THE LOCAL RIGIDITY OF THE MODULI SCHEME FOR CURVES

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Let \( Y \) be a smooth, quasi-projective scheme of finite type over an algebraically closed field of characteristic zero. Let \( X \) be the quotient of \( Y \) by a finite group of automorphisms. Assume that the branch locus of \( Y \) over \( X \) is of codimension at least 3. In this note, it is shown that \( X \) is locally rigid in the following sense: the singular locus of \( X \) is stratified and, given a point on a stratum, it is shown that there exists a locally algebraic transverse section to the stratum at the point which is rigid. This result is then applied to the coarse moduli scheme for curves of genus \( g \), where \( g > 4 \) (in characteristic zero).

1. Stratifying quotient schemes. Let \( k \) be an algebraically closed field. Let \( V \) be a smooth, irreducible quasi-projective algebraic \( k \)-scheme. By a quotient scheme, we mean a scheme \( V = V'/G \), where \( G \) is a finite group of automorphisms of \( V' \). In [3], Popp defines a stratification of such schemes.

Given a point \( P \in V \) and a point \( P' \in V' \) lying over \( P \), one may define the inertia group of \( P' \):

\[
I(P') = \{ \sigma \in G \mid \sigma x \equiv x \pmod{\mathcal{M}_P}, \text{ for all } x \in \mathcal{O}_{V', P} \}.
\]

If \( P'' \in V' \) is another point lying over \( P \), then \( I(P') \) and \( I(P'') \) are conjugate subgroups of \( G \).

Let \( Z_p \) denote the closed subscheme of \( \text{Spec } (\mathcal{O}_P) \) which is ramified in the covering \( f: V' \to V \) and let \( Z_p \) be the inverse image of \( Z_p \) in \( \text{Spec } (\mathcal{O}_P) \). Denote by \( Z'_1, \ldots, Z'_s \) those irreducible components of \( Z_p \) of dimension \( n - 1 \) (where \( n = \dim V \)). Let \( H_1, \ldots, H_s \) denote the inertia groups of the generic points of \( Z'_1, \ldots, Z'_s \) respectively and let \( H(P') \) denote the subgroup of \( I(P') \) generated by the \( H_i, i = 1, 2, \ldots, s \). (If \( s = 0 \), put \( H(P') = (1) \).) Let

\[
\bar{I}(P') = I(P')/H(P')
\]

and call this the small inertia group of \( P' \). Under the assumption that \( V' \) is smooth, Popp shows that \( \bar{I}(P') \) is independent of the cover; i.e.,
for any smooth cover \( V'' \to V \), if \( P'' \in V'' \) is a point lying over \( P \), then \( \overline{I}(P'') = \overline{I}(P') \). Thus, we may write \( \overline{I}(P) \) and speak of the small inertia group of \( P \).

Let \( W \) be an irreducible subscheme of \( V \) and suppose \( P \in W \). Then one says that \( V \) is equisingular at \( P \) along \( W \) if the following two conditions hold:

1. \( P \) is a smooth point of \( W \)
2. Suppose \( P' \) is a point lying over \( P \) and \( W' \) is the irreducible component of \( f^{-1}(W) \) containing \( P' \). Then the canonical homomorphism \( \overline{I}(W') \to \overline{I}(P') \) is a (surjective) isomorphism.

Let

\[
\text{Eqs} \left( \frac{V}{W} \right) = \{ P \in W \mid V \text{ is equisingular at } P \text{ along } W \}.
\]

Popp shows, under the assumption that \( k \) is of characteristic 0, that this notion of equisingularity satisfies the axioms which any good notion should (cf. [6]).

In particular, given \( Q \in V \), let \( M_Q \) denote the family of closed, irreducible subschemes \( W \) of \( V \) such that \( Q \in \text{Eqs} \left( \frac{V}{W} \right) \). Then the family \( \{ \text{Eqs} \left( \frac{V}{W} \right) \mid W \in M_Q \} \), for fixed \( Q \), has a greatest element called the stratum through \( Q \).

Another important property is that if \( E \) is a stratum and \( P \in E \), then there exists a neighborhood \( U \) of \( P \) in \( V \) and a minimal biholomorphic embedding \( \psi : U \to \mathbb{C}^e \) (where \( e = \dim \mathcal{M}_P / \mathcal{M}_P^2 \)) such that \( \psi(U) \) is topologically isomorphic to the direct product of \( \psi(U \cap E) = \mathcal{E} \) and a locally algebraic transverse section to \( \mathcal{E} \) at \( \psi(P) \) (see [3] for details).

The above straification, in characteristic 0, is really quite neat: if \( E \) is a stratum and \( P \in E \), then \( E = \{ Q \mid Q \text{ is analytically isomorphic to } P \} \).

2. The local rigidity of certain quotient schemes.

**Definition.** Let \( V \) be a quotient scheme in characteristic 0. Stratify \( V \) as in §1. Then we will say \( V \) is locally rigid if given a point \( P \) on a stratum \( E \), then there is a locally algebraic transverse section to \( E \) at \( P \) which is rigid.

**Proposition 1.** Let a finite group \( I \) act by holomorphic automorphisms of \( \mathbb{C}^n \), leaving the origin fixed. If \( I \) acts freely outside some \( I \)-invariant complex subspace \( W' \) (through the origin) of codimension \( \geq 3 \), then \( X = \mathbb{C}^n / I \) is rigid.

**Proof.** As is noted in [5], this is a valid generalization of Theorem 3 of [4].
THEOREM 1. Suppose $k$ is an algebraically closed field of characteristic 0. Let $Y$ be a smooth, quasi-projective algebraic $k$-scheme and let $G$ be a finite group of automorphisms of $Y$. Let $X = Y/G$. If the branch locus of $Y$ over $X$ is of codimension at least 3, then $X$ is locally rigid.

Proof. Suppose $x$ is a point of $X$. Let $I$ denote the inertia group of $x$. Note that since there is no ramification in codimension 1, we have $I = \bar{I}$. In a neighborhood of $x$, we can linearize the action of $I$ (cf. [1], [3]) so that $X$ at $x$ is locally analytically isomorphic to $\mathbb{C}^n/I$ at the point $Q$ which is the image of the origin under the canonical map $\mathbb{C}^n \to \mathbb{C}^n/I$.

Choose coordinates $z_1, \cdots, z_n$ in $\mathbb{C}^n$ such that $z_1, \cdots, z_r$ span the fixed space of $I$ (we may do this since the fixed space is linear). Then

$$\mathbb{C}^n/I \equiv \text{Spec}(\mathbb{C}[z_1, \cdots, z_r] \otimes \mathbb{C}[z_{r+1}, \cdots, z_n])'. $$

The stratum on which $Q$ lies is

$$E = \text{Spec}(\mathbb{C}[z_1, \cdots, z_r])$$

and the transverse section we desire is

$$S = \text{Spec}(\mathbb{C}[z_{r+1}, \cdots, z_n]).$$

Locally at $x$, the space $X$ is isomorphic to $E \times S$, not just topologically, but analytically as well. It follows from this and our hypotheses that the branch locus of the map $\text{Spec}(\mathbb{C}[z_{r+1}, \cdots, z_n]) \to S$ has codimension at least 3. Hence, applying Proposition 1, we may conclude that $S$ is rigid.

We may apply this theorem to $M_g$, the coarse moduli scheme for curves of genus $g$, in characteristic zero. $M_g$ is the quotient of the smooth, higher-level moduli scheme $J_{g,n}$, for $n$ sufficiently large, by the group $GL(2g, \mathbb{Z}/n)$ [2]. In [2], Popp computes the dimension of ramification points of the map $J_{g,n} \to M_g$. An inspection of his computations shows that, for $g > 4$, the branch locus of this map has codimension at least 3. Applying our theorem then yields:

PROPOSITION 2. $M_g$, the coarse moduli scheme for curves of genus $g$ in characteristic 0, is locally rigid if $g > 4$.

REFERENCES

1. H. Cartan, *Quotient d'un espace analytique par un groupe d'automorphismes*, in Algebraic


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