COMPLETELY OUTER GALOIS THEORY OF PERFECT RINGS

JAMES MICHAEL OSTERBURG
Let $G$ be a finite completely outer group of automorphisms of a perfect ring $R$. Let $\Delta$ be the crossed product of $R$ with $G$. Then $\Delta$ modules which are $R$ projective are $\Delta$ projective and $\Delta$ submodules which are $R$ direct summands are $\Delta$ direct summands.

Let $R$ be a ring with identity. All modules and homomorphisms are unital. Let $G = \{1, \sigma, \cdots, \tau\}$ be a finite group of automorphisms of $R$. By $Ru_\sigma$, $\sigma \in G$, we mean a bi-$R$ module where $r (r'u_\sigma) = (rr')u_\sigma$ and $(r'u_\sigma)r = (r'r^\sigma)u_\sigma$ for $r, r' \in R$ and $r^\sigma = \sigma(r)$. We call $G$ a completely outer group of automorphisms of $R$, if for each pair $\sigma \neq \tau$ of automorphisms in $G$, the bi-$R$ modules $Ru_\sigma$ and $Ru_\tau$ have no nonzero isomorphic subquotients. This notion was defined by T. Nakayama in [3, p. 203] and Y. Miyashita in [2, p. 126].

Let $S$ be the fixed ring of $R$ under $G$, i.e. $\{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}$. If $G$ is completely outer, then $R$ over $S$ is a $G$-Galois extension and the center of $R$ is the centralizer of $S$ in $R$. See [2, Proposition 6.4, p. 127]. Furthermore, if $R$ is a commutative ring and $R$ over $S$ is $G$-Galois, then $G$ is a completely outer group of automorphisms of $R$. See [2, Theorem 6.6, p. 128]. If $R$ is a simple ring and $G$ contains no inner automorphisms, then $G$ is a completely outer group of automorphisms of $R$ and conversely. See [2, Corollary, p. 128].

The crossed product $\Delta$ of $R$ with $G$ is $\bigoplus_{\sigma \in G} Ru_\sigma$ with $(xu_\sigma) \times (yu_\tau) = xy^\sigma u_{\sigma \tau}$ for any $x$ and $y$ in $R$. We can view $R$ as a left $\Delta$ module by defining $(xu_\sigma)r = xr^\sigma$ for $x$ and $r$ in $R$. Thus $R$ is a bi $\Delta$-$S$ module.

We now assume $R$ is a left perfect ring. These rings were studied by H. Bass [1].

Let $J(\Delta)$ (respectively $J(R)$) denote the Jacobson radical of $\Delta$ (respectively $R$).

**Lemma 1.** $J(\Delta) = J(R) \Delta = \Delta J(R)$. Thus for any left $\Delta$ module $M$, $J(R)M$ is a $\Delta$ submodule of $M$.

**Proof.** Because $\sigma(J(R)) = J(R)$ for all $\sigma \in G$, $J(R) \Delta = \Delta \cdot J(R)$. Thus for any simple, nonzero left $\Delta$ module $M$, $J(R)M$ is a $\Delta$ submodule. Now $M$ is a finitely generated $R$ module, since $\Delta$ is a finitely generated $R$ module. Nakayama's Lemma then shows
$J(R)M = 0$. Since $J(R)$ annihilates every simple left $\Delta$ module, $J(R) \subseteq J(\Delta)$.

Since $J(\Delta)$ is a bi-$R$ submodule of $\Delta$, $J(\Delta) = J(\Delta) \cap Ru_1 + J(\Delta) \cap Ru_2 + \cdots + J(\Delta) \cap Ru_n$. See [2, Proposition 6.1, p. 126]. Let $x = \delta = ru_\sigma \in J(\Delta) \cap Ru_\sigma$ for $\delta$ in $J(\Delta)$ and $r \in R$. Assume $\delta = \sum_{\sigma} \gamma_\sigma u_\sigma$, $y_\sigma \in R$, then $x = y_\sigma u_\sigma = ru_\sigma$, so $ru_1 = y_\sigma u_1 = xu_\sigma \sigma \in J(\Delta) \cap Ru_1$. It follows that $1 - y_\sigma s$ is right invertible in $R$ for all $s$ in $R$; since $u_1 - y_\sigma su_1$ in $J(\Delta)$ all $s$ in $R$. Thus $y_\sigma \in J(R)$; hence $x = y_\sigma u_\sigma \in J(R)u_\sigma$. Therefore $J(R)\Delta = \Delta(\Delta)$.

**REMARK.** Lemma 1 is true even if $R$ is not left perfect.

**LEMMA 2.** As a right $S$ module, $S$ is a direct summand of $R_S$. Let $J(S)$ (respectively $J(R)$) denote the Jacobson radical of $S$ (respectively of $R$), then $J(S) = S \cap J(R)$ and $J(R) = J(S)R = RJ(S)$. Thus if $M$ is a left $\Delta(R)$ module $J(S)M$ is a left $\Delta(R)$ submodule.

**Proof.** Let $\tilde{\Delta} = \Delta / J(\Delta)$ and $\tilde{R} = R / J(R)$. Because $R$ is perfect, $\tilde{R}$ is a semisimple, Artinian ring, which makes $\tilde{\Delta}$ a semisimple, Artinian ring. Thus $\tilde{\Delta}$ is a finitely generated, projective $\tilde{\Delta}$ module. By the Dual Basis Lemma, there exists $f_1, \cdots, f_n \in \text{Hom}_R(\tilde{R}, \tilde{\Delta})$ and $\tilde{x}_1, \cdots, \tilde{x}_n \in \tilde{R}$ such that $\tilde{x} = \sum_{i=1}^n f_i(\tilde{x})\tilde{x}_i$. Since $\text{Hom}_R(\tilde{R}, \tilde{\Delta}) \subseteq \text{Hom}_R(\tilde{\Delta}, \tilde{\Delta}) = \tilde{\Delta}$ we conclude $\text{Hom}_R(\tilde{R}, \tilde{\Delta}) = \sum_{\sigma} \gamma_{\sigma} u_\sigma \tilde{R}$. Thus each $f_i, i = 1, \cdots, n$, is of the form $\sum_{\sigma} \gamma_{\sigma} u_\sigma r_{\sigma}$, for some suitable $r_{\sigma} \in R$. Let $\tilde{x} \in \tilde{R}$, then $\tilde{x} = \sum_{i=1}^n f_i(\tilde{x})\tilde{x}_i = \sum_{i=1}^n (\sum_{\sigma} \gamma_{\sigma} u_\sigma r_{\sigma})\tilde{x}_i = \tilde{x} \sum_{i=1}^n \sigma u_\sigma (r_{\sigma})^\sigma$. Thus $1 = \sum_{\sigma} \gamma_{\sigma} (r_{\sigma})^\sigma$. Let $\tilde{d} = \sum_{i=1}^n r_{\sigma} \tilde{x}_i$, then $\text{tr} \tilde{d} = \tilde{1}$; hence $\text{tr} \tilde{d} - 1 \in J(\Delta) \cap S$.

Now $J(R) \cap S \subseteq J(S)$, for let $x = j = s, j \in J(R)$ and $s \in S$, then $1 - sy$ is right invertible in $R$ for any $y$ in $S$. Assume $(1 - sy)z = 1$, $z \in R$, then $(1 - sy)z^\sigma = 1$ for all $\sigma \in G$; hence $z \in S$. So $1 - sy$ is right invertible in $S$ for all $y$ in $S$, thus $x \in J(S)$ or $J(R) \cap S \subseteq J(S)$.

Thus $\text{tr} R + J(S) = S$, so by Nakayama’s Lemma $\text{tr} R = S$. Thus there is a $c$ in $R$ such that $\text{tr} c = 1$. Hence $\text{tr} : R_S \rightarrow S_S$ is onto and so splits. Thus $S_S$ is a direct summand of $R_S$.

The conclusion concerning the Jacobson radical of $S$ follows from [2, Theorem 7.10, p. 132].

**PROPOSITION 1.** A left $\Delta$ module is completely reducible as a $\Delta$ module if and only if it is completely reducible as an $R$ module. Moreover, a module is completely reducible as a left $R$-module if and only if it is completely reducible as an $S$ module.

**Proof.** A $\Delta$ module is annihilated by $J(\Delta)$ if and only if it is annihilated by $J(R)$ if and only if it is annihilated by $J(S)$.
PROPOSITION 2. Let $R$ be a left perfect ring and $G$ a completely outer group of automorphisms acting on $R$. Then $S$, the fixed ring of $R$ under $G$, and $\Delta$, the crossed product of $R$ with $G$ are left perfect.

Proof. Since $R$ is left perfect its Jacobson radical $J(R)$, is left $T$ nilpotent. So the Jacobson radical of $S, J(S)$, which is contained in $J(R)$ (by Lemma 2) is left $T$ nilpotent.

Also $S/J(S)$ is an Artinian ring since $S_S$ is a direct summand of $R_S$. See Lemma 2 and [2, Proposition 7.3, p. 130]. Thus $S$ is a left perfect; hence, $S$ is semiperfect.

Now $R$ as a right $S$ module is finitely generated and projective; moreover, $\Delta$ is isomorphic to $\text{End } R_S$ [2, p. 116]. Since $S$ is a direct summand of $R$, as a right $S$ module (Lemma 2) $R$ is an $S$ generator.

Let $e_1, \cdots, e_n$ be completely primitive idempotents orthogonal idempotents of $S$ such that $1 = e_1 + \cdots + e_n$. Furthermore, let $e_1, \cdots, e_k$ be a maximal family of mutually nonisomorphic idempotents. Then $\bar{R} = R/J(S)R$ is isomorphic, as a right $S$ module, to $\Sigma_{i=1}^k + (\tilde{e}_i \cdot \tilde{S})^{m_i}$, where $\tilde{e}_i = e_i + J(S)$, $\tilde{S} = S/J(S)$ and $m_i < \infty$ since $\bar{R}$ is finitely generated right $S$ module. Thus $R$ as a right $S$ module, is isomorphic to $\Sigma_{i=1}^k \oplus \Sigma_{i=1}^{m_i} \oplus P_{ij}$, where $P_{ij}$ is right $S$ isomorphic to $e_i S$, since idempotents can be lifted.

Let $f_{ij}$ be the projection of $R$ onto $P_{ij}$, then $f_{ij} \in \text{End } R_S = \Delta$ and the $f_{ij}$'s are orthogonal idempotents such that

$$1 = \sum_{i=1}^k \sum_{j=1}^{m_i} f_{ij}.$$  

Also $e_i S e_i = \text{End}_S(e_i S) = \text{End}_S(P_{ij}) = f_{ij} \Delta f_{ij}$.

Since $e_i S e_i$ is a local ring, $f_{ij} \Delta f_{ij}$ is a local ring. Hence $f_{ij}$ is a completely primitive idempotent; therefore $\Delta$ is semiperfect.

We know that $\Delta$ modulo its Jacobson radical, $J(\Delta)$, is semisimple and idempotents can be lifted modulo $J(\Delta)$. Let $M$ be a left $\Delta$ module, by [1, Lemma 2.6, p. 473] in order that $M$ have a projective cover it suffices that for any left $\Delta$ module $B$ requiring no more generators than $M, B = J(\Delta)B$ implies $B = 0$. But $B = J(\Delta)B = J(R)B$ and $R$ being left perfect implies $B = 0$. Thus every left $\Delta$ module $M$ has a projective cover and $\Delta$ is then left perfect.

Let $T$ be an arbitrary left perfect ring and let $J(T)$ denote the Jacobson radical of $T$. Then for any nonzero left $T$ module $M, J(T)M$ is a proper submodule. See [1, p. 473]. Hence the natural map $\pi : M \to M/J(T)M$ is a minimal epimorphism.

Let $M$ and $N$ be left $T$ modules and $f$ a left $T$ epimorphism from $M$ to $N$. By $\bar{f}$, we mean the induced map from $M/J(T)M \to N/J(T)N$ given by $\bar{f}(m + J(T)M) = f(m) + J(T)N$ for $m \in M$. 

LEMMA 3. The following are equivalent:

(1) \( f : M \rightarrow N \) is a minimal \( T \) epimorphism.

(2) \( \bar{f} : M/J(T)M \rightarrow N/J(T)N \) is an isomorphism. See [4, Proposition 8, p. 713].

PROPOSITION 3. Let \( M \) and \( N \) be left \( \Delta \) modules and \( f \) a minimal \( \Delta \) epimorphism form \( M \) to \( N \). Then \( f \) is a minimal \( R \) epimorphism and \( f \) is a minimal \( S \) epimorphism.

Proof. Certainly \( f \) is an \( R \) and an \( S \) epimorphism. Since \( J(\Delta)M = J(R)M = J(S)M \) and \( J(\Delta)N = J(R)N = J(S)N \), then \( \bar{f} \), which is a \( \Delta \) isomorphism, is an \( R \) and an \( S \) isomorphism.

PROPOSITION 4. Let \( M \) be a left \( \Delta \) module which is projective as an \( S \) module, then \( M \) is projective as a \( \Delta \) module.

Proof. Let \( P \) be the \( \Delta \) cover of \( M \) and \( f : P \rightarrow M \) a minimal \( \Delta \) epimorphism. Since \( M \) is \( S \) projective, \( f \) splits as an \( S \) epimorphism. Thus \( P \) as an \( S \) module is isomorphic to \( \ker f + X \), for some \( S \) submodule \( X \) of \( P \). But \( f \) is a minimal \( S \) epimorphism (Proposition 3), therefore \( \ker f = 0 \). So \( f \) is a \( \Delta \) isomorphism and \( M \) is \( \Delta \) projective.

PROPOSITION 5. Let \( M \) be a left \( \Delta \) module which is projective as an \( R \) module. Then \( M \) is projective as a \( \Delta \) module.

Proof. Let \( P \) be the \( \Delta \) projective cover of \( M \) and \( f \) a minimal \( \Delta \) epimorphism. Now \( f \) splits as an \( R \) map, and \( f \) is a minimal \( R \) epimorphism, so \( \ker f = 0 \).

PROPOSITION 6. Let \( M \) and \( N \) be left \( \Delta \) modules such that \( M/J(\Delta)M \) and \( N/J(\Delta)N \) are isomorphic as \( R \) modules. If \( M \) is \( R \) projective there exists a \( \Delta \) epimorphism \( \phi : M \rightarrow N \). Moreover, if \( N \) is \( R \) projective, then \( M \) and \( N \) are \( \Delta \) isomorphic. See [3, Lemma 5, p. 212].

Proof. We assume that \( M \) and \( N \) are completely reducible \( \Delta \) modules. Hence they are completely reducible \( R \)-modules, by Proposition 1.

Now Nakayama in [3, p. 214] has shown that if \( M \) and \( N \) are isomorphic as \( R \) modules, then they are isomorphic as \( \Delta \) modules.

If \( M \) and \( N \) are arbitrary left \( \Delta \) modules, then \( M/J(\Delta)M \) and \( N/J(\Delta)N \) are nonzero, completely reducible left \( \Delta \) modules. We have
assumed they are isomorphic as left $R$ modules; hence by the above argument, they are isomorphic as left $\Delta$ modules. Call the isomorphisms from $M/J(\Delta)M$ to $N/J(\Delta)N, f$.

Let $\pi : M \rightarrow M/J(\Delta)M$ and $\pi' : N \rightarrow N/J(\Delta)N$ be the natural maps. Since $M$ is $R$ projective, it is $\Delta$ projective. Thus we can find a $\Delta$ homomorphism $g$ from $M$ to $N$ such that $\pi'g = f\pi$. Now $g$ is an epimorphism since $\pi'$ is a minimal $\Delta$ map.

If $N$ is also $R$ projective, it is $\Delta$ projective. Thus $g$ is an isomorphism.

**Proposition 7.** Let $M$ and $N$ be left $\Delta$ modules and $f$ a $\Delta$ epimorphism from $M$ to $N$. If $f$ is a minimal epimorphism as an $R$ map, then $f$ is a minimal epimorphism as a $\Delta$ map. Furthermore, if $M$ and $N$ are $R$ projective, then $f$ is an isomorphism.

**Proof.** We know that $M/J(R)M$ and $N/J(R)N$ are isomorphic as $R$ modules and completely reducible. Hence they are isomorphic as $\Delta$ modules. Thus $f$ is a minimal $\Delta$ map.

If $M$ and $N$ are $R$ projective, then they are projective as $\Delta$ modules. Thus $f$ splits; let $g : N \rightarrow M$ be a $\Delta$ map such that $fg$ is the identity on $N$. Since the natural map $\pi : M \rightarrow M/J(\Delta)M$ is minimal $g$ is an isomorphism.

**Proposition 8.** Let $M, N$ be left $\Delta$ modules such that $M$ is a projective $R$ module. If $N$ is an $R$ direct summand of $M$, then $N$ is a $\Delta$ direct summand of $M$.

**Proof.** Since $M$ is $R$ projective and $N$ is a direct summand, then $N$ is $R$ projective. Hence $M$ and $N$ are $\Delta$ projective.

If $N$ is an $R$ direct summand of $M$, then $J(R)M \cap N = J(R)N$ so $J(\Delta)M \cap N = J(\Delta)N$. Thus $N/J(\Delta)N$ is a $\Delta$ submodule of the completely reducible $\Delta$ module $M/J(\Delta)M$.

Thus $N/J(\Delta)N$ is a $\Delta$ direct summand of $M/J(\Delta)M$, so there is a $\Delta$ epimorphism $\phi : M \rightarrow N/J(\Delta)N$. Let $\pi' : N \rightarrow N/J(\Delta)N$, be the natural map. Since $M$ is $\Delta$ projective there exists a $\Delta$ map $\psi : M \rightarrow N$ such that $\pi'\psi = \phi$. Now $\psi$ is an epimorphism since $\pi'$ is minimal. Thus $\psi$ splits as a $\Delta$ map and $N$ is then a $\Delta$ direct summand of $M$.

**Proposition 9.** Let $M$ and $N$ be left $\Delta$ modules such that $N/J(R)N$ is an $R$ homomorphic image of $M/J(R)M$ and $M$ is projective as an $R$ module. Then $N$ is a $\Delta$ homomorphic image of $M$. Moreover,
if $N$ is $R$ projective, then $N$ is a $\Delta$ direct summand of $M$.

**Proof.** Let $f$ be an $R$ epimorphism from $M/J(R)M$ to $N/J(R)N$. Since $R/J(R)$ is a semisimple, Artinian ring, $f$ splits; hence $N/J(R)N$ is a $R$ direct summand of $M/J(R)M$.

Now $G$ is a completely outer group of automorphisms of $R/J(R)$. The crossed product of $R/J(R)$ and $G$ is $\Delta/J(\Delta)$.

Applying Proposition 8 we see that $N/J(R)N$ is a $\Delta$ direct summand of $M/J(R)M$. Thus there is a $\Delta$ map $\phi$ from $M$ to $N/J(R)N$. Let $\pi': N \to N/J(R)N$ be the natural map. Since $M$ is $R$ projective, there is a $\Delta$ map $g: M \to N$ such that $\pi'g = \phi$. Now $g$ is an epimorphism, since $\pi'$ is a minimal $\Delta$ map.

If $N$ is $R$ projective, then $N$ is $\Delta$ projective. So $g$ splits.

**PROPOSITION 10.** Let $g = |G|$, then $R^g$ is $\Delta$ isomorphic to $\Delta$.

**Proof.** $(R/J(R))^g$ is $R$ isomorphic to $\Delta/J(R)\Delta$.

Thus Proposition 6 implies $R^g$ and $\Delta$ are isomorphic.

**PROPOSITION 11.** $R$ has a normal basis.

**Proof.** Proposition 10 imples $S^g$ is $S$ isomorphic to $R$ so $R$ has a normal basis by [2, Theorem 1.7, p. 118].

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Received November 19, 1973.

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