ON THE EXTREMAL ELEMENTS OF THE CONVEX CONE OF SUPERADDITIVE $n$-HOMOGENEOUS FUNCTIONS

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Let $P_n$ be the collection of finite-valued functions defined on the nonnegative orthant, $E^+_n$, of euclidean $n^2$-space such that for $p \in P_n$ it follows that $p : E^+_n \rightarrow \mathbb{E}$ and in addition

(a) $p$ is continuous,
(b) $p(\alpha x) = \alpha^n p(x)$, $\alpha \geq 0$,
(c) $p(x + y) \geq p(x) + p(y)$.

It follows readily that $P_n$ is closed with respect to addition and nonnegative scalar multiplication. Therefore, $P_n$ is a convex cone, whose vertex is the zero function, in the linear space of real functions defined on $E^+_n$. The purpose of this paper is to investigate the extremal elements of $P_n$.

1. Introduction. One well known member of $P_n$ is the permanent function. Recently functions that generalize the permanent function have been studied by Rothaus [9] and new representations for the permanent function have been sought (for example see [2] by Marcus and Newman). The interest in determining the extremal elements of $P_n$ comes from the fact that under certain circumstances it is possible to give an integral representation for any $p \in P_n$ in terms of the extremal elements of $P_n$ [1] (examples of similar studies may be found in papers by McLachlan [4], [5], [6] and Rakestraw [7]). In this paper it is shown that for $a \in E^+_n \setminus 0$, the functions $p_a(x) = \sup \{\lambda^n : x \equiv \lambda a\}$ are extremal elements of $P_n$. Replacing condition (b) by

(b) $p(\alpha x) = \alpha p(x)$, $\alpha \geq 0$,

gives the collection of monotone concave gauges, denoted by $P'_n$, defined on $E^+_n$ [8]. If for all $i = 1, \cdots, n$, $A_i \in P'_n$, then the function $A$ defined as $A(x) = \Pi_{i=1}^n A_i(x)$ is an element of $P_n$. If $S_n$ denotes all those $p \in P_n$ which are finite nonnegative linear combination of functions of this type, then clearly $S_n$ is a subcone of $P_n$ and $S_n$ contains the permanent function. It is shown here that for a function $p \in S_n$ to be an extremal element of $P_n$, then $p$ must be of the form $p(x) = [A(x)]^n$, where $A(x)$ is an extremal element of $P'_n$. 

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In the material to follow define \([p: \alpha] = \{x: p(x) = \alpha\}\). where \(p \in P_n\). It follows that \(\alpha[p: 1] = [p: \alpha^n]\) for all \(\alpha \geq 0\). Also, use will be made of the fact that for \(p \in P_n^\prime\) or \(p \in P_n\), then \(x \geq y\) (or \(x > y\)) implies that \(p(x) \geq p(y)\) (or \(p(x) > p(y)\)). Further if \(x \in \text{int} E_n^\ast\) and \(p \neq 0\), then \(p(x) > 0\).

2. **Extremal elements of** \(P_n\). The first theorem of this section gives some of the extremal elements of \(P_n\). It is conjectured that this set includes all the extremal elements of \(P_n\). The following lemmas will be needed.

**Lemma 1.1.** If \(p, q \in P_n\). Define \((p \wedge q)(x) = \min\{p(x), q(x)\}\). Then \(p \wedge q \in P_n\).

**Proof.** It follows readily from the definitions that \(p \wedge q\) is continuous and homogeneous. Also,

\[
(p \wedge q)(x + y) = \min\{p(x + y), q(x + y)\} \\
\geq \min\{p(x) + p(y), q(x) + q(y)\} \\
\geq \min\{p(x), q(x)\} + \min\{p(y), q(y)\} \\
= (p \wedge q)(x) + (p \wedge q)(y).
\]

For all \(k = 1, \ldots, n^2\), let \(p_k(x) = x_k^n, x = (x_1, \ldots, x_n) \in E_n^\ast\). Then \(p_k \in P_n\). With this in mind consider the following:

**Lemma 1.2.** Let \(a = (a_1, \ldots, a_n^\ast) \in E_n^\ast \setminus \{0\}\). Define \(p_a\) as follows:

\[
p_a(x) = \sup\{\lambda^n: x \geq \lambda a, \lambda \geq 0\}.
\]

Then \(p_a \in P_n\).

**Proof.** Without loss of generality, assume the nonzero coordinates of \(a\) are \(a_1, \ldots, a_k, k \leq n^2\). Let

\[
(1.1) \quad p(x) = \left(\frac{1}{a_1^n} p_1 \wedge \cdots \wedge \frac{1}{a_k^n} p_k\right)(x).
\]

Lemma 1.1 implies that \(p \in P_n\). Now for any given \(x \in E_n^\ast\) suppose that
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\[ p(x) = \frac{x_i^n}{a_i}, \]

\[ 1 \leq l \leq k. \] Then it follows readily that for each \( i \)

\[ x_i \geq \frac{x_l}{a_l} a_i \]

with equality when \( i = l \). Further there does not exist \( \lambda > x_i/a_i \) such that \( x \geq \lambda a \) since otherwise

\[ x_i > \lambda a_i > \frac{x_l}{a_l} a_i = x_i. \]

Hence, \( p_a(x) = p(x) \) for every \( x \in E_{n^*}^+ \), which implies that \( p_a = p \).

Notice that if \( a_i \) is a nonzero coordinate of \( a \) and \( x \in E_{n^*}^+ \) such that \( x_i = 0 \), then \( x \geq \lambda a \) implies that \( \lambda = 0 \). Thus, \( p_a(x) = 0 \). Also, if \( a = e_k \), where \( e_k \) is that vector having all zero coordinates except the \( k \)th coordinate which is 1, then \( p_a = p_e \).

In general, if \( p \in P_n \) the set \([p: 1]\) is difficult to characterize. For example a complete characterization of the set \([p : n!/n^*]\) (and hence \([p : 1]\)) where \( p \) is the permanent function is not known [3]. However, if \( p = p_a \) for some \( a \neq 0 \), then a characterization is possible. Let \( a = (a_1, \ldots, a_{n^2}) \in E_{n^*}^+ \), \( a \neq 0 \). For every \( i \in \{1, \ldots, n^2\} \), define

\[ R(a_i) = \{(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_{n^2}) : x_j \geq a_j \text{ for } j \neq i\}. \]

**Lemma 1.3.** If \( a \in E_{n^*}^+ \setminus \{0\} \), then

\[ [p_a : 1] = \cup \{R(a_i) : a_i \neq 0\}. \]

**Proof.** Let \( y \in R(a_i) \), where \( a_i \neq 0 \). Clearly, \( y \geq a \). Notice there does not exist \( \lambda > 1 \) such that \( y \geq \lambda a \), for otherwise \( a_i \geq \lambda a_i > a_i \). Hence, by definition \( p_a(y) = 1 \). This implies that

\[ \cup \{R(a_i) : a_i \neq 0\} \subseteq [p_a : 1]. \]

Now suppose \( y \in [p_a : 1] \). Considering (1.1), there exists \( k \in \{1, \ldots, n^2\} \) such that \( a_k > 0 \) and \( (y_k/a_k)^* = 1 \). This implies that \( y_k = a_k \), which implies that \( y_k = a_k \). For all other \( i \in \{1, \ldots, n^2\} \) such that \( a_i > 0 \),

\( (y_i/a_i)^* \geq 1 \) and hence \( y_i \geq a_i \). It follows that \( y \in R(a_k) \) and the proof is complete.

Using this result it is possible to show that \( p_a = p_b \) if and only if \( a = b \). Next using Lemma 1.3, \( p_a \) is shown to be an extremal element of \( P_n \).
THEOREM 1.1. The function $p_a$ is an extremal element of $P_n$.

Proof. Suppose $p_a = f + g$. Let $y \in R(a_i)$, where $a_i \neq 0$ and $i \in \{1, \cdots, n^2\}$, then

$$P_a(a) = P_a(y) = f(y) + g(y) \equiv f(a) + g(a) = P_a(a).$$

This implies $f(y) = f(a)$ and $g(y) = g(a)$, since $f(y) \equiv f(a)$ and $g(y) \equiv g(a)$. Also, $p_a(a) = f(a) + g(a)$ implies $p_a(a) \equiv f(a)$ and $p_a(a) \equiv g(a)$. Therefore, there exists $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha p_a(a) = f(a)$ and $\beta p_a(a) = g(a)$.

Again, without loss of generality, suppose the nonzero coordinates of $a$ are $a_1, \cdots, a_k$. Let $x \in E_n^+$ such that $x_1 > 0, \cdots, x_k > 0$. Then for every $i \in \{1, \cdots, k\}$ there exists $\lambda_i > 0$ such that $a_i = \lambda_i x_i$. Let $\lambda = \max \{\lambda_i : i \in \{1, \cdots, k\}\}$. Notice there exists a $j \in \{1, \cdots, k\}$ such that $\lambda = \lambda_j$. Hence, $\lambda x_i \geq a_i$ with equality when $i = j$. Clearly, if $i \in \{1, \cdots, n^2\} \setminus \{1, \cdots k\}$, then $\lambda x_i \geq a_i$. Therefore, $\lambda x \in R(a_i)$. Setting $y = \lambda x$, it follows that

$$f(x) = f \left( \frac{1}{\lambda} y \right) = \frac{1}{\lambda} f(y)$$

$$= \frac{1}{\lambda} f(a) = \frac{1}{\lambda} \alpha p_a(a)$$

$$= \frac{1}{\lambda} \alpha p_a(y) = \alpha p_a \left( \frac{1}{\lambda} y \right)$$

$$= \alpha p_a(x).$$

Clearly, if $x \in E_n^+$ such that $x_i = 0$ for some $i \in \{1, \cdots, k\}$, then $0 = p_a(x)$. This implies $f(x) = 0$, which in turn implies that $f(x) = \alpha p_a(x)$. In either case $f(x) = \alpha p_a(x)$. Hence, $f = \alpha p_a$. Likewise, $g = \beta p_a$. Therefore, $p_a$ is an extremal element of $P_n$.

By a somewhat similar proof it can be shown that the function $p_a$ is minimal in the set of all elements of $P_n$ which agree with $p_a(a)$ at $a$. Also, for $\alpha > 0$ the sets $[p_a : 1]$ have the property that $\alpha [p_a : 1] = [p_a : \alpha^n] = [p_a : 1]$.

Recall that $S_a \subset P_n$ is the set of finite nonnegative linear combinations of products of $n$ functions of $P_n'$. For $a \in E_n^+ \setminus \{0\}$, let $q_a(x) = \sup \{\lambda : x \equiv \lambda a\}$. Then, as in the case for $P_n$, $q_a$ is an extremal element of $P_n'$. Also, $p_a(x) = [q_a(x)]^a$, which implies that $p_a \in S_a$. Since $S_n$ is a subcone of $P_n$, then $p_a$ is an extremal element of $S_n$. It is conjectured that $\{p_a : a \in E_n^+\}$ represents all the extremal elements of $S_n$. 

Lemma 1.4. If \( p \neq 0 \) and \( p(x) = \prod_{i=1}^{n} A_i(x) \), where \( A_i \in P'_n \), is an extremal element of \( S_n \), then each \( A_i \) is an extremal element of \( P'_n \).

Proof. Suppose there exists a \( k = 1, \ldots, n \) such that \( A_k \) is not extremal in \( P'_n \). Then there exists \( f, g \in P'_n \) such that \( A_k = f + g \) and neither \( f \) or \( g \) is proportional to \( A_k \). Hence,

\[
p(x) = \prod_{i=1}^{n} A_i(x) = A_1(x) \cdots (f(x) + g(x)) \cdots A_n(x)
\]

Since \( p \) is extremal in \( S_n \), there exists \( \alpha \geq 0 \) and \( \beta \geq 0 \) such that \( A_1(x) \cdots f(x) \cdots A_n(x) = \alpha p(x) \) and \( A_1(x) \cdots g(x) \cdots A_n(x) = \beta p(x) \). Let \( x \in \text{int} E^*_n \). Then \( p(x) > 0 \). Also, it can be shown that each \( A_i(x) > 0 \), \( f(x) > 0 \) and \( g(x) > 0 \). Therefore,

\[
\alpha A_1(x) \cdots A_k(x) \cdots A_n(x) = \alpha p(x) = A_1(x) \cdots f(x) \cdots A_n(x),
\]

which implies that \( \alpha A_k(x) = f(x) \), for all \( x \in \text{int} E^*_n \). It follows from continuity that \( \alpha A_k(x) = f(x) \) for all \( x \in E^*_n \). This is a contradiction. Therefore, \( A_i \) is an extremal element of \( P'_n \) for each \( k \).

In any convex cone, if the sum of two nonzero elements is an extremal element, then the two elements are proportional. Hence, the only possible extremal elements of \( S_n \) are those elements of the form

\[
p(x) = \prod A_{i}^{(i)}(x),
\]

where \( l(i) \) is a nonnegative integer and \( \Sigma l(i) = n \). Moreover, Lemma 1.4 implies that the \( A_i \) must be extremal elements of \( P'_n \). The Lemma 1.4 and these comments give conditions that are necessary when \( p \) is an extremal element in \( S_n \). These conditions are not sufficient as will be seen in Proposition 1.1.

Attention will now be given to considering the extremal elements of \( P_n \).

Theorem 1.2. Let \( p \) be defined as in (1.2). Let \( k \) be the number of \( i \) for which \( l(i) > 0 \). If \( k > 1 \), then \( p \) is not an extremal element of \( P_n \).

Proof. Assume

\[
p(x) = \prod_{i=1}^{k} A_{i}^{(i)}(x),
\]
where each $l(i)$ is a positive integer, $\sum_{i=1}^{k} l(i) = n$ and the $A_i$ are distinct (pairwise nonproportional) extremal elements of $P'_n$. For each $i \in \{1, 2\}$ define

$$f_i(x) = \begin{cases} \frac{A_i(x)}{A_i(x) + A_2(x)} p(x), & A_i(x) + A_2(x) > 0 \\ 0 & A_i(x) + A_2(x) = 0. \end{cases}$$

It follows easily that $p = f_1 + f_2$. It will now be shown that each $f_i \in P_n$.

$n$-Homogeneity: Let $\alpha \geq 0$ and $x \in E^*_n$. If $\alpha = 0$, then $A_i(\alpha x) = A_2(\alpha x) = 0$ and hence $f_i(\alpha x) = 0 = \alpha^"f_i(x)$. Suppose $\alpha > 0$. If $0 = A_i(\alpha x) + A_2(\alpha x) = \alpha (A_i(x) + A_2(x))$, then $A_i(x) + A_2(x) = 0$ and hence $f_i(\alpha x) = \alpha^"f_i(x)$. Suppose $\alpha > 0$ and

$$\alpha (A_i(x) + A_2(x)) = A_i(\alpha x) + A_2(\alpha x) > 0,$$

then $A_i(x) + A_2(x) > 0$. Therefore,

$$f_i(\alpha x) = \frac{A_i(\alpha x)}{A_i(\alpha x) + A_2(\alpha x)} p(\alpha x) = \alpha^n \frac{A_i(x)}{A_i(x) + A_2(x)} p(x) = \alpha^"f_i(x).$$

So for all $\alpha \geq 0$ and $x \in E^*_n$, $f_i(\alpha x) = \alpha^"f_i(x)$.

Superadditivity: Let $x, y \in E^*_n$.

Case I. If $A_1(x + y) + A_2(x + y) = 0$, then

$$0 = A_1(x + y) + A_2(x + y) \geq A_1(x) + A_1(y) + A_2(x) + A_2(y) \geq 0$$

which implies that $A_i(x) + A_2(x) = 0$ and $A_i(y) + A_2(y) = 0$. Therefore, $f_i(x + y) = f_i(x) + f_i(y)$.

Case II. Suppose that $A_1(x + y) + A_2(x + y) > 0$, $A_1(x) + A_2(x) = 0$ and $A_i(y) + A_2(y) = 0$. Clearly, $f_i(x + y) \geq f_i(x) + f_i(y)$.

Case III. Suppose $A_1(x + y) + A_2(x + y) > 0$, $A_i(x) + A_2(x) > 0$ and $A_i(y) + A_2(y) = 0$. Then

$$f_i(x + y) = \frac{A_i(x + y)}{A_i(x + y) + A_2(x + y)} p(x + y),$$

$$f_i(x) = \frac{A_i(x)}{A_i(x) + A_2(x)} p(x)$$
and \( f_i(y) = 0 \). It must be shown that

\[
\frac{A_i(x + y)}{A_i(x + y) + A_2(x + y)} p(x + y) \geq \frac{A_i(x)}{A_i(x) + A_2(x)} p(x).
\]

This is true if and only if

\[
(A_i(x) + A_2(x)) A_i(x + y) \prod_{j=1}^{k} A_j^{(i)}(x + y) \geq (A_i(x + y) + A_2(x + y)) A_i(x) \prod_{j=1}^{k} A_j^{(i)}(x).
\]

(1.3)

It suffices to show that each term on the right hand side of (1.3) is less than or equal to the corresponding term on the left hand side of (1.3). Now for \( m = 1 \) (or \( m = 2 \))

\[
A_m(x) A_i(x + y) \prod_{j=1}^{k} A_j^{(i)}(x + y) = A_m(x + y) A_i(x + y) (A_m(x) A_{(m-1)}^{(i)}(x + y) \cdots A_k^{(i)}(x + y)) \geq A_m(x + y) A_i(x) \prod_{j=1}^{k} A_j^{(i)}(x).
\]

It follows that (1.3) is true.

**Case IV.** Suppose \( A_i(x + y) + A_2(x + y) > 0, A_i(x) + A_2(x) > 0 \) and \( A_i(y) + A_2(y) > 0 \). Then

\[
f_i(x + y) = \frac{A_i(x + y)}{A_i(x + y) + A_2(x + y)} p(x + y),
\]

\[
f_i(x) = \frac{A_i(x)}{A_i(x) + A_2(x)} p(x),
\]

and

\[
f_i(y) = \frac{A_i(y)}{A_i(y) + A_2(y)} p(y).
\]

It must be shown that

(a)

\[
\frac{A_i(x + y)}{A_i(x + y) + A_2(x + y)} \prod_{j=1}^{k} A_j^{(i)}(x + y)
\]

(b)

\[
\geq \frac{A_i(x)}{A_i(x) + A_2(x)} \prod_{j=1}^{k} A_j^{(i)}(x) + \frac{A_i(y)}{A_i(y) + A_2(y)} \prod_{j=1}^{k} A_j^{(i)}(y).
\]
This will be true if and only if

\[(c) \quad A_i(x + y)(A_1(x) + A_2(x))(A_1(y) + A_2(y)) \prod_{j=1}^{k} A_i^{(j)}(x + y)\]

\[(d) \quad \geq A_i(x)(A_1(y) + A_2(y))(A_1(x + y) + A_2(x + y)) \prod_{j=1}^{k} A_i^{(j)}(x)\]

\[(e) \quad + A_i(y)(A_1(x) + A_2(x))(A_1(x + y) + A_2(x + y)) \prod_{j=1}^{k} A_i^{(j)}(y).\]

Since

\[(f) \quad A_i(x + y)(A_1(x) + A_2(x))(A_1(y) + A_2(y)) \prod_{j=1}^{k} A_i^{(j)}(x + y)\]

\[(g) \quad \geq A_i(x)(A_1(x) + A_2(x))(A_1(y) + A_2(y)) \prod_{j=1}^{k} A_i^{(j)}(x + y)\]

\[(h) \quad + A_i(y)(A_1(x) + A_2(x))(A_1(y) + A_2(y)) \prod_{j=1}^{k} A_i^{(j)}(x + y),\]

it is sufficient to show that $(g) \geq (d)$ and $(h) \geq (e)$. An argument similar to the one in Case III shows that each term of $(g)$ or $(h)$ is greater than or equal to the corresponding term of $(d)$ or $(e)$. Thus, $(c) \geq (d) + (e)$ and hence $(a) \geq (b)$. Therefore, each $f_i$ is superadditive.

**Continuity:** Let $x \in E_{\ast}^n$ and $\{y_i\} \subset E_{\ast}^n$ such that $y_i \to x$. Suppose $A_1(x) + A_2(x) > 0$, then without loss of generality it may be assumed that $A_1(y_j) + A_2(y_j) > 0$ for each $j$. In this case

\[f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j) + A_2(y_j)} p(y_j) \to \frac{A_i(x)}{A_1(x) + A_2(x)} p(x) = f_i(x).\]

Suppose that $A_1(x) + A_2(x) = 0$, then $p(x) = f_i(x) = 0$. If there exists $m \in \{1, 2\}$ such that $A_m(y_j) = 0$, $f_i(y_j) = 0 = f_i(x)$. Suppose $A_m(y_j) > 0$ for $m = 1, 2$. Since each $A_m^{(m)}(y_j) \to 0$ and the expression

\[\frac{A_i(y_j)}{A_1(y_j) + A_2(y_j)}\]

is obviously bounded by 1, then $f_i(y_j) \to 0$. Therefore, $f_i$ is continuous. Hence, each $f_i \in P_n$.

It remains to be shown that the functions $f_i$ form a nonproportional decomposition of $p$. Suppose $f_i(x) = \alpha p(x)$, for all $x \in E_{\ast}^n$. Let $x \in E_{\ast}^n$. There exists a sequence $\{y_i\} \subset \text{int } E_{\ast}^n$ such that $y_i \to x$. Since $y_i \in \text{int } E_{\ast}^n$, then $A_1(y_i) > 0$, $A_2(y_i) > 0$ and $p(y_i) > 0$. Hence,
which implies that

\[ A_i(y_j) = \alpha (A_i(y_j) + A_2(y_j)). \]

Since \( A_1(y_j) \rightarrow A_1(x) \) and \( \alpha (A_i(y_j) + A_2(y_j)) \rightarrow \alpha (A_1(x) + A_2(x)) \), then \( A_1(x) = \alpha (A_1(x) + A_2(x)) \). Since \( A_1 \) and \( A_2 \) are pairwise nonproportional extremal elements in \( P_n \), this is a contradiction. Therefore, there does not exist \( \alpha \geq 0 \) such that \( f_i = \alpha p \). Hence, the decomposition is nonproportional, which implies that \( p \) is not an extremal element of \( P_n \).

Two questions immediately arise. First, is \( f_i \in S_n \)? Secondly, is every extremal element of \( P_\ast \) of the form \( q_n \), where \( a \in E_{n^2}^+ \setminus \{0\} \)? If both answers are affirmative, then every extremal element of \( S_n \) is of the form \( p_n \), where \( a \in E_{n^2}^+ \setminus \{0\} \). It is entirely possible that the functions \( f_i \) do not belong to \( S_n \).

The following is an example of a subcone of \( P_n \) that has as extremal elements some functions that are not extremal in \( P_n \).

**Example 1.1.** Let \( Q_n \) be the set of all \( p : E_{n^2}^+ \rightarrow E_1^+ \) such that

\[
p(x) = \sum_{i_1, \ldots, i_n \geq 1} \alpha_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n}
\]

where \( i_1 \leq \cdots \leq i_n \), \( \alpha_{i_1, \ldots, i_n} \geq 0 \) and \( x = (x_1, \ldots, x_n) \). Thus, \( Q_n \) is the set of nonnegative superadditive \( n \)-forms. Clearly, \( Q_n \) is a subcone of \( S_n \subset P_n \). Therefore, the functions \( p_{i_1}, \ldots, p_{i_n} \) are extremal elements of \( Q_n \). However, these are not all of the extremal elements of \( Q_n \). In fact without much difficulty it can be shown that the extremal elements of \( Q_n \) are the positive scalar multiples of functions of the form

\[
p(x) = x_{k_1} \cdots x_{k_n},
\]

where \( k_j \in \{1, \cdots, n^2\} \), for \( j = 1, \cdots, n \) and \( k_1 \leq \cdots \leq k_n \).

Now for every \( x = (x_1, \cdots, x_n) \in E_{n^2}^+ \), define \( p(x) \) as

\[
p(x) = x_1^{l(1)} \cdots x_n^{l(n^2)},
\]  

where \( (i) \) is a nonnegative integer and \( \sum_{i=1}^{n^2} l(i) = n \). Notice that \( l(i) > 0 \) for at most \( n \) values of \( i = 1, \cdots, n^2 \). Clearly, \( p \in Q_n \). In fact the preceding example shows that \( p \) is an extremal element of \( Q_n \). If \( k \) is the number of \( i \in \{1, \cdots, n^2\} \) for which \( l(i) > 0 \) and \( k > 1 \), Theorem 1.2
sais que $p$ n'est pas un élément extrémal de $P_n$. La proposition suivante montre que $p$ n'est pas un élément extrémal de $S_n$.

** Proposition 1.1.** *Le.p. défini comme en (1.4). Si $k > 1$, alors $p$ n'est pas un élément extrémal de $S_n$.*

*Proof.* Sans perte de généralité supposons

$$p(x) = x_1^{(1)} x_2^{(2)} \cdots x_k^{(k)}$$

où chaque $l(k) > 0$. Comme on a vu dans la démonstration du théorème 1.2, $p = f_1 + f_2$ où

$$f_i(x) = \begin{cases} \frac{x_i}{x_1 + x_2} x_1^{(1)} \cdots x_k^{(k)}, & x_1 + x_2 > 0 \\ 0, & x_1 + x_2 = 0. \end{cases}$$

Considérons $f_1$. Remarquons que

$$f_1(x) = \begin{cases} \frac{x_1 x_2}{x_1 + x_2} x_1^{(1)} x_2^{(2)} \cdots x_k^{(k)}, & x_1 + x_2 > 0 \\ 0, & x_1 + x_2 = 0. \end{cases}$$

Laissons $g(x) = \begin{cases} \frac{x_1 x_2}{x_1 + x_2}, & x_1 + x_2 > 0 \\ 0, & x_1 + x_2 = 0. \end{cases}$

Cette expression $f_1(x) = g(x) x_1^{(1)} x_2^{(2)} \cdots x_k^{(k)}$. Comme l'objectif est de montrer que $f_1 \in S_n$, il reste à montrer que $g \in P_n$. Comme dans le théorème 1.2, $g$ est continue et homogène de degré 1. Pour montrer la superadditivité, supposons que $x, y \in E_n$. Si $x_1 + x_2 = 0$ ou $y_1 + y_2 = 0$, alors il suinte facilement que $g(x + y) = g(x) + g(y)$. Supposez $x_1 + x_2 > 0$ et $y_1 + y_2 > 0$. Dans ce cas, il faut montrer que

$$\frac{(x_1 + y_1)(x_2 + y_2)}{x_1 + x_2 + y_1 + y_2} \geq \frac{x_1 x_2}{x_1 + x_2} + \frac{y_1 y_2}{y_1 + y_2},$$

qui est équivalent à la démonstration que

$$(x_1 + y_1)(x_2 + y_2)(x_1 + x_2)(y_1 + y_2) - [(x_1 x_2)(x_1 + y_1 + x_2 + y_2)(y_1 + y_2) + y_1 y_2(x_1 + y_1 + x_2 + y_2)(x_1 + x_2)] \geq 0.$$ 

Par calcul direct, le côté gauche de l'inégalité est égal à $(x_1 y_2 - x_2 y_1)^2$. La computation est laborieuse mais directe.
Hence, \( g \) is superadditive and \( f_1 \in S_n \). Likewise, \( f_2 \in S_n \). Hence, \( p \) is not an extremal element of \( S_n \).

Actually, it can be shown that if \( p \) is defined as in (1.2) and if at least two of the \( A_i \) are additive, then \( p \) is not an extremal element of \( S_n \).

**REFERENCES**


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