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**THE LOCAL RIGIDITY OF THE MODULI SCHEME FOR
CURVES**

ROBERT F. LAX

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Let Y be a smooth, quasi-projective scheme of finite type over an algebraically closed field of characteristic zero. Let X be the quotient of Y by a finite group of automorphisms. Assume that the branch locus of Y over X is of codimension at least 3. In this note, it is shown that X is locally rigid in the following sense: the singular locus of X is stratified and, given a point on a stratum, it is shown that there exists a locally algebraic transverse section to the stratum at the point which is rigid. This result is then applied to the coarse moduli scheme for curves of genus g , where $g > 4$ (in characteristic zero).

1. Stratifying quotient schemes. Let k be an algebraically closed field. Let V' be a smooth, irreducible quasi-projective algebraic k -scheme. By a *quotient scheme*, we mean a scheme $V = V'/G$, where G is a finite group of automorphisms of V' . In [3], Popp defines a stratification of such schemes.

Given a point $P \in V$ and a point $P' \in V'$ lying over P , one may define the inertia group of P' :

$$I(P') = \{ \sigma \in G \mid \sigma x \equiv x \pmod{\mathcal{M}_{P'}}, \text{ for all } x \in \mathcal{O}_{V', P'} \}.$$

If $P'' \in V'$ is another point lying over P , then $I(P')$ and $I(P'')$ are conjugate subgroups of G .

Let Z_p denote the closed subscheme of $\text{Spec}(\mathcal{O}_p)$ which is ramified in the covering $f: V' \rightarrow V$ and let $Z_{p'}$ be the inverse image of Z_p in $\text{Spec}(\mathcal{O}_{p'})$. Denote by Z'_1, \dots, Z'_s those irreducible components of $Z_{p'}$ of dimension $n - 1$ (where $n = \dim V$). Let H_1, \dots, H_s denote the inertia groups of the generic points of Z'_1, \dots, Z'_s respectively and let $H(P')$ denote the subgroup of $I(P')$ generated by the H_i , $i = 1, 2, \dots, s$. (If $s = 0$, put $H(P') = (1)$.) Let

$$\bar{I}(P') = I(P')/H(P')$$

and call this the *small inertia group* of P' . Under the assumption that V' is smooth, Popp shows that $\bar{I}(P')$ is independent of the cover; i.e.,

for any smooth cover $V'' \rightarrow V$, if $P'' \in V''$ is a point lying over P , then $\bar{I}(P'') \cong \bar{I}(P')$. Thus, we may write $\bar{I}(P)$ and speak of the small inertia group of P .

Let W be an irreducible subscheme of V and suppose $P \in W$. Then one says that V is *equisingular at P along W* if the following two conditions hold:

- (1) P is a smooth point of W
- (2) Suppose P' is a point lying over P and W' is the irreducible component of $f^{-1}(W)$ containing P' . Then the canonical homomorphism $\bar{I}(W') \rightarrow \bar{I}(P')$ is a (surjective) isomorphism.

Let

$$\text{Eqs}(V/W) = \{P \in W \mid V \text{ is equisingular at } P \text{ along } W\}.$$

Popp shows, under the assumption that k is of characteristic 0, that this notion of equisingularity satisfies the axioms which any good notion should (cf. [6]).

In particular, given $Q \in V$, let M_Q denote the family of closed, irreducible subschemes W of V such that $Q \in \text{Eqs}(V/W)$. Then the family $\{\text{Eqs}(V/W) \mid W \in M_Q\}$, for fixed Q , has a greatest element called the *stratum* through Q .

Another important property is that if E is a stratum and $P \in E$, then there exists a neighborhood U of P in V and a minimal biholomorphic embedding $\psi : U \rightarrow \mathbb{C}^e$ (where $e = \dim \mathcal{M}_P / \mathcal{M}_P^2$) such that $\psi(U)$ is topologically isomorphic to the direct product of $\psi(U \cap E) = \mathcal{E}$ and a locally algebraic transverse section to \mathcal{E} at $\psi(P)$ (see [3] for details).

The above stratification, in characteristic 0, is really quite neat: if E is a stratum and $P \in E$, then $E = \{Q \mid Q \text{ is analytically isomorphic to } P\}$.

2. The local rigidity of certain quotient schemes.

DEFINITION. Let V be a quotient scheme in characteristic 0. Stratify V as in §1. Then we will say V is *locally rigid* if given a point P on a stratum E , then there is a locally algebraic transverse section to E at P which is rigid.

PROPOSITION 1. *Let a finite group I act by holomorphic automorphisms of \mathbb{C}^m , leaving the origin fixed. If I acts freely outside some I -invariant complex subspace W' (through the origin) of codimension ≥ 3 , then $X = \mathbb{C}^m / I$ is rigid.*

Proof. As is noted in [5], this is a valid generalization of Theorem 3 of [4].

THEOREM 1. *Suppose k is an algebraically closed field of characteristic 0. Let Y be a smooth, quasi-projective algebraic k -scheme and let G be a finite group of automorphisms of Y . Let $X = Y/G$. If the branch locus of Y over X is of codimension at least 3, then X is locally rigid.*

Proof. Suppose x is a point of X . Let I denote the inertia group of x . Note that since there is no ramification in codimension 1, we have $I = \bar{I}$. In a neighborhood of x , we can linearize the action of I (cf. [1], [3]) so that X at x is locally analytically isomorphic to \mathbf{C}^n/I at the point Q which is the image of the origin under the canonical map $\mathbf{C}^n \rightarrow \mathbf{C}^n/I$.

Choose coordinates z_1, \dots, z_n in \mathbf{C}^n such that z_1, \dots, z_r span the fixed space of I (we may do this since the fixed space is linear). Then

$$\mathbf{C}^n/I \cong \text{Spec}(\mathbf{C}[z_1, \dots, z_r] \otimes \mathbf{C}[z_{r+1}, \dots, z_n]^I).$$

The stratum on which Q lies is

$$E = \text{Spec}(\mathbf{C}[z_1, \dots, z_r])$$

and the transverse section we desire is

$$S = \text{Spec}(\mathbf{C}[z_{r+1}, \dots, z_n]^I).$$

Locally at x , the space X is isomorphic to $E \times S$, not just topologically, but analytically as well. It follows from this and our hypotheses that the branch locus of the map $\text{Spec}(\mathbf{C}[z_{r+1}, \dots, z_n]) \rightarrow S$ has codimension at least 3. Hence, applying Proposition 1, we may conclude that S is rigid.

We may apply this theorem to M_g , the coarse moduli scheme for curves of genus g , in characteristic zero. M_g is the quotient of the smooth, higher-level moduli scheme $J_{g,n}$, for n sufficiently large, by the group $GL(2g, \mathbf{Z}/n)$ [2]. In [2], Popp computes the dimension of ramification points of the map $J_{g,n} \rightarrow M_g$. An inspection of his computations shows that, for $g > 4$, the branch locus of this map has codimension at least 3. Applying our theorem then yields:

PROPOSITION 2. *M_g , the coarse moduli scheme for curves of genus g in characteristic 0, is locally rigid if $g > 4$.*

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