product of primes is unique.*

COROLLARY 2.2. *In a Dedekind lattice every nonzero element is invertible in L^*.*

Proof. If $a \in L$, $a \neq 0$, then $a = \Pi_{i=1}^{n} p_i$ with p_i prime for all i. By Theorem 2.6 and Proposition 1.3, there exists $b_i \in L$ such that $p_i b_i$ is principal. Then, if $b = \Pi_{i=1}^{n} b_i$, ab is principal so a is invertible by Proposition 1.3.

COROLLARY 2.3. *A Dedekind lattice is a Noether lattice.*

Proof. If L is a Dedekind lattice then every nonzero element of L is invertible by Corollary 2.2. Thus, by Proposition 2.1 every element of L can be written as a finite join of principal elements. By using conditions (B) and (C) imposed on L one can prove exactly as in the ring theoretic case, that L then satisfies the *ACC*. Thus, L is a Noether lattice.

Dilworth [1] has noted a special case of the following theorem. By using Corollary 2.3 and Theorem 1.3, his proof can be extended to the present case. It is also possible to give a proof using Theorem 5 of **[4]**. We give a different proof.

THEOREM 2.2. *A multiplicative lattice, L, satisfying conditions* (A) *through* (D) *is a Dedekind lattice if and only if the nonzero elements of L^* form a group.*

Proof. If L is Dedekind every element of L is invertible so the nonzero elements of L^* form a group by Proposition 1.6.

If the nonzero elements of L^* form a group, every element of L is principal by Theorem 1.3. Thus L is a Noether lattice. Let S be the set of elements that cannot be written as a product of prime elements. If S is nonempty it contains a maximal element, a, since L is a Noether lattice. Now, a is not maximal in L since every maximal element of L is prime. Let m be a maximal element of L such that $a \leqq m$. Such an element exists since L is a Noether lattice.

Consider $a: m$. Clearly $a \leqq a: m$. Moreover $a \neq a: m$. For suppose $a = a: m$. Then, since m is principal and a is invertible in L^*,

$$m = a^{-1}am = a^{-1}(a: m)m = a^{-1}(a \cap m) = a^{-1}a = e.$$

Thus, $a < a: m$, so $a: m$ is a product of primes, that is, $a: m = p_1 \cdots p_n$, where each p_i is a prime. Then, since m is principal

$$a = a \cap m = (a: m)m = p_1 \cdots p_n m$$

is a representation of a as a product of prime elements. The contradiction establishes the Theorem.

The following result is an immediate consequence of the preceding theorem, Proposition 1.7, and Theorem 1.3.

COROLLARY 2.4. *A multiplicative lattice satisfying* (A) *through* (D) *is a Dedekind lattice if and only if it is an M-lattice.*

From the corollary to Theorem 6 of **[3]**, it follows that an M-lattice satisfying (A) through (D) also satisfies the *ACC*. In **[8]**, M. Ward has investigated M-lattices satisfying the *ACC*. By using the primary decomposition, he has shown that every element of such a lattice has a unique decomposition into a product of prime elements ([**8**], Theorem 5.2). This result, together with Proposition 1.5, could also be used to prove Theorem 2.2. Using Corollary 2.4, we also obtain the following restatement of Theorem 6.1 of **[8]**.

THEOREM 2.3. *A multiplicative lattice L is a Dedekind lattice if* *and only if it is a Noether lattice without zero divisors satisfying*

(i) *Every primary element of L is a power of a prime*;

(ii) *If p is prime, $p \leqq q$, and $p \neq q$, then $qp = p$.*

For the following theorem, we use the definition of integrally closed elements given in [5].

THEOREM 2.4. *A multiplicative lattice satisfying* (A) *through* (D) *is a Dedekind lattice if and only if L satisfies the following conditions*:

(1) *L is a Noether lattice*;

(2) *Every nonzero prime element of L is maximal*;

(3) *Every principal element of L is integrally closed.*

Proof. Assume L satisfies (1) through (3) and let p be a prime in L. Let $a \leqq p$ be a principal element. By (2) p is a minimal prime associated with a.

In [2], Furuyama has defined the nth symbolic primary power $q^{(n)}$ of a primary element q associated with p to be $(q^n)_p$, where $(q^n)_p$ denotes the p-primary component of q^n. He has then shown that if p is a prime associated with a principal integrally closed element, the only p-primary elements are the symbolic powers $p^{(n)}$. Thus, the symbolic powers $p^{(n)}$ are the only p-primary elements of L. Therefore, the quotient lattice L_p is totally ordered, the only elements of L_p being the powers $[p]^n$ of the maximal element $[p]$. By Theorem 6 of [6], this implies that L is an M-lattice. Thus, by Proposition 1.7 and Theorems 1.2 and 2.2, L is a Dedekind lattice.

Conversely, suppose L is Dedekind. By Corollary 2.3, L is a Noether lattice and by Theorem 2.1, every prime is maximal. Suppose a is a-dependent on b (for a definition of this relation, see [5]). Then there exists an integer n such that $(a \cup b)^{n+1} = b(a \cup b)^n$. Since L is Dedekind, every element of L is invertible in L^*. Thus,

$$a \cup b = (a \cup b)^{n+1}(a \cup b)^{-n} = b(a \cup b)^n(a \cup b)^{-n} = b.$$

Therefore, $a \leqq b$, so b is integrally closed.

Since, by Corollary 2.3, a Dedekind lattice is a Noether lattice, the following theorem is an obvious consequence of Theorem 5 of [4].

THEOREM 2.5. *A Dedekind lattice is isomorphic to the lattice of ideals of a Noetherian ring.*

The following result follows from the corresponding ring theoretic result by using Theorem 2.5. A lattice theoretic proof can also be given which is exactly analogous to the ring theoretic proof.

COROLLARY 2.5. *Let L be a Dedekind lattice. Then every element $\langle a, b \rangle$ of L^* can be written uniquely in the form*

$$\langle a, b \rangle = \prod_{\substack{p \in L \\ p \text{ prime}}} p^{n_p(\langle a,b \rangle)}$$

where $n_p(\langle a, b \rangle)$ is an integer and $n_p(\langle a, b \rangle) = 0$ for all but finitely many p in L. The following equations also hold:

(1) $n_p(\langle a, b \rangle \cup \langle c, d \rangle) = \min\{n_p(\langle a, b \rangle), n_p(\langle c, d \rangle)\}$

(2) $n_p(\langle a, b \rangle \cap \langle c, d \rangle) = \max\{n_p(\langle a, b \rangle), n_p(\langle c, d \rangle)\}$

(3) $n_p(\langle a, b \rangle \langle c, d \rangle) = n_p(\langle a, b \rangle) + n_p(\langle c, d \rangle)$

(4) $\langle a, b \rangle \leqq \langle c, d \rangle$ iff $n_p(\langle a, b \rangle) \geqq n_p(\langle c, d \rangle)$ *for all primes p in L.*

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