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If $\mathscr{A}$ is a class of open covers of a topological space $(X, \mathscr{T})$, then $(X, \mathscr{T})$ is said to be strongly $\mathscr{A}$-stable provided that for each $\mathscr{C} \in \mathscr{A}$ there is a homeomorphism h mapping X onto X, other than the identity homeomorphism, such that for each $C \in \mathscr{C}$, $h(C) = C$. This paper studies strongly $\mathscr{A}$-stable spaces. Although there are compact connected metric spaces that are not even strongly $\mathscr{A}$-stable with respect to the class $\mathscr{A}$ of all finite open covers, there is an extremely weak homogeneity condition that guarantees that a space $(X, \mathscr{T})$ is strongly $\mathscr{A}$-stable with respect to the class $\mathscr{A}$ of all locally finite open covers. If $H(X)$ is the full homeomorphism group of a space $(X, \mathscr{T})$ that is strongly $\mathscr{A}$-stable with respect to the class $\mathscr{A}$ of all finite open covers, then $H(X)$ is nonabelian and there is a nondiscrete Hausdorff topology $\mathscr{T}$ for $H(X)$ such that $(H(X), \mathscr{T})$ is a topological group.

There has been longstanding interest in topologizing groups of homeomorphisms. For example, in **[16]**, B. L. van der Waerden and D. van Dantzig showed that the group of isometries of a locally compact metric space with only finitely many components is a locally compact topological group under a natural topology; and in [**1**, Theorem 4] R. Arens showed that the full homeomorphism group of a locally compact, locally connected Hausdorff space forms a topological group under the compact open topology. We say that a topology τ on a group $(G, \circ)$ is compatible with $(G, \circ)$ provided that $(G, \circ, \tau)$ is a nondiscrete Hausdorff topological group. It has been known for some time that there exist nondegenerate Hausdorff spaces whose only homeomorphism is the identity. Such spaces are now called rigid spaces, and there are now several constructions of rigid spaces in the literature which differ from the earlier folklore example of a rigid metric continuum **[9]** and **[13]**. These examples show that none of the familiar topological properties on a space X guarantees that the homeomorphism group $H(X)$ admits a compatible topology. On the other hand A. Kertesz and T. Szele have shown that every infinite abelian group admits a compatible topology [**11**, Theorem 1]. The corresponding problem for infinite nonabelian groups is raised in **[11]** and to our knowledge remains unsolved.

In section two we define a sequence of increasingly restrictive topological properties for a space X, each of which is sufficient to

guarantee that $H(X)$ is an infinite nonabelian group that admits a compatible topology. These properties are of necessity quite different from the properties usually studied in general topology; however they interact with several standard topological properties and are, we believe, of interest in and of themselves.

In the third section, we introduce a method of constructing topologies on the homeomorphism group of an arbitrary topological space. Our construction makes use of quasi-uniformities and may be considered to be a generalization of a technique of J. Dieudonné; for, in the case that $(X, \mathcal{U})$ is a uniform space, our method yields the topology of uniform convergence. The topology constructed always makes $H(X)$ into a topological group; however, there is no assurance that this topology will be nondiscrete unless the given space $(X, \mathcal{T}_{\mathcal{U}})$ satisfies one of the properties of section two.

Throughout this paper we let i denote the identity element of a homeomorphism group $H(X)$ and use $H^*(X)$ to denote $H(X) - \{i\}$. If $\mathcal{C}$ is a cover of a space X, then $A_x^{\mathcal{C}} = \cap\{C \in \mathcal{C} : x \in C\}$.

2. Weak Galois spaces and Σ-stable spaces.

DEFINITION [6]. A topological space $(X, \mathcal{T})$ is a *Galois space* if, for each $U \in \mathcal{T}$ and each $p \in U$, there is $h \in H(X)$ such that $h|X - U = i|X - U$ and $h(p) \neq p$.

DEFINITION. A topological space $(X, \mathcal{T})$ is a *weak Galois space* if for each nonempty set $U \in \mathcal{T}$, there is $h \in H(X)$ and $p \in U$ such that $h|X - U = i|X - U$ and $h(p) \neq p$.

PROPOSITION 2.1. *Let $(X, \mathcal{T})$ be a Hausdorff weak Galois space. Then $H(X)$ is an infinite nonabelain group.*

Proof. Since a weak Galois space has no isolated points and since $(X, \mathcal{T})$ is Hausdorff, it is clear that X is an infinite set. Let $p \in X$. Then there is $h_1 \in H(X)$ and $x_1 \in X - \{p\}$ such that $h_1(x_1) \neq x_1$. Since $X - \{p, x_1\}$ is open, there is $h_2 \in H(X)$ and $x_2 \in X - \{p, x_1\}$ such that $h_2(p) = p$, $h_2(x_1) = x_1$ and $h_2(x_2) \neq x_2$. One proves by induction that $H(x)$ is infinite.

Let U be a nonempty open set. Then there is $h \in H(X)$ and $p \in U$ such that $h|X - U = i|X - U$ and $h(p) \neq p$. There are neighborhoods K of p and W of $h(p)$ such that $K \cap W = \varnothing$, and there is a neighborhood M of p such that $h(M) \subset W$. Let $V = U \cap k \cap M$. Then $p \in V$ and $V \cap h(V) = \varnothing$. There exists $f \in H(X)$ and $y \in h(V)$ such that $f|X - h(V) = i|X - h(V)$ and $f(y) \neq y$. Let $x = h^{-1}(y)$. Then $x \in V$ and $f(x) = x$. Consequently $h \circ f(x) = h(x) = y \neq f(x) = f \circ h(x)$.

DEFINITION [3]. An open cover $\mathscr{C}$ of a topological space $(X, \mathscr{T})$ is a *Q-cover* provided that for each $x \in X$, $A_x^{\mathscr{C}} \in \mathscr{T}$.

DEFINITION. Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{C}$ be an open cover of X. Then $(X, \mathscr{T})$ is (*strongly*) *$\mathscr{C}$-stable* provided that there exists $h \in H^*(X)$ such that for each $C \in \mathscr{C}$, $h(C) \subset C$ ($h(C) = C$). If Σ is a collection of open covers of X, then $(X, \mathscr{T})$ is (*strongly*) *Σ-stable* if for each $\mathscr{C} \in \Sigma$, $(X, \mathscr{T})$ is (strongly) $\mathscr{C}$-stable. If Σ is the collection of all finite [locally finite, point finite, Q-] open covers of $(X, \mathscr{T})$, we say that $(X, \mathscr{T})$ is (*strongly*) *finite* [*locally finite, point finite, Q-*] *stable.*

DEFINITION [7]. A collection $\mathscr{C}$ of subsets of a topological space $(X, \mathscr{T})$ is *locally finite somewhere* if there exists a nonempty open set U of X that intersects only finitely many members of $\mathscr{C}$.

The proof of the following theorem is similar to that of [4, Theorem 4].

THEOREM 2.2. *Let Σ be a collection of open covers of a weak Galois space $(X, \mathscr{T})$ and suppose that for each $\mathscr{C} \in \Sigma$, $\mathscr{C}$ is locally finite somewhere. Then $(X, \mathscr{T})$ is strongly Σ-stable.*

Proof. Let $\mathscr{C} \in \Sigma$. Since $\mathscr{C}$ is locally finite somewhere, there exists an open set V such that V intersects only finitely many members of $\mathscr{C}$. Let $\mathscr{F} = \{C \in \mathscr{C}: C \cap V \neq \varnothing\}$ and let

$$K = \{\mathscr{A} \subset \mathscr{F}: V \cap (\cap \mathscr{A}) \neq \varnothing\}.$$

The members of K are partially ordered by set inclusion, and since K is finite, there exists a maximal member $\mathscr{F}' \in K$. Clearly $V \cap (\cap \mathscr{F}')$ is open. Since $(X, \mathscr{T})$ is a weak Galois space, there exist $h \in H(X)$ and $x \in V \cap (\cap \mathscr{F}')$ such that $h | X - V \cap (\cap \mathscr{F}') = i | X - V \cap (\cap \mathscr{F}')$ and $h(x) \neq x$. Let $C \in \mathscr{C}$. If $C \notin \mathscr{F}'$, then by maximality of $\mathscr{F}'$, $C \cap V \cap (\cap \mathscr{F}') = \varnothing$. Hence $C \subset X - V \cap (\cap \mathscr{F}')$ and we have that $h(C) = C$. Let $C \in \mathscr{F}'$ and suppose that for some $p \in C$, $h(p) \neq p$. Since $h | X - V \cap (\cap \mathscr{F}') = i | X - V \cap (\cap \mathscr{F}')$, $h(p) \in V \cap (\cap \mathscr{F}') \subset C$. Thus $h(C) \subset C$. Evidently

$$h^{-1} | X - V \cap (\cap \mathscr{F}') = i | X - V \cap (\cap \mathscr{F}')$$

so that by an argument similar to that given for h, we have that for each $C \in \mathscr{C}$, $h^{-1}(C) \subset C$.

In the above theorem, if we take Σ to be the collection of all locally finite open covers or to be the collection of all point finite open covers, we obtain the following corollaries:

COROLLARY 2.3. *Every weak Galois space is strongly locally finite stable.*

COROLLARY 2.4. *Every weak Galois space that is of second category is strongly point finite stable.*

Proof. A topological space is of second category if and only if every point finite open cover is locally finite somewhere [**5**, Theorem 1].

DEFINITION. A topological space $(X, \mathcal{T})$ is *locally compact at a point* $p \in X$ if there exists an open set U containing p such that $\bar{U}$ is compact.

THEOREM 2.5. *Let $(X, \mathcal{T})$ be a metrizable weak Galois space that is locally compact at some point. Then $(X, \mathcal{T})$ is a strongly Q-stable space.*

Proof. Let $\mathscr{C}$ be a Q-cover and let $\mathscr{K} = \{A_x^{\mathscr{C}} : x \in X\}$. In order to show that $(X, \mathcal{T})$ is $\mathscr{C}$-stable, it suffices to show that $(X, \mathcal{T})$ is $\mathscr{K}$-stable. For if $(X, \mathcal{T})$ is $\mathscr{K}$-stable, then for each $C \in \mathscr{C}$ and each $x \in C$, $h(x) \in h(A_x^{\mathscr{C}}) \subset A_x^{\mathscr{C}} \subset C$. We proceed to show that $(X, \mathcal{T})$ is $\mathscr{K}$-stable.

We first claim that there exists $x \in X$ such that

$$x \notin \overline{\cup \mathscr{A}_y^{\mathscr{C}} \mid x \notin A_y^{\mathscr{C}}}.$$

Assume the contrary. Then for each $x \in X$, there exists a sequence $\{s_{x,n}\}_{n=1}^{\infty}$ such that $\{s_{x,n}\}$ converges to x and such that

$$\{s_{x,n}\} \subset \cup \{A_y^{\mathscr{C}} \mid x \notin A_y^{\mathscr{C}}\}.$$

Consequently for each positive integer n, $x \notin A_{s_{x,n}}^{\mathscr{C}}$.

Let x_1 be a point at which $(X, \mathcal{T})$ is locally compact and select an open ball $B_{r_1}(x_1)$ of radius r_1 such that $r_1 < 1$ and $\overline{B_{r_1}(x_1)}$ is compact. By the previous argument, there exists a sequence $\{s_{x_1,n}\}$ which converges to s_1 such that for each positive integer n, $x_1 \notin A_{s_{x_1,n}}^{\mathscr{C}}$. Thus there exists $s_{x_1,n} \in B_{r_1}(x_1)$ such that $x \notin A_{s_{x_1,n}}^{\mathscr{C}}$. Let $x_2 = s_{x1,n}$. Let $B_{r_2}(x_2)$ be an open ball about x_2 of radius r_2 such that

$$B_{r_1}(x_1) \cap A_{x_2}^{\mathscr{C}} \supset \overline{B_{r_2}(x_2)}$$

and $r < \frac{1}{2}$. By the previous argument, there exists $x_3 \in B_{r_2}(x_2)$ with $x_2 \notin A^{\mathscr{C}}_{x_3}$. Let $B_{r_3}(x_3)$ be an open ball about x_3 of radius r_3 such that $B_{r_2}(x_2) \cap A^{\mathscr{C}}_{x_3} \supset \overline{B_{r_3}(x_3)}$ and $r_3 < \frac{1}{3}$. Proceeding as above we obtain by induction a sequence $\{B_{r_i}(x_i)\}$ such that

$$B_{r_1}(x_1) \supset B_{r_1}(x_1) \cap A^{\mathscr{C}}_{x_2} \supset \overline{B_{r_2}(x_2)} \supset B_{r_2}(x_2) \cap A^{\mathscr{C}}_{x_3} \supset \overline{B_{r_3}(x_3)} \supset \cdots$$

and such that for each positive integer i, $x_i \notin A^{\mathscr{C}}_{x_{i+1}}$ and $r_i < 1/i$.

Since $\overline{B_{r_1}(x_1)}$ is compact, there exists a point $a \in \cap_{i=1}^{\infty} \overline{B_{r_i}(x_i)}$. It follows that $a \in \cap_{i=1}^{\infty} A^{\mathscr{C}}_{x_i}$, so that $A^{\mathscr{C}}_a \subset \cap_{i=1}^{\infty} A^{\mathscr{C}}_{x_i}$. But $\{x_i\} \cap (\cap_{i=1}^{\infty} A^{\mathscr{C}}_{x_i}) = \varnothing$. Thus $\{x_i\} \cap A^{\mathscr{C}}_a = \varnothing$. Since $A^{\mathscr{C}}_a$ is open and $\{x_i\}$ converges to a, it is impossible that $\{x_i\} \cap A^{\mathscr{C}}_a = \varnothing$.

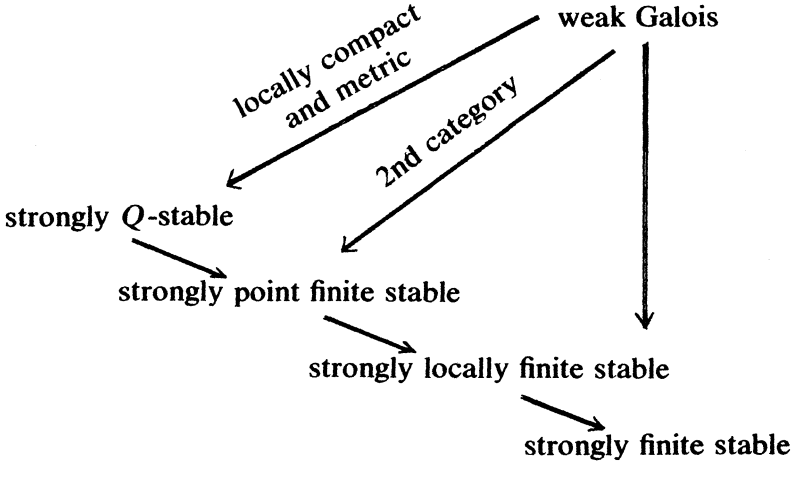
Therefore there exists a point $p \in X$ such that

$$p \notin \overline{\cup \{A^{\mathscr{C}}_y \mid p \notin A^{\mathscr{C}}_y\}}.$$

Let U be an open set about p such that $U \cap (\cup \{A^{\mathscr{C}}_y \mid p \notin A^{\mathscr{C}}_y\}) = \varnothing$ and let $V = U \cap A^{\mathscr{C}}_p$. Since $(X, \mathscr{T})$ is a weak Galois space there exists $h \in H^*(X)$ with $h \mid X - V = i \mid X - V$. Let $A^{\mathscr{C}}_y \in \mathscr{K}$ and suppose that $p \notin A^{\mathscr{C}}_y$. Then $V \cap A^{\mathscr{C}}_y = \varnothing$ and so $h(A^{\mathscr{C}}_y) = A^{\mathscr{C}}_y$. If $p \in A^{\mathscr{C}}_y$, then $V \subset A^{\mathscr{C}}_p \subset A^{\mathscr{C}}_y$. For each $x \in A^{\mathscr{C}}_y$ such that $h(x) \neq x$, $x \in V$, and therefore $h(x) \in V \subset A^{\mathscr{C}}_p \subset A^{\mathscr{C}}_y$. Hence h is $\mathscr{K}$-stable. A similar argument shows that h^{-1} is also $\mathscr{K}$-stable.

In the proof of Theorem 2.5, the hypothesis that $(X, \mathscr{T})$ is locally compact at a point cannot be omitted, for it has been shown in [7] that the space of rationals with usual topology is not a point finite stable space. (The condition that $(X, \mathscr{T})$ is a weak Galois space cannot be omitted either, since there exist rigid compact metric spaces [9]. Clearly such spaces cannot be Q-stable spaces.)

The results of Theorems 2.2 and 2.5 may be summarized by the following diagram:



3. The construction of topologies on homeomorphism groups.

DEFINITION. Let X be a nonempty set. A *quasi-uniformity* on X is a filter $\mathcal{U}$ or $X \times X$ such that the following hold.

(i) For each $U \in \mathcal{U}$, $\Delta = \{(x, x) : x \in X\} \subset U$.

(ii) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V \circ V \subset U$.

In this section we assume familiarity with those basic concepts concerning quasi-uniformities that have natural analogues in the study of uniform spaces; these concepts are defined and discussed in detail in **[14]**. We also make use of the following elementary results.

Let $(X, \mathcal{T})$ be a topological space and let Σ be a collection of Q-covers of X such that for each $A \in \mathcal{T}$ and each $x \in A$, there exists $\mathcal{C} \in \Sigma$ such that $A^{\mathcal{C}}_x \subset A$. For each $\mathcal{C} \in \Sigma$ let

$$U_{\mathcal{C}} = \cup\{\{x\} \times A^{\mathcal{C}}_x : x \in X\}.$$

Then $\{U_{\mathcal{C}} : \mathcal{C} \in \Sigma\}$ is a subbase for a compatible quasi-uniformity $\mathcal{U}_\Sigma$ on $(X, \mathcal{T})$.

If Σ is the collection of all Q- [point finite, locally finite, finite] open covers of X, then $\mathcal{U}_\Sigma$ is called the *fine transitive*, [*point finite covering*, *locally finite covering*, *Pervin*] *quasi-uniformity*. For each $A \in \mathcal{T}$ let $S_A = (A \times A) \cup [(X - A) \times X]$. Then the Pervin quasi-uniformity also has a subbase $\{S_A : A \in \mathcal{T}\}$. If S is any subbase for $\mathcal{T}$, then $\{S_A : A \in S\}$ is a subbase for a compatible quasi-uniformity on X **[14]**.

DEFINITION. Let $(X, \mathcal{U})$ be a quasi-uniform space and for each $U \in \mathcal{U}$ let $W(U) = \{(f, g) \in H(X) \times H(X)$: for each $x \in X$, $(f(x), g(x)) \in U\}$. Then $\{W(U) : U \in \mathcal{U}\}$ is a base for $W(\mathcal{U})$, *the quasi-uniformity of quasi-uniform convergence on* $H(X)$.

THEOREM 3.1 [**2**, Proposition 4]. *Let $(X, \mathcal{U})$ be a quasi-uniform space and let G be a subgroup of $H(X)$ such that each member of G is a $\mathcal{U}$-quasi-uniformly continuous function. Let G have the topology of quasi-uniform convergence with respect to $\mathcal{U}$. Then G is a topological semigroup.*

The following two examples show that under the hypotheses of Theorem 3.1, $H(X)$ need not be a topological group. We omit the verification of the second example, which, though tedious, is straightforward.

EXAMPLE 3.2. *A T_0 weak Galois space $(X, \mathcal{T})$ which is not T_1 whose homeomorphism group under the topology of quasi-uniform*

convergence with respect to the Pervin quasi-uniformity forms a topological semigroup that is not a topological group.

Construction. Let X be the set of all real numbers and let the topology on X be

$$\mathcal{T} = \{(a, b) \subset X : (a, b) \text{ is an open interval containing } 0\} \cup \{\varnothing, X\}.$$

Then clearly $(X, \mathcal{T})$ is a T_0 space which is not T_1. Let h be a homeomorphism of E^1 with the usual topology such that $h(0) = 0$. For each nonempty $G \in \mathcal{T}$, $h^{-1}(G)$ is an open interval containing 0. Hence $h \in H(X)$. It now follows easily that $(X, \mathcal{T})$ is a weak Galois space.

For each $G \in \mathcal{T}$ let $S_G = G \times G \cup (X - G) \times X$. We recall that $S = \{S_G : G \in \mathcal{T}\}$ is a subbase for the Pervin quasi-uniformity $\mathcal{U}$ on $(X, \mathcal{T})$. Since every $h \in H(X)$ is $\mathcal{U}$-quasi-uniformly continuous, by the previous theorem, $H(X)$ forms a topological semigroup under the topology of quasi-uniform convergence with respect to $\mathcal{U}$.

Define $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ by $f(x) = -x$. Since $f \in H(E^1)$ and $f(0) = 0$, $f \in H(X)$. Clearly $f^{-1} = f$. We claim that the inverse map $\varnothing : H(X) \to H(X)$ defined by $\varnothing(h) = h^{-1}$ is not continuous at f. Consider the neighborhood $W(S_{(-1,1)})f^{-1}$ and let $W(S_{(a,b)})f$ be a basic neighborhood of f in $H(X)$. Consider the function $g : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined by $g(x) = -\frac{1}{2}x$. Then $g \in H(X)$. Moreover we now show that $g \in W(S_{(a,b)})f$. Let $x \in X$.

If $-x \in (a, b)$, then $a < \dfrac{a}{2} < \dfrac{-x}{2} < \dfrac{b}{2} < b$. Thus $(f(x), g(x)) = \left(-x, \dfrac{-x}{2}\right) \in (a, b) \times (a, b) \subset S_{(a,b)}$.

If $-x \notin (a, b)$, then

$$(f(x), g(x)) = \left(-x, \frac{-x}{2}\right) \in (X - (a, b)) \times X \subset S_{(a,b)}.$$

Therefore $g \in W(S_{(a,b)})f$. We note that $g^{-1}(x) = -2x$ and that $g^{-1} \notin W(S_{(-1,1)})f^{-1}$, since

$$(f^{-1}(3/4), g^{-1}(3/4)) = \left(\frac{-3}{4}, \frac{-3}{2}\right) \notin (-1, 1) \times (-1, 1) \cup (X - (-1, 1)) \times X = S_{(-1,1)}.$$

EXAMPLE 3.3. *A quasi-uniformity $\mathcal{U}$ compatible with the usual topology on the closed interval $I = [-1, 1]$ which has the property that every $h \in H(I)$ is $\mathcal{U}$-quasi-uniformly continuous and $H(I)$ is not a topological group under the topology of quasi-uniform convergence with respect to $\mathcal{U}$.*

Construction. Let $\mathcal{U}$ be the quasi-uniformity on I having subbase $S = \{S_G : G = (a, b) \text{ or } (a, 1] \text{ or } [-1, b)\}$ where $-1 < a < b < 1$. It may be seen that the inverse map is not continuous at $-i$.

DEFINITION. A collection Σ of Q-covers of a topological space X is a *topological collection* provided that if $\mathscr{C} \in \Sigma$, then for all $h \in H(X)$, $\{h(C) : C \in \mathscr{C}\} \in \Sigma$.

LEMMA. *Let $(X, \mathcal{T})$ be a topological space and let Σ be a topological collection of Q-covers such that if $x \in A \in \mathcal{T}$, there is $\mathscr{C} \in \Sigma$ with $A_x^{\mathscr{C}} \subset A$. Then $H(X)$ is a 0-dimensional topological group under the topology of uniform convergence with respect to the uniformity $\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1}$.*

Proof. We note that every $h \in H(X)$ is $\mathcal{U}_\Sigma$-quasi-uniformly continuous so that every $h \in H(X)$ is $\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1}$-uniformly continuous. Thus $H(X)$ is a topological group [**2,** Proposition 4]. Since for each $\mathscr{C} \in \Sigma$, $U_{\mathscr{C}} \circ U_{\mathscr{C}} = U_{\mathscr{C}}$, $\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1}$ has a base of equivalence relations. Thus $\mathcal{T}_{W(\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1})}$ is 0-dimensional.

THEOREM 3.4. *Let $(X, \mathcal{T})$ be a topological space and let Σ be the collection of all Q-* [*point finite, locally finite, finite open*] *covers of $(X, \mathcal{T})$. Let $(X, \mathcal{T})$ be strongly Σ-stable. Then $H(X)$ is a nondiscrete 0-dimensional topological group under the topology of uniform convergence with respect to $\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1}$.*

Proof. Since one easily sees that Σ is a topological collection of Q-covers, by the preceding lemma it suffices to show that $\mathcal{T}_W(\mathcal{U}_\Sigma \vee \mathcal{U}_\Sigma^{-1})$ is nondiscrete.

Let G be an open set about i. Then there exist basic members K and V of $\mathcal{U}_\Sigma$ such that $W(K \cap V^{-1})i \subset G$. Let $\mathscr{A}_j$, $j = 1, 2, \cdots, n$, and $\mathscr{B}_j$, $j = 1, 2, \cdots, m$ be members of Σ such that $K = \cap_{j=1}^n U_{\mathscr{A}_j}$ and $V = \cap_{j=1}^m U_{\mathscr{B}_j}$ and let $\mathscr{C}_1 = \cup_{j=1}^n \mathscr{A}_j$, $\mathscr{C}_2 = \cup_{j=1}^m \mathscr{B}_j$. Then $\mathscr{C}_1$ and $\mathscr{C}_2$ are members of Σ. We have that $K = \cap_{j=1}^n U_{\mathscr{A}_j} = U_{\cap_{j=1}^n \mathscr{A}_j} = U_{\mathscr{C}_1}$ and that $V = \cap_{j=1}^m U_{\mathscr{B}_j} = U_{\cup_{j=1}^m \mathscr{B}_j} = U_{\mathscr{C}_2}$. Since $(X, \mathcal{T})$ is strongly Σ-stable, there exists $h \in H^*(X)$ such that both h and h^{-1} are $\mathscr{C}_1 \cup \mathscr{C}_2$-stable.

Since h is $\mathscr{C}_1$-stable, h is $\{A_x^{\mathscr{C}_1} : x \in X\}$-stable. For each $x \in X$, $h(x) \subset A_x^{\mathscr{C}_1}$. Therefore for each $x \in X$, $(x, h(x)) \in \{x\} \times A_x^{\mathscr{C}_1} \subset U_{\mathscr{C}_1} = K$. Hence $h \in W(K)i$. Let $x \in X$ and let $y = h(x)$. Then $x \in A_y^{\mathscr{C}_2}$. Otherwise $h^{-1}(y) = x \notin A_y^{\mathscr{C}_2}$ whereas h^{-1} is $\mathscr{C}_2$-stable. Therefore $(h(x), x) \in \{y\} \times A_y^{\mathscr{C}_2} \subset U_{\mathscr{C}_2} = V$ and $(x, h(x)) \in V^{-1}$. Hence $h \in W(V^{-1})i$. Finally we have that

$$h \in W(K)i \cap W(V^{-1})i = [W(K) \cap W(V^{-1})]i = W(K \cap V^{-1})i \subset G.$$

Thus $H(X)$ is a nondiscrete topological group.

COROLLARY. *Let $(X, \mathcal{T})$ be a T_0 weak Galois space. Then $H(X)$ admits a compatible topology $\mathcal{T}'$ such that $(H(X), \mathcal{T}')$ is a Galois space.*

Proof. Let $\mathcal{U}$ be the locally finite covering quasi-uniformity for $(X, \mathcal{T})$. Then by Corollary 2,3, $(X, \mathcal{T})$ is strongly locally finite stable. Thus $H(X)$ is a nondiscrete topological group under the topology of uniform convergence with respect to $\mathcal{U} \vee \mathcal{U}^{-1}$. Since $(X, \mathcal{T})$ is T_0, $H(X)$ is a T_0 space with respect to the compact open topology $\hat{\mathcal{T}}$ on $H(X)$. As $\hat{\mathcal{T}} \subset \mathcal{T}_{W(\mathcal{U})} \subset \mathcal{T}_{W(\mathcal{U} \vee \mathcal{U}^{-1})}$ and $(H(X), \mathcal{T}_{W(\mathcal{U} \vee \mathcal{U}^{-1})})$ is a topological group, $\mathcal{T}_{W(\mathcal{U} \vee \mathcal{U}^{-1})}$ is Hausdorff. By Theorem 3.4, $H(X)$ is 0-dimensional and homogeneous so that $H(X)$ is a Galois space [7, Page 529].

COROLLARY. *Let $(X, \mathcal{T})$ be a second countable T_0 weak Galois space. Then $H(X)$ admits a compatible metric topology.*

Proof. Let $\mathcal{B}$ be a countable base for $\mathcal{T}$ and let $\mathcal{U}$ be the quasi-uniformity on $(X, \mathcal{T})$ having subbasis

$$S = \{(G \times G) \cup [(X - G) \times X] : G \in \mathcal{B}\}.$$

Then $\mathcal{U}$ has a countable base and so $\mathcal{U} \vee \mathcal{U}^{-1}$ also has a countable base. Let $\mathcal{L}$ be the locally finite covering quasi-uniformity for $(X, \mathcal{T})$. Since $\mathcal{U} \subset \mathcal{L}$, the topology of uniform convergence on $H(X)$ with respect to $\mathcal{U} \vee \mathcal{U}^{-1}$ is nondiscrete. The argument that $\mathcal{T}_{W(\mathcal{U} \vee \mathcal{U}^{-1})}$ is Hausdorff is the same as that of the previous corollary.

If $(X, \mathcal{T})$ is a weak Galois completely regular Hausdorff space, then the topology of uniform convergence with respect to the fine uniformity is compatible with $H(X)$ [12]. This result and the first corollary above motivate the following questions:

Is every infinite nonabelian group isomorphic to the full homeomorphism group of a T_0 weak Galois space? If so, can the T_0 weak Galois space also be taken to be completely regular?

The result of [8] that every group is isomorphic to the full homeomorphism group of a complete, connected, locally connected metric space gives some reason to hope that the questions raised above can be answered affirmatively. Evidently, however, the techniques employed in [8] are not applicable here.

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