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Let G be a finite completely outer group of automorphisms of a perfect ring R. Let Δ be the crossed product of R with G. Then Δ modules which are R projective are Δ projective and Δ submodules which are R direct summands are Δ direct summands.

Let R be a ring with identity. All modules and homomorphisms are unital. Let $G = \{1, \sigma, \cdots, \tau\}$ be a finite group of automorphisms of R.

By $Ru_\sigma, \sigma \in G$, we mean a bi-R module where $r(r'u_\sigma) = (rr')u_\sigma$ and $(r'u_\sigma)r = (r'r^\sigma)u_\sigma$ for $r, r' \in R$ and $r^\sigma = \sigma(r)$. We call G a completely outer group of automorphisms of R, if for each pair $\sigma \neq \tau$ of automorphisms in G, the bi-R modules Ru_σ and Ru_τ have no nonzero isomorphic subquotients. This notion was defined by T. Nakayama in [**3**, p. 203] and Y. Miyashita in [**2**, p. 126].

Let S be the fixed ring of R under G, i.e. $\{r \in R \mid \sigma(r) = r$ for all $\sigma \in G\}$. If G is completely outer, then R over S is a G-Galois extension and the center of R is the centralizer of S in R. See [**2**, Proposition 6.4, p. 127]. Furthermore, if R is a commutative ring and R over S is G-Galois, then G is a completely outer group of automorphisms of R. See [**2**, Theorem 6.6, p. 128]. If R is a simple ring and G contains no inner automorphisms, then G is a completely outer group of automorphisms of R and conversely. See [**2**, Corollary, p. 128].

The crossed product Δ of R with G is $\Sigma_{\sigma \in G} \oplus Ru_\sigma$ with $(xu_\sigma) \times (yu_\tau) = xy^\sigma u_{\sigma\tau}$ for any x and y in R. We can view R as a left Δ module by defining $(xu_\sigma)r = xr^\sigma$ for x and r in R. Thus R is a bi Δ-S module.

We now assume R is a left perfect ring. These rings were studied by H. Bass [**1**].

Let $J(\Delta)$ (respectively $J(R)$) denote the Jacobson radical of Δ (respectively R).

LEMMA 1. $J(\Delta) = J(R)\Delta = \Delta J(R)$. *Thus for any left Δ module M, $J(R)M$ is a Δ submodule of M.*

Proof. Because $\sigma(J(R)) = J(R)$ for all $\sigma \in G$, $J(R)\Delta = \Delta \cdot J(R)$. Thus for any simple, nonzero left Δ module $M, J(R)M$ is a Δ submodule. Now M is a finitely generated R module, since Δ is a finitely generated R module. Nakayama's Lemma then shows

$J(R)M = 0$. Since $J(R)$ annihilates every simple left Δ module, $J(R) \subseteq J(\Delta)$.

Since $J(\Delta)$ is a bi-R submodule of $\Delta, J(\Delta) = J(\Delta) \cap Ru_1 + J(\Delta) \cap Ru_\sigma + \cdots + J(\Delta) \cap Ru_\tau$. See [2, Proposition 6.1, p. 126]. Let $x = \delta = ru_\sigma \in J(\Delta) \cap Ru_\sigma$ for δ in $J(\Delta)$ and $r \in R$. Assume $\delta = \Sigma_{\sigma \in G}\, y_\sigma u_\sigma$, $y_\sigma \in R$, then $x = y_\sigma u_\sigma = ru_\sigma$, so $ru_1 = y_\sigma u_1 = xu_{\sigma^{-1}} \in J(\Delta) \cap Ru_1$. It follows that $1 - y_\sigma s$ is right invertible in R for all s in R; since $u_1 - y_\sigma s u_1$ in $J(\Delta)$ all s in R. Thus $y_\sigma \in J(R)$; hence $x = y_\sigma u_\sigma \in J(R)u_\sigma$. Therefore $J(R)\Delta = J(\Delta)$.

REMARK. Lemma 1 is true even if R is not left perfect.

LEMMA 2. *As a right S module, S is a direct summand of R_S. Let $J(S)$* (respectively $J(R)$) *denote the Jacobson radical of S* (respectively of R), *then $J(S) = S \cap J(R)$ and $J(R) = J(S)R = RJ(S)$. Thus if M is a left $\Delta(R)$ module $J(S)M$ is a left $\Delta(R)$ submodule.*

Proof. Let $\bar{\Delta} = \Delta/J(\Delta)$ and $\bar{R} = R/J(R)$. Because R is perfect, $\bar{R}$ is a semisimple, Artinian ring, which makes $\bar{\Delta}$ a semisimple, Artinian ring. Thus $\bar{R}$ is a finitely generated, projective $\bar{\Delta}$ module. By the Dual Basis Lemma, there exists $f_1, \cdots, f_n \in \mathrm{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta})$ and $\bar{x}_1, \cdots, \bar{x}_n \in \bar{R}$ such that $\bar{x} = \Sigma_{i=1}^n f_i(\bar{x})\bar{x}_i$. Since $\mathrm{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta}) \subseteq \mathrm{Hom}_{\bar{R}}(\bar{R}, \bar{\Delta}) = \bar{\Delta}$ we conclude $\mathrm{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta}) = \Sigma_{\sigma \in G} u_\sigma \bar{R}$. Thus each f_i, $i = 1, \cdots, n$, is of the form $\Sigma_{\sigma \in G}\, u_\sigma \bar{r}_i$, for some suitable $\bar{r}_i \in R$. Let $\bar{x} \in \bar{R}$, then $\bar{x} = \Sigma_{i=1}^n f_i(\bar{x})\bar{x}_i = \Sigma_{i=1}^n (\Sigma_{\sigma \in G}\, \bar{x} u_\sigma \bar{r}_i)\bar{x}_i = \bar{x}\, \Sigma_{i=1}^n \Sigma_{\sigma \in G}\, (\bar{r}_i \bar{x}_i)^\sigma$. Thus $\bar{1} = \Sigma_{\sigma \in G}\, \Sigma_{i=1}^n (\bar{r}_i \bar{x}_i)^\sigma$. Let $\bar{d} = \Sigma_{i=1}^n \overline{r_i x_i}$, then tr $\bar{d} = \bar{1}$; hence tr $d - 1 \in J(R) \cap S$.

Now $J(R) \cap S \subseteq J(S)$, for let $x = j = s$, $j \in J(R)$ and $s \in S$, then $1 - sy$ is right invertible in R for any y in S. Assume $(1 - sy)z = 1$, $z \in R$, then $(1 - sy)z^\sigma = 1$ for all $\sigma \in G$; hence $z \in S$. So $1 - sy$ is right invertible in S for all y in S, thus $x \in J(S)$ or $J(R) \cap S \subseteq J(S)$.

Thus tr $R + J(S) = S$, so by Nakayama's Lemma tr $R = S$. Thus there is a c in R such that tr $c = 1$. Hence $\mathrm{tr}: R_S \to S_S$ is onto and so splits. Thus S_S is a direct summand of R_S.

The conclusion concerning the Jacobson radical of S follows from [2, Theorem 7.10, p. 132].

PROPOSITION 1. *A left Δ module is completely reducible as a Δ module if and only if it is completely reducible as an R module. Moreover, a module is completely reducible as a left R-module if and only if it is completely reducible as an S module.*

Proof. A Δ module is annihilated by $J(\Delta)$ if and only if it is annihilated by $J(R)$ if and only if it is annihilated by $J(S)$.

PROPOSITION 2. *Let R be a left perfect ring and G a completely outer group of automorphisms acting on R. Then S, the fixed ring of R under G, and Δ, the crossed product of R with G are left perfect.*

Proof. Since R is left perfect its Jacobson radical $J(R)$, is left T nilpotent. So the Jacobson radical of $S, J(S)$, which is contained in $J(R)$ (by Lemma 2) is left T nilpotent.

Also $S/J(S)$ is an Artinian ring since S_S is a direct summand of R_S. See Lemma 2 and [2, Proposition 7.3, p. 130]. Thus S is a left perfect; hence, S is semiperfect.

Now R as a right S module is finitely generated and projective; moreover, Δ is isomorphic to End R_S [2, p. 116]. Since S is a direct summand of R, as a right S module (Lemma 2) R is an S generator.

Let $e_1, \cdots, e_n$ be completely primitive idempotents orthogonal idempotents of S such that $1 = e_1 + \cdots + e_n$. Furthermore, let $e_1, \cdots, e_k$ be a maximal family of mutually nonisomorphic idempotents. Then $\bar{R} = R/J(S)R$ is isomorphic, as a right S module, to $\Sigma_{i=1}^{k} + (\bar{e}_i \cdot \bar{S})^{m_i}$, where $\bar{e}_i = e_i + J(S)$, $\bar{S} = S/J(S)$ and $m_i < \infty$, since $\bar{R}$ is finitely generated right S module. Thus R as a right S module, is isomorphic to $\Sigma_{i=1}^{k} \oplus \Sigma_{i=1}^{m_i} \oplus P_{ij}$, where P_{ij} is right S isomorphic to e_iS, since idempotents can be lifted.

Let f_{ij} be the projection of R onto P_{ij}, then $f_{ij} \in$ End $R_S = \Delta$ and the f_{ij}'s are orthogonal idempotents such that

$$1 = \sum_{i=1}^{k} \sum_{j=1}^{m_i} f_{ij}. \quad \text{Also } e_iSe_i = \text{End}_S(e_iS) = \text{End}_S(P_{ij}) = f_{ij}\,\Delta f_{ij}.$$

Since e_iSe_i is a local ring, $f_{ij}\,\Delta f_{ij}$ is a local ring. Hence f_{ij} is a completely primitive idempotent; therefore Δ is semiperfect.

We know that Δ modulo its Jacobson radical, $J(\Delta)$, is semisimple and idempotents can be lifted modulo $J(\Delta)$. Let M be a left Δ module, by [1, Lemma 2.6, p. 473] in order that M have a projective cover it suffices that for any left Δ module B requiring no more generators than $M, B = J(\Delta)B$ implies $B = 0$. But $B = J(\Delta)B = J(R)B$ and R being left perfect implies $B = 0$. Thus every left Δ module M has a projective cover and Δ is then left perfect.

Let T be an arbitrary left perfect ring and let $J(T)$ denote the Jacobson radical of T. Then for any nonzero left T module $M, J(T)M$ is a proper submodule. See [1, p. 473]. Hence the natural map $\pi : M \to M/J(T)M$ is a minimal epimorphism.

Let M and N be left T modules and f a left T epimorphism from M to N. By $\bar{f}$, we mean the induced map from $M/J(T)M \to N/J(T)N$ given by $\bar{f}(m + J(T)M) = f(m) + J(T)N$ for $m \in M$.

LEMMA 3. *The following are equivalent:*

(1) *$f: M \to N$ is a minimal T epimorphism.*

(2) *$\bar{f}: M/J(T)M \to N/J(T)N$ is an isomorphism. See* [4, Proposition 8, p. 713].

PROPOSITION 3. *Let M and N be left Δ modules and f a minimal Δ epimorphism form M to N. Then f is a minimal R epimorphism and f is a minimal S epimorphism.*

Proof. Certainly f is an R and an S epimorphism. Since $J(\Delta)M = J(R)M = J(S)M$ and $J(\Delta)N = J(R)N = J(S)N$, then $\bar{f}$, which is a Δ isomorphism, is an R and an S isomorphism.

PROPOSITION 4. *Let M be a left Δ module which is projective as an S module, then M is projective as a Δ module.*

Proof. Let P be the Δ cover of M and $f: P \to M$ a minimal Δ epimorphism. Since M is S projective, f splits as an S epimorphism. Thus P as an S module is isomorphic to $\ker f + X$, for some S submodule X of P. But f is a minimal S epimorphism (Proposition 3), therefore $\ker f = 0$. So f is a Δ isomorphism and M is Δ projective.

PROPOSITION 5. *Let M be a left Δ module which is projective as an R module. Then M is projective as a Δ module.*

Proof. Let P be the Δ projective cover of M and f a minimal Δ epimorphism. Now f splits as an R map, and f is a minimal R epimorphism, so $\ker f = 0$.

PROPOSITION 6. *Let M and N be left Δ modules such that $M/J(\Delta)M$ and $N/J(\Delta)N$ are isomorphic as R modules. If M is R projective there exists a Δ epimorphism $\phi: M \to N$. Moreover, if N is R projective, then M and N are Δ isomorphic. See* [3, Lemma 5, p. 212].

Proof. We assume that M and N are completely reducible Δ modules. Hence they are completely reducible R-modules, by Proposition 1.

Now Nakayama in [3, p. 214] has shown that if M and N are isomorphic as R modules, then they are isomorphic as Δ modules.

If M and N are arbitrary left Δ modules, then $M/J(\Delta)M$ and $N/J(\Delta)N$ are nonzero, completely reducible left Δ modules. We have assumed they are isomorphic as left R modules; hence by the above argument, they are isomorphic as left Δ modules. Call the isomorphisms from $M/J(\Delta)M$ to $N/J(\Delta)N, f$.

Let $\pi: M \to M/J(\Delta)M$ and $\pi': N \to N/J(\Delta)N$ be the natural maps. Since M is R projective, it is Δ projective. Thus we can find a Δ homomorphism g from M to N such that $\pi' g = f\pi$. Now g is an epimorphism since π' is a minimal Δ map.

If N is also R projective, it is Δ projective. Thus g is an isomorphism.

PROPOSITION 7. *Let M and N be left Δ modules and f a Δ epimorphism from M to N. If f is a minimal epimorphism as an R map, then f is a minimal epimorphism as a Δ map. Furthermore, if M and N are R projective, then f is an isomorphism.*

Proof. We know that $M/J(R)M$ and $N/J(R)N$ are isomorphic as R modules and completely reducible. Hence they are isomorphic as Δ modules. Thus f is a minimal Δ map.

If M and N are R projective, then they are projective as Δ modules. Thus f splits; let $g: N \to M$ be a Δ map such that fg is the identity on N. Since the natural map $\pi: M \to M/J(\Delta)M$ is minimal g is an isomorphism.

PROPOSITION 8. *Let M, N be left Δ modules such that M is a projective R module. If N is an R direct summand of M, then N is a Δ direct summand of M.*

Proof. Since M is R projective and N is a direct summand, then N is R projective. Hence M and N are Δ projective.

If N is an R direct summand of M, then $J(R)M \cap N = J(R)N$ so $J(\Delta)M \cap N = J(\Delta)N$. Thus $N/J(\Delta)N$ is a Δ submodule of the completely reducible Δ module $M/J(\Delta)M$.

Thus $N/J(\Delta)N$ is a Δ direct summand of $M/J(\Delta)M$, so there is a Δ epimorphism $\phi: M \to N/J(\Delta)N$. Let $\pi': N \to N/J(\Delta)N$, be the natural map. Since M is Δ projective there exists a Δ map $\psi: M \to N$ such that $\pi'\psi = \phi$. Now ψ is an epimorphism since π' is minimal. Thus ψ splits as a Δ map and N is then a Δ direct summand of M.

PROPOSITION 9. *Let M and N be left Δ modules such that $N/J(R)N$ is an R homomorphic image of $M/J(R)M$ and M is projective as an R module. Then N is a Δ homomorphic image of M. Moreover,* *if N is R projective, then N is a Δ direct summand of M.*

Proof. Let f be an R epimorphism from $M/J(R)M$ to $N/J(R)N$. Since $R/J(R)$ is a semisimple, Artinian ring, f splits; hence $N/J(R)N$ is a R direct summand of $M/J(R)M$.

Now G is a completely outer group of automorphisms of $R/J(R)$. The crossed product of $R/J(R)$ and G is $\Delta/J(\Delta)$.

Applying Proposition 8 we see that $N/J(R)N$ is a Δ direct summand of $M/J(R)M$. Thus there is a Δ map ϕ from M to $N/J(R)N$. Let $\pi' : N \to N/J(R)N$ be the natural map. Since M is R projective, there is a Δ map $g : M \to N$ such that $\pi' g = \phi$. Now g is an epimorphism, since π' is a minimal Δ map.

If N is R projective, then N is Δ projective. So g splits.

PROPOSITION 10. *Let $g = |G|$, then R^g is Δ isomorphic to Δ.*

Proof. $(R/J(R))^g$ is R isomorphic to $\Delta/J(R)\Delta$.

Thus Proposition 6 implies R^g and Δ are isomorphic.

PROPOSITION 11. *R has a normal basis.*

Proof. Proposition 10 imples S^g is S isomorphic to R so R has a normal basis by [2, Theorem 1.7, p. 118].

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