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C. Martin Edwards and Peter John Stacey

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ON GROUP ALGEBRAS OF CENTRAL GROUP EXTENSIONS

C. M. EDWARDS AND P. J. STACEY

If A and G are separable locally compact topological groups with A abelian, a central group extension G^f, itself a separable locally compact topological group, of A by G can be defined for each Borel 2-cocycle f from G to A. The structure of the group algebras of G^f has been studied for the case of compact A. In this paper structure theorems for these group algebras are obtained in the general situation.

For compact A it is shown in **[9]** that for each element α of the dual group $\hat{A}$ of A there exists an idempotent R_α in the centralizer $\Delta(L_1(G^f))$ of the L_1-group algebra $L_1(G^f)$ of G^f. In **[8]** it is shown that R_α possesses a unique extension, also denoted by R_α, to an idempotent in the centralizer $\Delta(C^*(G^f))$ of the C^*-group algebra $C^*(G^f)$ of G^f. Moreover the family $\{R_\alpha : \alpha \in \hat{A}\}$ satisfies the conditions $R_\alpha R_\beta = \delta_{\alpha\beta} R_\alpha\ \forall\ \alpha, \beta \in \hat{A}$ and $\Sigma_{\alpha \in \hat{A}} R_\alpha = 1$, the identity operator and where the sum is the strong limit of the family of finite partial sums. However, it is shown in **[3]** that $\Delta(C^*(G^f))$ is a C^*-algebra *-isomorphic to the ideal centre $\mathfrak{Z}(C^*(G^f))$ of $C^*(G^f)$ (see **[6]**). Since G^f, and hence $C^*(G^f)$, is separable $\mathfrak{Z}(C^*(G^f))$ is contained in the centre $Z(C^*(G^f)^\mu)$ of the Baire *(or monotone σ-) envelope $C^*(G^f)^\mu$ of $C^*(G^f)$ (see **[1]**). Denoting the image of R_α under the isomorphism by r_α, it follows that $\{r_\alpha : \alpha \in \hat{A}\}$ is a family of mutually orthogonal projections in $Z(C^*(G^f)^\mu)$ such that $\Sigma_{\alpha \in \hat{A}} r_\alpha = 1$, the identity in $C^*(G^f)^\mu$ where the sum is the least upper bound of the family of finite partial sums. Moreover for each $\alpha \in \hat{A}$, $r_\alpha \cdot L_1(G^f) = L_1(G^f, \alpha) \subseteq L_1(G^f)$ and $r_\alpha \cdot C^*(G^f) = C^*(G^f, \alpha) \subseteq C^*(G^f)$. Hence direct sum decompositions of $L_1(G^f)$, $C^*(G^f)$, $C^*(G^f)^\mu$ and $W^*(G^f)$, the W^*-group algebra of G^f, are defined.

The crucial observation allowing a theory to be developed for noncompact A is that in the compact case $\hat{A} \subset L_1(A)$. Therefore in general, instead of studying the mapping $\alpha \to r_\alpha$ from $\hat{A}$ to $Z(C^*(G^f)^\mu)$, a mapping $\phi \to r(\phi)$ from $L_1(A)$ into $Z(C^*(G^f)^\mu)$ should be constructed. Since in general $L_1(A)$ does not contain idempotents, it then becomes less obvious how direct sum decompositions can be defined. The main result (Theorem 3.1) shows that such a mapping r exists and has a unique extension, also denoted by r, to a σ-normal *-isomorphism from $C^*(A)^\mu$ into $Z(C^*(G^f)^\mu)$. Direct sum decompositions of $C^*(G^f)^\mu$ and $W^*(G^f)$ result from the abundance of idempotents in $C^*(A)^\mu$. Indeed the Fourier transform leads to a σ-

isomorphism between the Boolean σ-algebra of idempotents in $C^*(A)^\mu$ and the σ-algebra of Borel sets in $\hat{A}$. Therefore every Borel set E in $\hat{A}$ defines a central projection in $C^*(G^f)^\mu$ and hence direct sum decompositions of $C^*(G^f)^\mu$ and $W^*(G^f)$. In particular the projections $\{r_\alpha : \alpha \in \hat{A}\}$ constructed in the compact case are those arising from the Borel sets in $\hat{A}$ consisting of single points.

The range $r(C^*(A)^\mu)$ of r is a commutative Baire *-algebra. Therefore the range $\Pi(r(C^*(A)^\mu))$ of the restriction of a σ-normal essential representation Π of $C^*(G^f)^\mu$ on separable Hilbert space is a commutative W^*-algebra (see **[12]**). Using this fact it is shown in §4 that every such representation possesses an essentially unique direct integral decomposition over $\hat{A}$. There exists a bijection between the set of such representations Π of $C^*(G^f)^\mu$ and the set of continuous unitary representations π of G^f on separable Hilbert spaces. The second main result (Theorem 4.3) shows that almost all the terms in the corresponding direct integral decomposition of π are of the form $(a, g) \to \alpha(a)\pi_\alpha(g)$ for some $\alpha \in \hat{A}$, where π_α is a projective representation of G with multiplier $\alpha \circ f$.

Finally in §5 certain results associated with the compactness of A are proved. In particular it is shown that $\Sigma_{\alpha \in \hat{A}} r_\alpha = 1$ if and only if A is compact.

Results related to those in this paper, but of a rather different nature have been obtained by Insel **[11]**.

2. Preliminaries. Throughout this paper G denotes a separable locally compact topological group with unit element e and m denotes a left invariant Haar measure on G. Let $M(G)$ denote the measure algebra of G, let δ_e denote its identity and let $L_1(G)$ denote the L_1-group algebra of G. For the definitions of these and related terms the reader is referred to **[10]**. $L_1(G)$ is isometrically *-isomorphic to the closed two-sided *-ideal $M_a(G)$ of elements of $M(G)$ absolutely continuous with respect to m, by means of the mapping $\eta \to m_\eta$ defined for $\eta \in L_1(G)$ by $dm_\eta = \eta dm$. Let $C^*(G)$ denote the C^*-envelope of $L_1(G)$, the C^*-group algebra of G, and let $W^*(G)$ denote the W^*-envelope of $C^*(G)$, the W^*-group algebra of G. For these definitions the reader is referred to **[4, 5, 17]**. $C^*(G)$ will be identified throughout with its universal representation and therefore will be regarded as a weak* dense subalgebra of $W^*(G)$. The measure algebra $M(G)$ will also be identified with a subalgebra of $W^*(G)$ **[18]**.

Let $C^*(G)^{h\mu}$ be the smallest subset of $W^*(G)$ containing the set $C^*(G)^h$ of self-adjoint elements of $C^*(G)$ and which contains the least upper bounds and greatest lower bounds of its uniformly bounded monotone sequences. Then $C^*(G)^{h\mu} + iC^*(G)^{h\mu}$ is a C^*-algebra, known as the *Baire* envelope* of $C^*(G)$ and denoted by $C^*(G)^\mu$. For details see **[14]**.

There exist bijections between the families of essential representations of $L_1(G)$, essential representations of $C^*(G)$, essential σ-normal representations of $C^*(G)^\mu$ and essential normal representations of $W^*(G)$, the bijections being defined by restricting a given essential normal representation of $W^*(G)$ to $L_1(G)$, $C^*(G)$ and $C^*(G)^\mu$ respectively. Moreover there exists a bijection $\pi \to \Pi$ from the set of continuous unitary representations of G onto the set of essential representations of $L_1(G)$ defined for $\eta \in L_1(G)$, $\xi_1, \xi_2 \in H_\pi$, the representation space of π, by

$$\langle \Pi(\eta)\xi_1, \xi_2\rangle = \int_G \eta(g)\langle \pi(g)\xi_1, \xi_2\rangle dm(g). \tag{2.1}$$

Each of these bijections maps primary and irreducible representations into primary and irreducible representations respectively and preserves unitary equivalence.

Let A be a separable locally compact abelian group with unit element 0, let n be an invariant Haar measure on A and let $\hat{A}$ be the dual group of A. $\hat{A}$ is discrete if and only if A is compact. The Fourier transform F on $L_1(A)$ is defined for $\phi \in L_1(A)$, $\alpha \in \hat{A}$ by

$$(F\phi)(\alpha) = \int_A \alpha(a)\phi(a)dn(a).$$

F extends to an isometric *-isomorphism from $C^*(A)$ onto $C_0(\hat{A})$, the algebra of continuous functions on $\hat{A}$ which take arbitrarily small values outside compact sets, equipped with the supremum norm **[16]**. F also extends uniquely to a σ-normal isometric *-isomorphism from $C^*(A)^\mu$ onto $F_{\mathfrak{B}}(\hat{A})$, the algebra of bounded Borel functions on $\hat{A}$ **[12]**. Both these extensions will be denoted by the same symbol F.

A Borel function f from $G \times G$ to A satisfying

$$f(g,e) = f(e,g) = 0 \qquad \forall\ g \in G,$$

$$f(g_1, g_2) + f(g_1 g_2, g_3) = f(g_1, g_2 g_3) + f(g_2, g_3) \qquad \forall\ g_1, g_2, g_3 \in G$$

is said to be a *Borel* 2*-cocycle* from G to A. In the special case $A = T$, the multiplicative group of complex numbers of unit modulus, a Borel 2-cocycle is said to be a *multiplier* on G. For each Borel 2-cocycle f from G to A and each $\alpha \in \hat{A}$, $\alpha \circ f$ is a multiplier on G.

To each multiplier ω on G there exists a 'twisted' convolution and involution on $L_1(G)$ with respect to which it forms a Banach *-algebra $L_1(G, \omega)$ with bounded approximate identity. $C^*(G, \omega)$, $C^*(G, \omega)^\mu$

and $W^*(G,\omega)$ respectively denote the C^*, Baire* and W^*-envelopes of $L_1(G,\omega)$. There exist bijections between the families of essential representations of $L_1(G,\omega)$, essential representations of $C^*(G,\omega)$, essential σ-normal representations of $C^*(G,\omega)^\mu$ and essential normal representations of $W^*(G,\omega)$. In this case (2.1) sets up a bijection between the set of essential representations of $L_1(G,\omega)$ acting on a separable Hilbert space and the set of projective representations of G with multiplier ω acting on a separable Hilbert space. Each of the bijections maps primary and irreducible representations into primary and irreducible representations respectively and preserves unitary equivalence. See **[7, 8, 9]** for details.

Let f be a Borel 2-cocycle from G to A and for (a_1,g_1), $(a_2,g_2)\in A\times G$, let

$$(a_1,g_1)(a_2,g_2)=(a_1+a_2+f(g_1,g_2),g_1g_2).$$

With this multiplication $A\times G$ is a group which possesses a separable locally compact topology, the Borel structure of which coincides with the product Borel structure and with respect to which $A\times G$ is a topological group. This group is said to be the *central group extension* of A by G corresponding to f and is denoted by G^f. The measure $n\times m$ is a left invariant Haar measure on G^f **[13]**.

If $\mathfrak{A}$ is a complex Banach algebra, the set $\Delta(\mathfrak{A})$ of bounded linear operators W on $\mathfrak{A}$ satisfying

$$W(\psi_1\psi_2)=(W\psi_1)\psi_2=\psi_1(W\psi_2) \qquad \forall\ \psi_1,\psi_2\in\mathfrak{A}$$

is said to be the *centralizer algebra* of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{A}^\mu$ be its Baire* envelope and let $\mathfrak{A}^{**}$ be its W^*-envelope. With $\mathfrak{A}$, $\mathfrak{A}^\mu$ regarded as being embedded in $\mathfrak{A}^{**}$, the *idealizer* $\mathfrak{M}(\mathfrak{A})$ of $\mathfrak{A}$ is the largest C^*-subalgebra of $\mathfrak{A}^{**}$ in which $\mathfrak{A}$ is an ideal. Let $\mathfrak{A}^m$ denote the set of self-adjoint elements of $\mathfrak{A}^{**}$ which can be reached by increasing nets from $\mathfrak{A}^\sim$ the C^*-subalgebra of $\mathfrak{A}^{**}$ obtained by adjoining the identity 1 of $\mathfrak{A}^{**}$ to $\mathfrak{A}$. If $\mathfrak{A}_m=-\mathfrak{A}^m$, then the self-adjoint part of $\mathfrak{M}(\mathfrak{A})$ equals $\mathfrak{A}^m\cap\mathfrak{A}_m$ (1). Further $\Delta(\mathfrak{A})$ is a commutative C^*-algebra with identity and the mapping $W\to W^{**}1$ is a *-isomorphism from $\Delta(\mathfrak{A})$ onto the centre $Z(\mathfrak{M}(\mathfrak{A}))$ of $\mathfrak{M}(\mathfrak{A})$. Moreover $Z(\mathfrak{M}(\mathfrak{A}))=\mathfrak{Z}(\mathfrak{A})$, the *ideal centre* of $\mathfrak{A}$ **[2, 3, 15]**. If $\mathfrak{A}$ is separable, $\mathfrak{A}^m\subseteq\mathfrak{A}^\mu$, $1\in\mathfrak{A}^\mu$ and hence $\mathfrak{M}(\mathfrak{A})\subseteq\mathfrak{A}^\mu$, $\mathfrak{Z}(\mathfrak{A})\subseteq Z(\mathfrak{A}^\mu)$, the centre of $\mathfrak{A}^\mu$.

Throughout the paper the multiplication and involution in $W^*(G^f)$ and, for $\alpha\in\hat{A}$, in $W^*(G,\alpha\circ f)$ are denoted by $\cdot$, * respectively.

3. The structure theorem. In this section the main theorem concerning the structure of the group algebras of G^f is proved. It is shown that $C^*(A)^\mu$ can be embedded in the centre of $C^*(G^f)^\mu$. Since $C^*(A)^\mu$ possesses many idempotents, this result leads to direct sum decompositions of $C^*(G^f)^\mu$ and $W^*(G^f)$. The conditions under which similar decompositions of $L_1(G^f)$ and $C^*(G^f)$ also exist are examined in §5.

The section begins with a statement of the main theorem and its corollaries.

THEOREM 3.1. *For* $\phi \in L_1(A)$ *define* $r(\phi) = n_\phi \times \delta_e$ *where* $n_\phi \in M(A)$ *is defined by* $dn_\phi = \phi dn$ *and* δ_e *is the identity in* $M(G)$. *If* $C^*(G^f)$ *and* $M(G^f)$ *are regarded as subalgebras of* $W^*(G^f)$, *then the mapping* $r: \phi \to r(\phi)$ *extends uniquely from* $L_1(A)$ *to a* σ*-normal* $*$*-isomorphism from* $C^*(A)^\mu$ *into the centre* $Z(C^*(G^f)^\mu)$ *of* $C^*(G^f)^\mu$.

The extension of r to $C^*(A)^\mu$ will also be denoted by r.

COROLLARY 3.2. *For* $E \in \mathfrak{B}(\hat{A})$, *the* σ*-algebra of Borel subsets of* $\hat{A}$, *define* $\tilde{r}(E) = r(F^{-1}\chi_E)$ *where* χ_E *is the characteristic function of* E, F^{-1} *is the inverse Fourier transform and* r *is defined above. Then* $\tilde{r}: E \to \tilde{r}(E)$ *is a* σ*-isomorphism from* $\mathfrak{B}(\hat{A})$ *into the Boolean* σ*-algebra of central projections in* $C^*(G^f)^\mu$.

COROLLARY 3.3 (i) *For each Borel subset* E *of* $\hat{A}$ *with complement* E^c *there exist monotone sequentially closed two-sided ideals* $\tilde{r}(E)\cdot C^*(G^f)^\mu$, $\tilde{r}(E^c)\cdot C^*(G^f)^\mu$ *in* $C^*(G^f)^\mu$ *such that* $C^*(G^f)^\mu = (\tilde{r}(E)\cdot C^*(G^f)^\mu) \oplus (\tilde{r}(E^c)\cdot C^*(G^f)^\mu)$.

(ii) *For each Borel subset* E *of* $\hat{A}$ *with complement* E^c *there exist weak* closed two-sided ideals* $\tilde{r}(E)\cdot W^*(G^f)$, $\tilde{r}(E^c)\cdot W^*(G^f)$ *in* $W^*(G^f)$ *such that* $W^*(G^f) = (\tilde{r}(E)\cdot W^*(G^f)) \oplus (\tilde{r}(E^c)\cdot W^*(G^f))$.

(iii) *The algebraic direct sums*

$$\bigoplus_{\alpha\in\hat{A}} (\tilde{r}(\{\alpha\})\cdot C^*(G^f)^\mu), \qquad \bigoplus_{\alpha\in\hat{A}} (\tilde{r}(\{\alpha\})\cdot W^*(G^f))$$

are two-sided ideals in $C^*(G^f)^\mu$, $W^*(G^f)$ *respectively.*

The proof of Theorem 3.1 depends upon several results, some of which are of independent interest.

PROPOSITION 3.4. *For* $\mu \in M(A)$ *let* $R(\mu)$ *be the linear operator on* $L_1(G^f)$ *defined by*

$$R(\mu)\Psi = (\mu \times \delta_e)\cdot\Psi \qquad \forall\ \Psi \in L_1(G^f).$$

Then the mapping $R: \mu \to R(\mu)$ *is an isometric *-isomorphism from* $M(A)$ *into* $\Delta(L_1(G^f))$.

Proof. The mapping $\mu \to \mu \times \delta_e$ is an isometric *-isomorphism from $M(A)$ into the centre $Z(M(G^f))$ of $M(G^f)$. But, by Theorem 6.1 of **[9]**, there exists an isometric *-isomorphism $X \to W_X$ from $Z(M(G^f))$ onto $\Delta(L_1(G^f))$ defined by $W_X\Psi = X\cdot\Psi$, $\forall\ \Psi \in L_1(G^f)$.

COROLLARY 3.5 *For* $\phi \in L_1(A)$ *let* $R(\phi)$ *be the linear operator on* $L_1(G^f)$ *defined by*

$$R(\phi)\Psi = (n_\phi \times \delta_e)\cdot\Psi \qquad \forall\ \Psi \in L_1(G^f).$$

Then the mapping $R: \phi \to R(\phi)$ *is an isometric *-isomorphism from* $L_1(A)$ *into* $\Delta(L_1(G^f))$.

Proof. This follows immediately from Proposition 3.4 by regarding $L_1(A)$ as an ideal in $M(A)$.

LEMMA 3.6 *For* $\phi \in L_1(A)$ *let* $R(\phi) \in \Delta(L_1(G^f))$ *be defined as above. Then* $R(\phi)$ *extends uniquely to an element, also denoted by* $R(\phi)$, *of* $\Delta(C^*(G^f))$ *such that, when* $C^*(G^f)$ *and* $M(G^f)$ *are regarded as subalgebras of* $W^*(G^f)$,

$$R(\phi)^{**}\Psi = (n_\phi \times \delta_e)\cdot\Psi \qquad \forall\ \Psi \in W^*(G^f).$$

Proof. Let π be an irreducible representation of G^f on the Hilbert space H and let Π be the representation of $M(G^f)$ defined for $X \in M(G^f)$ by

$$\langle \Pi(X)\xi_1, \xi_2\rangle = \int_{G^f} \langle \pi(a,g)\xi_1, \xi_2\rangle dX(a,g) \qquad \forall\ \xi_1, \xi_2 \in H. \tag{3.0}$$

The irreducibility of π implies that there exists $\alpha \in \hat{A}$ such that $\pi(a,e) = \alpha(a)1_H\ \forall\, a \in A$. Therefore, by (3.0)

$$\langle (n_\phi \times \delta_e)\xi_1, \xi_2\rangle = (F\phi)(\alpha)\langle \xi_1, \xi_2\rangle \qquad \forall\ \xi_1, \xi_2 \in H$$

from which it follows that

$$\|\Pi(n_\phi \times \delta_c)\| = |(F\phi)(\alpha)|. \tag{3.1}$$

Therefore, for $\Psi \in L_1(G^f)$,

$$\begin{aligned}\|\Pi(R(\phi)\Psi)\| &\leqq \|\Pi(n_\phi \times \delta_e)\| \ \|\Pi(\Psi)\| \\ &= |(F\phi)(\alpha)| \ \|\Pi(\Psi)\| \\ &\leqq \|\phi\|_{C^*(A)} \|\Psi\|_{C^*(G^f)}\end{aligned}$$

since F is an isometry from $C^*(A)$ onto $C_0(\hat{A})$. By taking the supremum over all irreducible representations Π of $L_1(G^f)$, it follows that

$$\|R(\phi)\Psi\|_{C^*(G^f)} \leqq \|\phi\|_{C^*(A)} \|\Psi\|_{C^*(G^f)}. \tag{3.2}$$

Therefore $R(\phi)$ extends uniquely to a bounded linear operator, denoted by the same symbol, on $C^*(G^f)$ such that $\|R(\phi)\| \leqq \|\phi\|_{C^*(A)}$. Simple limit arguments show that $R(\phi) \in \Delta(C^*(G^f))$.

The double adjoint $R(\phi)^{**}$ of $R(\phi)$ acting on $W^*(G^f)$ is the unique weak* continuous extension of $R(\phi)$ from $L_1(G^f)$ to $W^*(G^f)$. However, by 1.7.8 of **[17]**, the multiplication in $W^*(G^f)$ is weak *-continuous and so the mapping $\Psi \to (n_\phi \times \delta_e) \cdot \Psi$ is also a weak *-continuous extension of $R(\phi)$ to $W^*(G^f)$. It follows that $R(\phi)^{**}\Psi = (n_\phi \times \delta_e) \cdot \Psi$, $\forall \Psi \in W^*(G^f)$.

LEMMA 3.7 *The mapping $R: \phi \to R(\phi)$ from $L_1(A)$ into $\Delta(C^*(G^f))$ defined in Lemma* 3.6 *possesses a unique extension to an isometric *-isomorphism from $C^*(A)$ into $\Delta(C^*(G^f))$.*

Proof. (3.2) shows that R possesses a unique extension to a norm nonincreasing mapping from $C^*(A)$ into $\Delta(C^*(G^f))$. Simple limit arguments show that the extension, also denoted by R, is a *-homomorphism. For $\phi \in L_1(A)$,

$$\begin{aligned}\|\phi\|_{C^*(A)} &= \sup\{|(F\phi)(\alpha)|: \ \alpha \in \hat{A}\} \\ &= \sup\{\|\Pi(n_\phi \times \delta_e)\|: \ \Pi \in \mathrm{Irr}(G^f)\}\end{aligned}$$

by (3.1), where $\mathrm{Irr}(G^f)$ denotes the set of irreducible normal representations of $W^*(G^f)$,

$$\leqq \|n_\phi \times \delta_e\|_{W^*(G^f)} = \|R(\phi)^{**}1\|_{W^*(G^f)}$$

by Lemma 3.6,

$$\leqq \|R(\phi)^{**}\| = \|R(\phi)\|.$$

Hence R is isometric on $L_1(A)$.

Let $\phi' \in C^*(A)$ satisfy $R(\phi')=0$ and let $\{\phi_\lambda\}$ be a net in $L_1(A)$ such that, relative to the C^*-norm, $\lim \phi_\lambda = \phi'$. Then, from above,

$$\|\phi_\lambda\|_{C^*(A)} = \|R(\phi_\lambda)\| = \|R(\phi_\lambda - \phi')\| \leqq \|\phi_\lambda - \phi'\|_{C^*(A)} \to 0.$$

It follows that $\phi'=0$ and hence that R is a *-isomorphism from the C^*-algebra $C^*(A)$ into the C^*-algebra $\Delta(C^*(G^f))$. Therefore, using 1.8.1 of **[4]**, R is an isometry from $C^*(A)$ into $\Delta(C^*(G^f))$.

LEMMA 3.8. *For $\alpha \in \hat{A}$, $\Psi \in L_1(G^f)$, let*

$$(P_\alpha \Psi)(g) = \int_A \alpha(a)\Psi(a,g)dn(a) \qquad \forall\ g \in G.$$

*Then P_α is a norm nonincreasing *-homomorphism from $L_1(G^f)$ onto $L_1(G, \alpha \circ f)$ and P_α possesses a unique extension to a *-homomorphism from $C^*(G^f)$ onto $C^*(G, \alpha \circ f)$.*

Proof. The calculations used in **[9]** to show, for the case of compact A, that P_α is a norm nonincreasing *-homomorphism from $L_1(G^f)$ into $L_1(G, \alpha \circ f)$ also apply here. To show that P_α has range $L_1(G, \alpha \circ f)$, let $\psi \in L_1(G)$, $\phi \in L_1(A)$ with

$$\int_A \phi(a)dn(a) = 1.$$

The function Ψ defined for $(a,g) \in G^f$ by

$$\Psi(a,g) = \overline{\alpha(a)}\phi(a)\psi(g)$$

is an element of $L_1(G^f)$ such that $P_\alpha \Psi = \psi$.

The calculations used in **[8]** to show that, for the case of compact A, P_α extends uniquely to a *-homomorphism, also denoted by P_α, from $C^*(G^f)$ into $C^*(G, \alpha \circ f)$ also apply here. However, $P_\alpha C^*(G^f)$ is closed in $C^*(G, \alpha \circ f)$ (see 1.8.3 of **[4]**) and contains $L_1(G, \alpha \circ f)$. It follows that $P_\alpha C^*(G^f) = C^*(G, \alpha \circ f)$.

Proof of Theorem 3.1. It follows from Lemma 3.7 and the remarks at the end of §2 that the mapping $\phi \to R(\phi)^{**}1$ is an isometric *-isomorphism from $C^*(A)$ into $Z(C^*(G^f)^\mu)$. Further, Lemma 3.6 shows that for $\phi \in L_1(A)$

$$R(\phi)^{**}1 = n_\phi \times \delta_e = r(\phi).$$

Since $L_1(A)$ is dense in $C^*(A)$, the mapping $\phi \to R(\phi)^{**}1$ is the unique extension of r to $C^*(A)$ and will be denoted by the same symbol r.

Since $W^*(G^f)$ can be regarded as an algebra of operators on the universal representation space of $C^*(G^f)$, r can be regarded as a faithful representation of $C^*(A)$ and therefore possesses a unique extension to a σ-normal representation (also denoted by r) of $C^*(A)^\mu$. It remains to show that this extension is faithful and that its range lies inside $Z(C^*(G^f)^\mu)$.

Recall that the Fourier transform F on $L_1(A)$ possesses a unique extension to a σ-normal *-isomorphism (denoted by the same symbol) from $C^*(A)^\mu$ onto the algebra $F_{\mathfrak{B}}(\hat{A})$ of bounded Borel functions on $\hat{A}$. For $E \in \mathfrak{B}(\hat{A})$, the σ-algebra of Borel subsets of $\hat{A}$, let

$$\tilde{r}(E) = r(F^{-1}\chi_E) \tag{3.3}$$

where χ_E is the characteristic function of E. Since both r and F^{-1} are σ-normal it follows that $\tilde{r}$ is a σ-homomorphism into the complete Boolean algebra of central projections in $W^*(G^f)$. It will first be shown that $\tilde{r}$ is a σ-isomorphism. To this end let $E \in \mathfrak{B}(\hat{A})$ and let (ϕ_λ) be a net in $L_1(A)$ converging to $F^{-1}\chi_E$ in the weak* topology of $W^*(A)$. Then χ_E is the pointwise limit on $\hat{A}$ of the net $(F\phi_\lambda)$. For $\alpha \in \hat{A}$, $\Psi \in L_1(G^f)$, $g \in G$,

$$(P_\alpha(r(\phi_\lambda)\cdot\Psi))(g) = (P_\alpha R(\phi_\lambda)\Psi)(g) = (F\phi_\lambda)(\alpha)(P_\alpha\Psi)(g). \tag{3.4}$$

Notice that r possesses a unique extension to a weak* continuous *-homomorphism (denoted by the same symbol) from $W^*(A)$ into $Z(W^*(G^f))$. Using this fact, the weak* continuity of P_α^{**} and the weak* continuity of multiplication in $W^*(G^f)$, it follows from (3.4) that, for $k \in C^*(G, \alpha\circ f)^*$,

$$\begin{aligned}\langle P_\alpha^{**}(\tilde{r}(E)\cdot\Psi), k\rangle &= \lim\langle P_\alpha^{**}(r(\phi_\lambda)\cdot\Psi), k\rangle = \lim(F\phi_\lambda)(\alpha)\langle P_\alpha\Psi, k\rangle \\ &= \chi_E(\alpha)\langle P_\alpha\Psi, k\rangle.\end{aligned} \tag{3.5}$$

Now suppose that $E_1, E_2 \in \mathfrak{B}(\hat{A})$ satisfy $\tilde{r}(E_1) = \tilde{r}(E_2)$. Let $\alpha \in E_1$, $\alpha \notin E_2$. Then, from (3.5), for $\Psi \in L_1(G^f)$,

$$P_\alpha\Psi = P_\alpha^{**}(\tilde{r}(E_1)\cdot\Psi) = P_\alpha^{**}(\tilde{r}(E_2)\cdot\Psi) = 0$$

and since, by Lemma 3.8, P_α maps $L_1(G^f)$ onto $L_1(G, \alpha\circ f)$ this yields a contradiction. Hence $E_1 \subseteq E_2$ and similarly $E_2 \subseteq E_1$. Thus $E_1 = E_2$

and $\tilde{r}$ is a σ-isomorphism.

To show that r is an isomorphism suppose that $\phi \in C^*(A)^\mu$, $0 \leqq \phi \leqq 1$, $r(\phi) = 0$. Then $\psi = F\phi \in F_{\mathfrak{B}}(\hat{A})$, $0 \leqq \psi \leqq 1$ and the sequence $(1-(1-\psi)^n)$ is monotone increasing with least upper bound $\chi_{E'}$ where $E' = \{\alpha : \alpha \in \hat{A}, \psi(\alpha) > 0\}$. By the σ-normality of r and F^{-1} it follows that $\tilde{r}(E') = 0$ and therefore, from above, that $E' = \varnothing$. Hence $\psi = 0$ and, since F is an isomorphism, $\phi = 0$. Suppose next that $\phi \in C^*(A)^{\mu h}$, $\|\phi\| \leqq 1$, $r(\phi) = 0$. Then $\psi = F\phi \in F^r_{\mathfrak{B}}(\hat{A})$, the algebra of bounded real-valued Borel functions on $\hat{A}$, $\|\psi\| \leqq 1$ and

$$|\psi| = (\psi^2)^{\frac{1}{2}} = \sup\left\{1 - \sum_{r=1}^{n} \frac{(2r-3)(2r-1)\cdots 3\cdot 1}{(2r)(2r-2)\cdots 4\cdot 2}(1-\psi^2)^r\right\}.$$

By the σ-normality of r and F^{-1} it follows that $r(F^{-1}(|\psi|)) = 0$ and, as above, that $|\psi| = 0$, $\psi = 0$, $\phi = 0$. If ϕ is an arbitrary element of $C^*(A)^\mu$ such that $r(\phi) = 0$, applying the above result to its real and imaginary part proves that $\phi = 0$. Therefore r is an isomorphism.

It remains to show that $r(C^*(A)^\mu) \subseteq Z(C^*(G^f)^\mu)$. To this end let

$$L = \{\phi : \phi \in C^*(A)^\mu, r(\phi) \in Z(C^*(G^f)^\mu)\}.$$

Let $(\phi_n) \subset L$ be a uniformly bounded monotone increasing sequence with least upper ϕ. Then, by the σ-normality of r, $(r(\phi_n)) \subset Z(C^*(G^f)^\mu)$ is a uniformly bounded monotone increasing sequence with least upper bound $r(\phi)$. But $Z(C^*(G^f)^\mu)$ is monotone sequentially closed and hence $r(\phi) \in Z(C^*(G^f)^\mu)$, $\phi \in L$. Therefore L is monotone sequentially closed and contains $C^*(A)$. Hence $C^*(A)^\mu = L$ and the proof is complete.

Notice that Corollary 3.2 was proved in the course of the above proof. Corollary 3.3 is an immediate consequence of the fact that $\{\tilde{r}(E) : E \in \mathfrak{B}(\hat{A})\}$ is a Boolean σ-algebra of projections in $Z(C^*(G^f)^\mu)$.

4. Representations. Let $\mathrm{Rep}(G^f)$ and $\mathrm{Rep}(G, \alpha \circ f)$, $\alpha \in \hat{A}$ respectively denote the sets of essential representations of $L_1(G^f)$ and $L_1(G, \alpha \circ f)$ on separable Hilbert spaces; let $\mathrm{Fac}(G^f)$ and $\mathrm{Fac}(G, \alpha \circ f)$ respectively denote the subsets of $\mathrm{Rep}(G^f)$ and $\mathrm{Rep}(G, \alpha \circ f)$ consisting of primary representations; let $\mathrm{Irr}(G^f)$ and $\mathrm{Irr}(G, \alpha \circ f)$ respectively denote the subsets of $\mathrm{Fac}(G^f)$ and $\mathrm{Fac}(G, \alpha \circ f)$ consisting of irreducible representations.

If $\Pi_\alpha \in \mathrm{Rep}(G, \alpha \circ f)$, then the mapping $\Psi \to \Pi_\alpha(P_\alpha \Psi)$, where P_α is defined in Lemma 3.8, on $L_1(G^f)$ is an element of $\mathrm{Rep}(G^f)$. The corresponding continuous unitary representation of G^f is $(a, g) \to \alpha(a)\pi_\alpha(g)$, where π_α is the projective representation of G

corresponding to Π_α under (2.1). In the sequel the essential representation $\Psi \to \Pi_\alpha(P_\alpha \Psi)$ of $L_1(G^f)$ is denoted by $\alpha \otimes \Pi_\alpha$ and the corresponding continuous unitary representation of G^f by $\alpha \otimes \pi_\alpha$. Let $\mathrm{Rep}(G^f, \alpha)$, $\mathrm{Fac}(G^f, \alpha)$ and $\mathrm{Irr}(G^f, \alpha)$ respectively denote the images of $\mathrm{Rep}(G, \alpha \circ f)$, $\mathrm{Fac}(G, \alpha \circ f)$ and $\mathrm{Irr}(G, \alpha \circ f)$ under the bijection $\Pi_\alpha \to \alpha \otimes \Pi_\alpha$.

In **[7]** it is shown how, for compact A, every element of $\mathrm{Rep}(G^f)$ can be written as a direct sum of elements of the family $\{\mathrm{Rep}(G^f, \alpha)\colon \alpha \in \hat{A}\}$. The generalization relies on the theory of direct integrals, for details of which the reader is referred to **[4, 5]**. Throughout this section the commutative Baire* algebra $r(C^*(A)^\mu)$ will be denoted by Z.

LEMMA 4.1. *Let $\Pi \in \mathrm{Rep}(G^f)$, let π be the corresponding continuous unitary representation of G^f and let π_e be the continuous unitary representation $a \to \pi(a, e)$ of A. Then $\Pi(Z) = \pi_e(A)''$, the Von Neumann algebra generated by $\pi_e(A)$.*

Proof. Let Π_e be the element of $\mathrm{Rep}(A)$ associated with π_e and recall that $\Pi_e(C^*(A)^\mu) = \Pi_e(A)''$ (see **[4]**, 13.3.5, **[12]**, p. 322). A simple calculation shows that for $\phi \in L_1(A)$, $\Pi_e(\phi) = \Pi(r(\phi))$ and hence $\Pi_e = \Pi \circ r$. This completes the proof of the lemma.

The first preliminary result concerning the structure of $\mathrm{Rep}(G^f)$ is the following.

PROPOSITION 4.2. (i) *For $\Pi \in \mathrm{Rep}(G^f)$, $\Pi \in \mathrm{Rep}(G^f, \alpha)$ for some $\alpha \in \hat{A}$ if and only if $\Pi(Z) = \mathbf{C}1_H$ where 1_H is the identity operator on the representation space H of Π.*

(ii) *If $\alpha \neq \beta$ then* $\mathrm{Rep}(G^f, \alpha) \cap \mathrm{Rep}(G^f, \beta) = \varnothing$.

(iii) $\mathrm{Fac}(G^f) = \bigcup_{\alpha \in \hat{A}} \mathrm{Fac}(G^f, \alpha)$.

(iv) $\mathrm{Irr}(G^f) = \bigcup_{\alpha \in \hat{A}} \mathrm{Irr}(G^f, \alpha)$.

Proof. (i) Lemma 4.1 shows that $\Pi(Z)$ is trivial if and only if for all $a \in A$, $\pi_e(a) = \alpha(a)1_H$ for some $\alpha \in \hat{A}$. It follows that $\Pi(Z)$ is trivial if and only if $\pi = \alpha \otimes \pi_\alpha$ for some projective representation π_α of G with multiplier $\alpha \circ f$ or equivalently if and only if $\Pi \in \mathrm{Rep}(G^f, \alpha)$ for some $\alpha \in \hat{A}$.

(ii) If $\Pi \in \mathrm{Rep}(G^f, \alpha) \cap \mathrm{Rep}(G^f, \beta)$ and if π is the corresponding continuous unitary representation of G^f then, for $a \in A$, $\alpha(a)1_H = \pi(a, e) = \beta(a)1_H$ and so $\alpha = \beta$.

(iii) If $\Pi \in \mathrm{Fac}(G^f)$ then $\Pi(Z) \subseteq \Pi(Z(W^*(G^f))) = \mathbf{C}1_H$ and hence, by (i), $\Pi \in \mathrm{Rep}(G^f, \alpha)$ for some $\alpha \in \hat{A}$. Therefore $\Pi = \alpha \otimes \Pi_\alpha$ for some $\Pi_\alpha \in \mathrm{Rep}(G, \alpha \circ f)$ and, since Π is primary, it follows that Π_α is also primary. It follows that $\mathrm{Fac}(G^f) \subseteq \cup_{\alpha \in \hat{A}} \mathrm{Fac}(G^f, \alpha)$ and the reverse inclusion is trivial.

(iv) The proof is similar to that of (iii).

The main result about the structure of $\mathrm{Rep}(G^f)$ is the following.

THEOREM 4.3. *For $\Pi \in \mathrm{Rep}(G^f)$ there exists a positive measure $\mu \in M(\hat{A})$, unique up to measure class, and a family $\{\Pi^\alpha : \alpha \in \hat{A}\}$, where $\Pi^\alpha \in \mathrm{Rep}(G^f, \alpha)$ for μ-almost all $\alpha \in \hat{A}$, such that Π is unitarily equivalent to $\int_{\hat{A}}^{\oplus} \Pi^\alpha d\mu(\alpha)$.*

Proof. $\Pi \circ r \circ F^{-1}$ is a σ-normal representation of $F_{\mathfrak{B}}(\hat{A})$ with range $\Pi(Z)$ which is a Von Neumann algebra since the representation space is separable. Moreover it is the unique σ-normal extension of its restriction to $C_0(\hat{A})$. By standard representation theory for $C_0(\hat{A})$ there exists a positive measure $\mu \in M(\hat{A})$, unique up to measure class, such that $\Pi(Z)$ is *-isomorphic to $L_\infty(\hat{A}, \mu)$. Using **[4]**, 8.2.2, 8.3.2, **[5]** App. IV, there exists a family $\{\Pi^\alpha : \alpha \in \hat{A}, \Pi^\alpha \in \mathrm{Rep}(G^f)\}$ such that Π is unitarily equivalent to $\int_{\hat{A}}^{\oplus} \Pi^\alpha d\mu(\alpha)$ and $\Pi(Z)$ is isomorphic to the algebra of diagonalizable operators. It remains to prove that $\Pi^\alpha \in \mathrm{Rep}(G^f, \alpha)$ for μ-almost all $\alpha \in \hat{A}$. It follows from Lemma 4.1 and Proposition 4.2 that this is achieved once it has been proved that, if π^α is the continuous unitary representation of G^f corresponding to Π^α, then the continuous unitary representation $(\pi^\alpha)_e$ of A is primary for μ-almost all $\alpha \in \hat{A}$. If $\pi' = \int_{\hat{A}}^{\oplus} \pi^\alpha d\mu(\alpha)$, then by 18.7.4 of **[4]**, π' is unitarily equivalent to the representation π of G^f associated with Π. But, by Lemma 4.1, $\Pi(Z) = \pi_e(A)''$ and therefore the decomposition $\pi_e = \int_{\hat{A}}^{\oplus} (\pi^\alpha)_e d\mu(\alpha)$ is the central decomposition of π_e. Using 8.4.1 of **[4]**, it follows that $(\pi^\alpha)_e$ is primary for μ-almost all $\alpha \in \hat{A}$.

REMARK. Let $K = \{k : k \in C^*(G^f)^*,\ k \geqq 0,\ \|k\| \leqq 1\}$, let $k \in K$ and let Π_k be the cyclic representation of $C^*(G^f)$ on H_k associated with k (see **[4]**, 2.4.4.). Then, according to **[17]**, §3.1, a decomposition of Π_k over K corresponding to $\Pi_k(Z)$ can be obtained by means of a unique positive Radon measure ν_k. Theorem 4.3 also defines a decomposition of Π_k corresponding to $\Pi_k(Z)$, given by the measure μ_k on $\hat{A}$. An application of the uniqueness theorem (see **[4]**, 8.2.4) then establishes the existence of a Borel isomorphism from $\hat{A} \setminus E$, for some Borel set E satisfying $\mu_k(E) = 0$, into K which transforms μ_k into ν_k. From

Theorem 4.3, the images under this isomorphism of μ_k-almost all of the points of $\hat{A} \setminus E$ lie in the set $\partial^Z_{pr}(K) = \{k : k \in K, \Pi_k(Z) = \mathbf{C}1_{H_k}\}$, the set of Z-primary points of K. A corollary of Theorem 4.3 is therefore that the measure ν_k on K is pseudo-concentrated on $\partial^Z_{pr}(K)$. Further discussion of this and related topics is not within the scope of this paper (cf. **[17]**, §3.1).

5. The compact case. In this section the following two criteria which exhibit the compactness of A are proved.

THEOREM 5.1. *If the family* $\{\tilde{r}(\{\alpha\}) : \alpha \in \hat{A}\}$ *of mutually orthogonal central projections in* $C^*(G^f)^\mu$ *is defined by* (3.3), *then* $\Sigma_{\alpha \in \hat{A}} \tilde{r}(\{\alpha\}) = 1$ *if and only if A is compact.*

THEOREM 5.2. *If the family* $\{\tilde{r}(\{\alpha\}) : \alpha \in \hat{A}\}$ *of mutually orthogonal central projections in* $C^*(G^f)^\mu$ *is defined by* (3.3), *then*

(i) $\tilde{r}(\{\alpha\}) \cdot L_1(G^f) \subseteq L_1(G^f)$ *for some* $\alpha \in \hat{A}$ *if and only if A is compact*

and

(ii) $\tilde{r}(\{\alpha\}) \cdot C^*(G^f) \subseteq C^*(G^f)$ *for some* $\alpha \in \hat{A}$ *if and only if A is compact.*

If A is compact, the mapping Q_α defined for $\alpha \in \hat{A}$, $\eta \in L_1(G, \alpha \circ f)$ by

$$(5.1) \qquad (Q_\alpha \eta)(a, g) = \overline{\alpha(a)}\eta(g) \qquad \forall\ (a, g) \in G^f$$

is an isometric *-isomorphism onto a norm closed two-sided *-ideal $L_1(G^f, \alpha)$ in $L_1(G^f)$ **[9]**. Further, $P_\alpha Q_\alpha = 1$, the identity operator on $L_1(G, \alpha \circ f)$ and, if $R_\alpha = Q_\alpha P_\alpha$, the family $\{R_\alpha : \alpha \in \hat{A}\}$ of projections in $\Delta(L_1(G^f))$ satisfies $R_\alpha R_\beta = \delta_{\alpha\beta} R_\alpha$. A simple calculation shows that, since $\hat{A} \subset L_1(A)$, for $\Psi \in L_1(G^f)$

$$(5.2) \qquad R_\alpha \Psi = R(\bar{\alpha})\Psi = \tilde{r}(\{\alpha\}) \cdot \Psi$$

using the notation of §3.

The map Q_α defined by (5.1) extends uniquely to a *-homomorphism Q_α from $C^*(G, \alpha \circ f)$ onto a norm closed two-sided *-ideal $C^*(G^f, \alpha)$ in $C^*(G^f)$. Further, if P_α is extended, as in Lemma 3.8, to a *-homomorphism P_α from $C^*(G^f)$ onto $C^*(G, \alpha \circ f)$, then $P_\alpha Q_\alpha = 1$ the identity operator on $C^*(G, \alpha \circ f)$ and $R_\alpha = Q_\alpha P_\alpha$ is a projection onto $C^*(G^f, \alpha)$ **[8]**. By means of simple limit arguments it can be deduced from (5.2) that, if the extension of $R(\bar{\alpha})$ to an element of

$\Delta(C^*(G^f))$ is denoted by the same symbol, then, for $\alpha \in \hat{A}$, $\Psi \in C^*(G^f)$,

$$R_\alpha \Psi = R(\bar{\alpha})\Psi = \tilde{r}(\{\alpha\}) \cdot \Psi. \tag{5.3}$$

LEMMA 5.3. *If A is compact then $\bigoplus_{\alpha \in \hat{A}} \tilde{r}(\{\alpha\}) \cdot W^*(G^f)$ is weak* dense in $W^*(G^f)$.*

Proof. It is shown in Theorem 5.5 of **[9]** that $\bigoplus_{\alpha \in \hat{A}} R_\alpha L_1(G^f)$ is norm dense in $L_1(G^f)$ and hence weak* dense in $W^*(G^f)$. However, by (5.2), $\bigoplus_{\alpha \in \hat{A}} R_\alpha L_1(G^f) = L_1(G^f) \cap (\bigoplus_{\alpha \in \hat{A}} \tilde{r}(\{\alpha\}) \cdot W^*(G^f))$, from which the result follows.

Proof of Theorem 5.1. Let $\Sigma_{\alpha \in \hat{A}} \tilde{r}(\{\alpha\})$, defined to be the least upper bound in $W^*(G^f)$ of the family $\{\Sigma_{\alpha \in \Lambda} \tilde{r}(\{\alpha\}): \Lambda \subseteq \hat{A}, \Lambda \text{ finite}\}$ be denoted by u. If A is compact then, by Lemma 5.3, there exists a net (Ψ_λ) of elements of $\bigoplus_{\alpha \in \hat{A}} \tilde{r}(\{\alpha\}) \cdot W^*(G^f)$ with weak* limit 1. The weak* continuity of multiplication in $W^*(G^f)$ then implies that $(1-u) \cdot \Psi_\lambda \to 1-u$. However, $(1-u)\cdot\Psi_\lambda = 0 \ \forall\lambda$ and thus $u = 1$.

Conversely, assume that $u = 1$ and let μ be a positive normalised regular Borel measure on $\hat{A}$. Let $H = L_2(\hat{A}, L_2(G), \mu)$ and for $(a, g) \in G^f$, $\xi \in H$, $h \in G$, $\alpha \in \hat{A}$, let

$$(\pi(a,g)\xi)_\alpha(h) = \alpha(a)(\alpha \circ f)(g, g^{-1}h)\xi_\alpha(g^{-1}h). \tag{5.4}$$

Then π is easily seen to be a continuous unitary representation of G^f. If Π is the corresponding element of $\operatorname{Rep}(G^f)$ a simple calculation shows that for $\Psi \in L_1(G^f)$, $\xi \in H$, $\alpha \in \hat{A}$,

$$(\Pi(\Psi)\xi)_\alpha = L_\alpha(P_\alpha \Psi)\xi_\alpha$$

where L_α is the left regular representation of $L_1(G, \alpha \circ f)$ defined for $\eta \in L_1(G, \alpha \circ f)$, $\eta' \in L_2(G)$ by

$$L_\alpha(\eta)\eta' = \eta \cdot \eta'.$$

Since Π possesses a unique normal extension to $W^*(G^f)$ and since, for each $\alpha \in \hat{A}$, L_α possesses a unique normal extension to $W^*(G, \alpha \circ f)$ it follows that for $\Psi \in W^*(G^f)$, $\xi \in H$, $\alpha \in \hat{A}$,

$$(\Pi(\Psi)\xi)_\alpha = L_\alpha(P_\alpha^{**}\Psi)\xi_\alpha.$$

Using (5.4) and Lemma 4.1 it is clear that $\Pi(Z)$ is *-isomorphic to $L_\infty(\hat{A}, \mu)$ and therefore μ is the measure on $\hat{A}$ corresponding to Π through Theorem 4.3. For $\alpha \in \hat{A}$ define $\Pi^\alpha \in \operatorname{Rep}(G^f)$ on $\Pi(\tilde{r}(\{\alpha\}))H = H_\alpha$ for $\Psi \in W^*(G^f)$ by $\Pi^\alpha(\Psi) = \Pi(\tilde{r}(\{\alpha\}) \cdot \Psi)$ and notice that the hypothesis $u = 1$ leads to

(5.5) $$\Pi = \bigoplus_{\alpha \in \hat{A}} \Pi^\alpha.$$

But, for $\alpha \in \hat{A}$,

$$\Pi^\alpha(Z) = \Pi^\alpha(r(F^{-1}\chi_{\{\alpha\}}) \cdot Z) = \Pi^\alpha(rF^{-1}(\chi_{\{\alpha\}} \cdot F_{\mathfrak{P}}(\hat{A})))$$
$$= \{\lambda\, \Pi^\alpha(\tilde{r}(\{\alpha\})) \colon \lambda \in \mathbf{C}\} = \mathbf{C}1_{H_\alpha}.$$

It follows from Proposition 4.2 and (3.5) that for each $\alpha \in \hat{A}$, $\Pi^\alpha \in \operatorname{Rep}(G^f, \alpha)$. Therefore (5.5) describes a decomposition of Π into a direct sum over $\hat{A}$ of elements of $\operatorname{Rep}(G^f, \alpha)$. Theorem 4.3 shows that μ is discrete. Hence $\hat{A}$ is discrete and A is compact.

Proof of Theorem 5.2. (i) If A is compact it follows immediately from (5.2) that

$$\tilde{r}(\{\alpha\}) \cdot L_1(G^f) = R_\alpha L_1(G^f) \subseteq L_1(G^f) \qquad \forall\ \alpha \in \hat{A}.$$

Conversely, assume that A is noncompact and thus that $\hat{A}$ is nondiscrete. It will be shown that

$$L_1(G^f) \cap (\tilde{r}(\{\alpha\}) \cdot L_1(G^f)) = \{0\} \qquad \forall\ \alpha \in \hat{A}$$

which, because of (3.5), is a stronger result than that to be proved. For some $\alpha \in \hat{A}$, let $\Psi \in L_1(G^f))$ and define the mapping d_Ψ on $\hat{A}$ by $d_\Psi(\beta) = P_\beta \Psi\ \forall\ \beta \in \hat{A}$. It follows from (3.5) that either $P_\beta \Psi = 0$ $\forall \beta \in \hat{A}$ or $d_\psi^{-1}(0) = \hat{A} \setminus \{\alpha\}$. However, by Proposition 2.4 of **[11]**, d_Ψ is continuous and thus, if $d_\psi^{-1}(0) = \hat{A} \setminus \{\alpha\}$, $\{\alpha\}$ is open. By 15.8 and 15.17(b) of **[10]** this implies that $\hat{A}$ is discrete, contradicting the assumption that A is noncompact. Hence $P_\beta \Psi = 0\, \forall \beta \in \hat{A}$ and, by the injective property of the Fourier transform, $\Psi = 0$.

(ii) If A is compact it follows immediately from (5.3) that

$$\bar{r}(\{\alpha\}) \cdot C^*(G^f) = R_\alpha C^*(G^f) \subseteq C^*(G^f) \qquad \forall \alpha \in \hat{A}.$$

Conversely, assume that $\bar{r}(\{\alpha\}) \cdot C^*(G^f) \subseteq C^*(G^f)$ for some $\alpha \in \hat{A}$ and choose $\Psi \in C^*(G^f)$ such that $P_\alpha \Psi \neq 0$. It follows from (3.5) that for $\beta \in \hat{A}$, $P_\beta(\bar{r}(\{\alpha\}) \cdot \Psi) = \delta_{\alpha\beta} P_\alpha \Psi$ and so, as in the proof of (i) above, it suffices to show that the mapping $\beta \to P_\beta(\bar{r}(\{\alpha\}) \cdot \Psi)$ is continuous. However, given $\epsilon > 0$ there exists $\Psi' \in L_1(G^f)$ such that $\| \bar{r}(\{\alpha\}) \cdot \Psi - \Psi' \| < \epsilon/4$. Then, for $\beta, \gamma \in \hat{A}$,

$$\begin{aligned} \| P_\beta(\bar{r}(\{\alpha\}) \cdot \Psi) - P_\gamma(\bar{r}(\{\alpha\}) \cdot \Psi) \|_{C^*(G^f)} &\leqq 2 \| \bar{r}(\{\alpha\}) \cdot \Psi - \Psi' \|_{C^*(G^f)} \\ &\quad + \| P_\beta \Psi' - P_\gamma \Psi \|_{C^*(G^f)} \\ &< \epsilon/2 + \| P_\beta \Psi' - P_\gamma \Psi' \|_1 . \end{aligned}$$

The result thus follows from the continuity of the mapping $\beta \to P_\beta \Psi'$.

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THE QUEEN'S COLLEGE, OXFORD, ENGLAND
AND
LA TROBE UNIVERSITY, VICTORIA, AUSTRALIA.

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