

# Pacific Journal of Mathematics

**MODEL-COMPLETENESS IN A FIRST ORDER LANGUAGE  
WITH A GENERALIZED QUANTIFIER**

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## MODEL-COMPLETENESS IN A FIRST ORDER LANGUAGE WITH A GENERALIZED QUANTIFIER

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**The concept of Model-Completeness is defined in a first order language with a generalized quantifier. A necessary and sufficient condition is given for that Model-Completeness and its relation to categoricity is discussed.**

Some results of this paper were obtained in the author's thesis [12] and were announced in [11]. They, together with other results of [12] were improved independently by the author and by S. Shelah. A suggestion of S. Shelah made some proofs simpler and due to it, better results were obtained in Theorem 1.5. The author wishes to thank S. Shelah for his remarks.

Let  $L$  be a first order language with equality and let  $L(Q)$  be the language obtained from  $L$  by adding a new quantifier  $Q$ . Let  $\alpha, \beta$  denote infinite cardinals. We define  $\alpha$ -satisfaction for  $L(Q)$  by interpreting  $Q$  as "there exist at least  $\alpha$  elements". If a sentence  $\phi$  of  $L(Q)$  is  $\alpha$ -satisfied in a model  $\mathfrak{A}$  for  $L$  we write  $\mathfrak{A} \models_{\alpha} \phi$  and we say that  $\mathfrak{A}$  is an  $\alpha$ -model for  $\phi$ . Let  $\mathfrak{A}, \mathfrak{B}$  be two models for  $L$ ,  $|\mathfrak{A}| \geq \alpha$  and  $\mathfrak{A} \subseteq \mathfrak{B}$ . Write  $\mathfrak{A} <_{\alpha} \mathfrak{B}$  if for every  $n$ , every formula  $\phi(x_1, \dots, x_n)$  in  $L(Q)$  and every  $a_1, \dots, a_n$  in  $\mathfrak{A}$ :  $\mathfrak{A} \models_{\alpha} \phi[a_1, \dots, a_n]$  iff  $\mathfrak{B} \models_{\alpha} \phi[a_1, \dots, a_n]$ . Let  $T$  be an ordinary first order theory (namely a theory in  $L$ ) that has infinite models. Define  $T(Q) = T \cup \{Qx[x = x]\}$ . Call  $T$   $\alpha$ -model-complete if for every  $\mathfrak{A}, \mathfrak{B}$  which are  $\alpha$ -models for  $T(Q)$  and  $\mathfrak{A} \subseteq \mathfrak{B}$  also  $\mathfrak{A} <_{\alpha} \mathfrak{B}$ . A necessary and sufficient condition for  $T$  to be  $\alpha$ -model-complete for  $\alpha > \aleph_0$  is given in section 1.

Let  $T$  be as before. Define  $T(\alpha) = \{\phi : \phi \text{ is a sentence in } L(Q) \text{ and for every } \mathfrak{A}, \text{ if } \mathfrak{A} \models_{\alpha} T(Q) \text{ then } \mathfrak{A} \models_{\alpha} \phi\}$ . Call  $T$   $\alpha$ -complete if for every sentence  $\phi$  in  $L(Q)$  either  $\phi \in T(\alpha)$  or  $\neg\phi \in T(\alpha)$ . In §2, it is shown that if  $T$  is categorical in one uncountable power, it is  $\alpha$ -complete and for every  $\alpha \geq \aleph_0$ :  $T(\alpha) = T(\aleph_0)$ . If  $T$  is also model-complete (in the usual sense) then it is  $\alpha$ -model-complete for every  $\alpha \geq \aleph_0$  and  $T(\alpha)$  is decidable provided  $T$  is axiomatic.

### 1. $\alpha$ -Model-Completeness.

**DEFINITION 1.1.** Let  $\phi(x, x_1, \dots, x_m)$  be a formula in  $L$  such that  $x, x_1, \dots, x_m$  are exactly all its free variables. Let  $\mathfrak{A}$  be a model for  $L$  and let  $a_1, \dots, a_m$  be elements in  $\mathfrak{A}$ . Define:

$$\phi(\mathfrak{A}, a_1, \dots, a_m) = \{a : \mathfrak{A} \models \phi[a, a_1, \dots, a_m]\}.$$

Let  $T$  be a theory in  $L$  and observe the following condition in which  $T$  is involved.

*Condition 1.1.* For every  $m \geq 0$  and every  $\phi(x, x_1, \dots, x_m)$  in  $L$  there exists an integer  $n_\phi$  such that for every  $\alpha$ -model  $\mathfrak{A}$  of  $T(Q)$  and every  $a_1, \dots, a_m$  in  $\mathfrak{A}$ : if  $|\phi(\mathfrak{A}, a_1, \dots, a_m)| > n_\phi$  then  $|\phi(\mathfrak{A}, a_1, \dots, a_m)| \geq \alpha$ .

**LEMMA 1.1.** *Let  $T$  be a theory in  $L$ ,  $\alpha \geq \aleph_0$ . If  $T$  fulfills Condition 1.1 then for every formula  $\psi$  in  $L(Q)$  there exists a formula  $\phi$  in  $L$  such that  $T(Q) \models_\alpha \psi \leftrightarrow \phi$ . (The meaning of the notation " $\models_\alpha$ " is "semantically valid in  $\alpha$ ".)*

*Proof.* Use induction on the structure of  $\psi$ . The lemma is true for formulae in  $L$  and it is clear that if it is true for  $\psi, \psi_1$  in  $L(Q)$  it is also true for  $\neg\psi, \psi \wedge \psi_1, \exists v\psi$  (for every individual variable  $v$ ). We now prove the lemma for  $Qv\psi$  assuming it is true for  $\psi$ . Suppose  $\psi$  is  $\psi(x, x_1, \dots, x_m)$  and  $v$  is  $x$ . Let  $\phi(x, x_1, \dots, x_m)$  be a formula in  $L$  such that  $T(Q) \models_\alpha \psi \leftrightarrow \phi$ . Let  $n$  be an integer the existence of which is assumed in Condition 1.1. Let  $\exists^{\geq n+1} x\phi(x, x_1, \dots, x_m)$  be a formula of  $L$  "saying" that there are at least  $n+1$  different elements  $x$  such that  $\phi(x, x_1, \dots, x_m)$  (here we use the assumption that  $L$  contains the equality sign). It is easy to see that for every model  $\mathfrak{A}$ , if  $\mathfrak{A}$  is a model of  $T(Q)$  and  $a_1, \dots, a_m \in \mathfrak{A}$  then

$$\mathfrak{A} \models_\alpha Qx\psi(x, a_1, \dots, a_m) \leftrightarrow \exists^{\geq n+1} x\phi(x, a_1, \dots, a_m).$$

Hence

$$T(Q) \models_\alpha Qx\psi(x, a_1, \dots, a_m) \leftrightarrow \exists^{\geq n+1} x\phi(x, a_1, \dots, a_m).$$

Therefore  $\exists^{\geq n+1} x\phi$  is the required formula for  $Qx\psi$ .

Note that Lemma 1.1 is true also when  $L$  is uncountable.

*An Example.* Let  $T$  be the first order theory of a dense linear ordering having neither first nor last element. Using the well known elimination of quantifiers (e.g. Kreisel and Krivine [6]) it is easy to see that  $T$  fulfills Condition 1.1 for  $\alpha = \aleph_0$  but not for  $\alpha > \aleph_0$ .

Now again let  $T$  be a theory in  $L$  but suppose  $\alpha > \aleph_0$  and observe the following condition involving  $T$ .

*Condition 1.2.* For every  $m \geq 0$ , every formula  $\phi(x, x_1, \dots, x_m)$  in  $L$ , every  $\alpha$ -model  $\mathfrak{A}$  of  $T(Q)$  and every  $a_1, \dots, a_m \in \mathfrak{A}$  either  $|\phi(\mathfrak{A}, a_1, \dots, a_m)| < \aleph_0$  or  $|\phi(\mathfrak{A}, a_1, \dots, a_m)| \geq \alpha$ .

The following lemma settles the relation between Condition 1.1 and Condition 2.2.

**LEMMA 1.2.** *Let  $T$  be a theory in a language  $L$  (possibly uncountable). Then for every  $\alpha > |L|$  Condition 1.1 is equivalent to Condition 1.2.*

*Proof.* It is clear that if Condition 1.1 holds, then also Condition 1.2 holds. Choose any cardinal  $\mu$  such that  $2^\mu > |L|$ . By Keisler [4] (Theorem 3.3 (iii), p. 121) if  $D$  is a regular ultra filter on  $\mu$  there exist natural numbers  $n_\nu$ ,  $\nu < \mu$ , such that  $D$ -Prod  $\lambda_\nu n_\nu = 2^\mu$ . Suppose that Condition 1.2 holds but Condition 1.1 does not hold. Hence there exists a formula  $\phi(x, x_1, \dots, x_m)$  in  $L$  such that for every  $n_\nu$ ,  $\nu < \mu$ , it is possible to find an  $\alpha$ -model  $\mathfrak{A}_\nu$  of  $T(Q)$  and elements  $a_{\nu_1}, \dots, a_{\nu_m}$  in  $\mathfrak{A}_\nu$  such that  $n_\nu < |\phi(\mathfrak{A}_\nu, a_{\nu_1}, \dots, a_{\nu_m})| < \alpha$ . Since Condition 1.2 holds we obtain:  $n_\nu < |\phi(\mathfrak{A}_\nu, a_{\nu_1}, \dots, a_{\nu_m})| < \aleph_0$ . By Skolem-Lowenheim Theorem we are allowed to suppose that  $|\mathfrak{A}_\nu| = 2^{2^{\aleph_0}}$  (where  $2^{K,0} = K$ ,  $2^{K,n+1} = 2^{2^{K,n}}$  and  $2^{K,\aleph_0} = \sup\{2^{K,n} : n < \aleph_0\}$  for every infinite cardinal  $K$ ). Observe now the structures  $(\mathfrak{A}_\nu, \phi(\mathfrak{A}_\nu, a_{\nu_1}, \dots, a_{\nu_m}))$  and take the ultra product  $D$ -Prod  $\lambda_\nu (\mathfrak{A}_\nu, \phi(\mathfrak{A}_\nu, a_{\nu_1}, \dots, a_{\nu_m}))$ . Denote it by  $(\mathfrak{B}, \phi(\mathfrak{B}, b_1, \dots, b_m))$ . Then:

$$|\mathfrak{B}| \cong 2^{2^{\aleph_0}} > |\phi(\mathfrak{B}, b_1, \dots, b_m)| = 2^\mu > |L|.$$

Therefore we can use Vaught [10] (the generalization of Corollary 4.2, p. 401). Hence, there exists an  $\alpha$ -model  $\mathfrak{C}$  of  $T(Q)$  and elements  $c_1, \dots, c_m$  in  $\mathfrak{C}$  such that  $|\phi(\mathfrak{C}, c_1, \dots, c_m)| = \aleph_0$ , a contradiction to the assumption that Condition 1.2 holds.

In some applications it is simpler to deal with Condition 1.2 than with Condition 1.1, so there is also a practical purpose in Lemma 1.2.

**LEMMA 1.3.** *Let  $T$  be any first order theory,  $\alpha \cong \aleph_0$ . If  $T$  is model-complete (in the usual sense) and  $T$  fulfills Condition 1.1 then  $T$  is  $\alpha$ -model-complete.*

*Proof.* Use Lemma 1.1.

**LEMMA 1.4.** *Let  $T$  be a theory in  $L$  and suppose  $\alpha > |L|$ . If  $T$  is  $\alpha$ -model-complete then  $T$  fulfills Condition 1.1.*

*Proof.* Suppose that  $T$  does not fulfill Condition 1.1. Then by Lemma 1.2 it also does not fulfill Condition 1.2. Therefore there exists an  $\alpha$ -model  $\mathfrak{A}$  of  $T(Q)$ , a formula  $\phi(x, x_1, \dots, x_m)$  in  $L$  and elements  $a_1, \dots, a_m$  in  $\mathfrak{A}$  such that  $\aleph_0 \leq |\phi(\mathfrak{A}, a_1, \dots, a_m)| < \alpha$ . Let  $C$  be any set

of power  $\alpha$  such that  $C$  and the domain of  $\mathfrak{A}$  are disjoint. Denote by  $D(\mathfrak{A})$  the diagram of  $\mathfrak{A}$  and let  $T'$  be the following set of sentences:

$T \cup D(\mathfrak{A}) \cup \{\phi(c, a_1, \dots, a_m) : c \in C\} \cup \{c_1 \neq c_2 : \text{for every two different elements } c_1, c_2, \text{ in } C\}$ .

$T'$  is a first order theory and every finite subset of  $T'$  has a model. Hence, by the Compactness Theorem,  $T'$  has a model  $\mathfrak{A}'$ . Since  $\mathfrak{A} \subseteq \mathfrak{A}'$  and  $T$  is  $\alpha$ -model-complete then  $\mathfrak{A} <_\alpha \mathfrak{A}'$ . But  $\mathfrak{A} \vDash_\alpha \neg Qx\phi(x, a_1, \dots, a_m)$  while  $\mathfrak{A}' \vDash_\alpha Qx\phi(x, a_1, \dots, a_m)$ , a contradiction.

For a theory  $T$  in  $L$  such that  $\alpha > |L|$  Lemmas 1.3 and 1.4 yield the following:

**THEOREM 1.5.** *Let  $T$  be a theory in  $L$ . Suppose  $T$  is model-complete (in the usual sense) and  $\alpha > |L|$ . Then a sufficient and necessary condition for  $T$  to be  $\alpha$ -model-complete is Condition 1.1.*

It is possible to look at Theorem 1.5 also from the aspect of definability. Let  $\mathfrak{A}$  be a model for  $L$ ,  $|\mathfrak{A}| \cong \alpha$ . Suppose  $A_1 \subseteq \mathfrak{A}$ . Call  $A_1$   $\alpha$ -parametrically definable in  $\mathfrak{A}$  if there exist a formula  $\phi(x, x_1, \dots, x_m)$  in  $L(Q)$  and elements  $a_1, \dots, a_m$  in  $\mathfrak{A}$  such that for every  $a$  in  $\mathfrak{A}$ ,  $a \in A_1$  iff  $\mathfrak{A} \vDash_\alpha \phi(a, a_1, \dots, a_m)$ . By Lemmas 1.1–1.4 we obtain at once:

**THEOREM 1.5\*.** *Let  $T$  be a theory in  $L$  which is also model-complete. Suppose  $\alpha > |L|$ . Then  $T$  is  $\alpha$ -model complete iff for every  $\alpha$ -model  $\mathfrak{A}$  of  $T(Q)$  and for every set  $A_1 \subseteq \mathfrak{A}$ , if  $A_1$  is  $\alpha$ -parametrically definable in  $\mathfrak{A}$  then  $|A_1| < \aleph_0$  or  $|A_1| \cong \alpha$ .*

We proceed with this section by relating to some known model-complete theories. The theory of totally discrete linear ordering having neither first nor last element is model-complete (in the usual sense) but for every  $\alpha \cong \aleph_0$  it is not  $\alpha$ -model-complete.

The theory of dense linear ordering having neither first nor last element is  $\aleph_0$ -model complete but for every  $\alpha \cong \aleph_1$  it is not  $\alpha$ -model complete. For the theory of algebraically closed fields and the theory of real closed fields we have the following theorem:

**THEOREM 1.6.** *Let  $T$  be the theory of algebraically closed fields or the theory of real closed fields. Let  $\phi(x_1, \dots, x_n)$  be any formula in  $L(Q)$  (where  $L$  is the language of  $T$ ). Then there exists a quantifier free formula  $\psi(x_1, \dots, x_n)$  such that  $T(Q) \vDash_\alpha \phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$  for every  $\alpha \cong \aleph_0$ .*

*Proof.* The proof is similar to the usual elimination of quantifiers for these theories (e.g. Kreisel and Krivine [6]).

**COROLLARY 1.7.** *The theory of algebraically closed fields and the theory of real closed fields are  $\alpha$ -model-complete for every  $\alpha \geq \aleph_0$ .*

The last theorem of this section gives a partial answer to a natural question, that is, what conclusions about  $\beta$ -model-completeness can be made assuming  $\alpha$ -model-completeness? Using Fuhrken [2] and Keisler [3], one can prove in a straightforward manner that:

**THEOREM 1.8.** *Let  $T$  be a countable first order theory.*

(1) *If  $T$  is  $\alpha$ -model-complete,  $\alpha > \aleph_0$ , then it is also  $\aleph_0$ -model-complete.*

(2) *If  $T$  is  $\aleph_1$ -model-complete, then  $T$  is  $\alpha$ -model-complete for every regular  $\alpha$ .*

(3) (G.C.H) *If  $T$  is  $\alpha$ -model-complete where  $\alpha$  is a successor of a regular cardinal, then  $T$  is  $\beta$ -model-complete for every regular  $\beta$ .*

(4) *If  $T$  is  $\alpha$ -model-complete where  $\alpha$  is a singular cardinal, then  $T$  is  $\beta$ -model-complete for every strong limit cardinal  $\beta$ .*

(5) (G.C.H) *If  $T$  is  $\alpha$ -model-complete where  $\alpha$  is a singular cardinal then  $T$  is  $\beta$ -model-complete for every singular  $\beta$ .*

*Proof.* All the parts of the theorem are proved in the same way so it will be enough if we prove for example part (1).

Assume that  $T$  is  $\alpha$ -model-complete but not  $\aleph_0$ -model complete. Then there exist two  $\aleph_0$ -models  $\mathfrak{A}, \mathfrak{B}$  for  $T(Q)$ ,  $\mathfrak{A} \subseteq \mathfrak{B}$ , and there exist a formula  $\phi(x_1, \dots, x_m)$  in  $L(Q)$  and elements  $a_1, \dots, a_m$  in  $\mathfrak{A}$  such that  $\mathfrak{A} \vdash_{\tau_0} \phi[a_1, \dots, a_m]$  while  $\mathfrak{B} \vdash_{\tau_0} \neg \phi[a_1, \dots, a_m]$ . By Fuhrken [2] we may assume that  $|\mathfrak{A}| = |\mathfrak{B}| = \aleph_0$ . We also may assume for the sake of simplicity that none of the elements  $a_1, \dots, a_m$  is an interpretation of an individual constant in the language of  $T$  and also that this language does not contain functions symbols. Let  $c_1, \dots, c_m, P(x)$  be  $m$  new individual constants and a new unary predicate, respectively. Let  $\psi$  be any formula in  $L(Q)$ . Write  $\psi^P$  for the formula obtained from  $\psi$  by relativizing all the quantifiers of  $\psi$  to  $P$  (the relativisation of  $Q$  is exactly as the relativisation of the existential quantifier). Denote:  $T^P = \{\psi^P : \psi \in T\}$ . Let  $S$  be the following set of sentences:

$$T \cup T^P \cup \{QxP(x), \bigwedge_{1 \leq i \leq m} P(c_i), \phi^P(c_1, \dots, c_m), \neg \phi(c_1, \dots, c_m)\}.$$

It is easy to see that a suitable expansion of  $\mathfrak{B}$  is an  $\aleph_0$ -model of  $S$ . By Fuhrken [2] it follows that there exists an  $\alpha$ -model  $\mathfrak{D}'$  of  $S$ . Define  $C = \{d : \mathfrak{D}' \vdash P[d]\}$ . Let  $\mathfrak{D}$  be the model obtained from  $\mathfrak{D}'$  by reducing

$\mathfrak{D}'$  to the language of  $T$ . Let  $\mathfrak{C}$  be the submodel of  $\mathfrak{D}$  built on  $C$  ( $\mathfrak{C}$  is a submodel since we assumed that our language does not contain functions symbols). It follows immediately that  $\mathfrak{C}, \mathfrak{D} \models_{\alpha} T(Q)$ . Denote by  $d_1, \dots, d_m$  the elements of  $\mathfrak{D}$  which correspond to the individual constants  $c_1, \dots, c_m$ . Then  $\mathfrak{C} \models_{\alpha} \phi[d_1, \dots, d_m]$ ,  $\mathfrak{D} \models_{\alpha} \neg\phi[d_1, \dots, d_m]$ , a contradiction to the assumption that  $T$  is  $\alpha$ -model complete.

A similar theorem, concerning the connections between  $\alpha$ -completeness and  $\beta$ -completeness, can be formulated.

It is unknown whether this result is the best result one can obtain.

**2.  $\alpha$ -Completeness, categoricity and  $\alpha$ -model-completeness.** Recall now the notions  $T(\alpha)$  and  $\alpha$ -completeness in the beginning. As an analogue to Vaught's Theorem about the connection between categoricity and completeness we have here:

**THEOREM 2.1.** *Let  $T$  be a countable first order theory categorical in an uncountable power. Then, for every  $\alpha \geq \aleph_0$ ,  $T$  is  $\alpha$ -complete and  $T(\alpha) = T(\aleph_0)$ .*

*Proof.* If  $T$  is not  $\aleph_0$ -complete then there exist two  $\aleph_0$ -models  $\mathfrak{A}, \mathfrak{B}$  of  $T(Q)$  and there exists a sentence  $\phi$  in  $L(Q)$  such that  $\mathfrak{A} \models_{\aleph_0} \phi$  and  $\mathfrak{B} \models_{\aleph_0} \neg\phi$ . By Fuhrken [2] there exist two  $\aleph_1$ -models  $\mathfrak{A}_1, \mathfrak{B}_1$  such that  $\mathfrak{A}_1 \models_{\aleph_1} \phi$ ,  $\mathfrak{B}_1 \models_{\aleph_1} \neg\phi$  and  $|\mathfrak{A}_1| = |\mathfrak{B}_1| = \aleph_1$ . If  $T$  is not  $\alpha$ -complete for  $\alpha > \aleph_0$ , then there exist two  $\alpha$ -models  $\mathfrak{A}_1, \mathfrak{B}_1$  of  $T(Q)$  and a sentence  $\phi$  in  $L(Q)$  such that  $\mathfrak{A}_1 \models_{\alpha} \phi$ ,  $\mathfrak{B}_1 \models_{\alpha} \neg\phi$  and  $|\mathfrak{A}_1| = |\mathfrak{B}_1| = \alpha$ . So whether  $\alpha = \aleph_0$  or  $\alpha > \aleph_0$  the assumption that  $T$  is not  $\alpha$ -complete leads us to two uncountable models of  $T$  that have the same power and are not isomorphic, a contradiction to Morley [7]. Suppose now that there exists  $\alpha$  such that  $T(\alpha) \neq T(\aleph_0)$ . Since  $T$  is  $\aleph_0$ -complete and also  $\alpha$ -complete there exists  $\phi$  in  $L(Q)$  such that  $\phi \in T(\aleph_0)$  and  $\neg\phi \in T(\alpha)$ . By Fuhrken [2] there exists an  $\alpha$ -model  $\mathfrak{A}$  for  $T(Q)$  such that  $\mathfrak{A} \models_{\alpha} \phi$ , a contradiction to the assumption that  $\neg\phi \in T(\alpha)$ .

**REMARK.** If  $T$  is categorical in  $\aleph_0$  then  $T$  is also  $\aleph_0$ -complete but it is not necessarily  $\alpha$ -complete for  $\alpha > \aleph_0$ . One can easily see that by taking  $T$  as the theory of dense linear ordering (having neither first nor last element). Again as in the previous section arises the question about the connection between  $\alpha$ -completeness and  $\beta$ -completeness and the answer here is the same as there. Another question about  $\alpha$ -completeness is to find a sufficient and necessary condition on formulae in  $L$  so that  $T$  will be  $\alpha$ -complete; but what we know about  $\alpha$ -completeness are Theorems 2.2 and 2.3.

Let  $\phi(x)$  be a formula in  $L(Q)$  having  $x$  as its only free variable. Let  $\mathfrak{A}$  be a model for  $L$ . Denote  $\phi(\mathfrak{A}, \alpha) = \{a : \mathfrak{A} \models_{\alpha} \phi[a]\}$ .

**THEOREM 2.2.** *Let  $T$  be a countable first order theory. Assume  $T$  is  $\alpha$ -complete,  $\alpha > \aleph_0$ . Then for every formula  $\phi(x)$  in  $L(Q)$  (having  $x$  as its only free variable) there exists a cardinal  $m_\phi$ , finite or equal to  $\alpha$ , such that for every model  $\mathfrak{A}$  for  $T$  of power  $\alpha$ ,  $|\phi(\mathfrak{A}, \alpha)| = m_\phi$ .*

*Proof.* Let  $\mathfrak{A}$  be any countable model for  $T$  (there exists such a model since by the definition of  $\alpha$ -completeness  $T$  has infinite models. It has also a countable model because it is countable). It is clear that either  $\mathfrak{A} \models_{\aleph_0} Qx\phi(x)$  or  $\mathfrak{A} \models_{\aleph_0} \neg Qx\phi(x)$ . In the first case define  $\Sigma_1 = T(Q) \cup \{Qx\phi(x)\}$ . By Fuhrken [2] there exists an  $\alpha$ -model  $\mathfrak{A}_1$  for  $\Sigma_1$ . Since  $\mathfrak{A}_1 \models_\alpha Qx\phi(x)$  and  $T$  is  $\alpha$ -complete then for every  $\alpha$ -model  $\mathfrak{A}$  for  $T(Q)$ ,  $\mathfrak{A} \models_\alpha Qx\phi(x)$ . Hence  $m_\phi = \alpha$  in this case. In the second case there exists a finite number  $k$  such that  $\mathfrak{A} \models_{\aleph_0} \exists^k !x\phi(x)$ , where  $\exists^k !x\phi(x)$  is a formula in  $L(Q)$  "saying" that there are exactly  $k$  elements  $x$  such that  $\phi(x)$ . Define  $\Sigma_2 = T(Q) \cup \{\exists^k !x\phi(x)\}$ . By the same argument as before there exists an  $\alpha$ -model  $\mathfrak{A}_2$  for  $\Sigma_2$ . Because of the  $\alpha$ -completeness of  $T$  we obtain  $\mathfrak{A} \models_\alpha \exists^k !x\phi(x)$  for every  $\alpha$ -model  $\mathfrak{A}$  of  $T(Q)$ . Hence, in this case,  $m_\phi = k$ .

**THEOREM 2.3.** *Let  $T$  be a complete theory in  $L$  which is also  $\alpha$ -model-complete,  $\alpha \cong \aleph_0$ . Then  $T$  is also  $\alpha$ -complete.*

*Proof.* Suppose on the contrary that  $T$  is not  $\alpha$ -complete. Then there exist two  $\alpha$ -models  $\mathfrak{A}, \mathfrak{B}$  for  $T(Q)$  and a sentence  $\phi$  in  $L(Q)$  such that  $\mathfrak{A} \models_\alpha \phi$  and  $\mathfrak{B} \models_\alpha \neg\phi$ . Since  $T$  is complete then  $\mathfrak{A}$  is elementary equivalent to  $\mathfrak{B}$ . By Bell and Slomson [1] (p. 161), there exists a model  $\mathfrak{D}$  which is an elementary extension of  $\mathfrak{A}$  and  $\mathfrak{B}$ . Since  $T$  is  $\alpha$ -model-complete and  $\mathfrak{A} \models_\alpha \phi$  it follows that  $\mathfrak{D} \models_\alpha \phi$ . By the same argument we obtain also  $\mathfrak{D} \models_\alpha \neg\phi$ , a contradiction.

**DEFINITION 2.1.** Let  $L(Q)$  be recursive and let  $T$  be a theory in  $L$ . Call  $T$   $\alpha$ -decidable if  $T(\alpha)$  is recursive (more precisely, the set of Gödel-Numbers of all the sentences in  $T(\alpha)$  is recursive).

**THEOREM 2.4.** *Let  $T$  be a theory in  $L$  categorical in an uncountable power. Suppose  $L(Q)$  and  $T$  are recursive. Then  $T$  is  $\alpha$ -decidable for every  $\alpha \cong \aleph_0$ .*

*Proof.* By Theorem 2.1 we have:  $T(\alpha) = T(\aleph_0)$  for every  $\alpha \cong \aleph_0$ . So it is sufficient to show that  $T(\aleph_1)$  is recursive. By Keisler [5] we know that  $T(\aleph_1)$  is recursively enumerable. Since  $T$  is  $\aleph_1$ -complete then for every  $\phi$  in  $L(Q)$ ,  $\phi \in T(\aleph_1)$  iff  $\neg\phi \notin T(\aleph_1)$ . This means that also the complement of  $T(\aleph_1)$  is recursively enumerable. Hence  $T(\aleph_1)$  is recursive.



LEMMA 2.5. *Let  $T$  be any theory in a countable first order language  $L$  such that  $T$  is categorical in an uncountable power. Let  $\mathfrak{A}$  be a model for  $T$  and let  $a_1, \dots, a_n$  be elements in  $\mathfrak{A}$ . Suppose  $|\mathfrak{A}| = \alpha > \aleph_0$  and  $\phi(x, x_1, \dots, x_n)$  is a formula in  $L$  having exactly  $x, x_1, \dots, x_n$  as free variables. Then  $|\phi(\mathfrak{A}, a_1, \dots, a_n)| = \alpha$  or  $|\phi(\mathfrak{A}, a_1, \dots, a_n)| < \aleph_0$ .*

*Proof.* Since  $\alpha > \aleph_0$  then  $T$  is categorical in  $\alpha$ , by Morley [7]. Denote by  $T((\mathfrak{A}, a_1, \dots, a_n))$  the (first order) theory of  $(\mathfrak{A}, a_1, \dots, a_n)$ . Again by Morley [7] it is easy to see that  $T((\mathfrak{A}, a_1, \dots, a_n))$  is categorical in  $\alpha$  so  $(\mathfrak{A}, a_1, \dots, a_n)$  is a saturated model. It is well known (see for instance Morley and Vaught [8], Theorem 3.7) that in a saturated model each infinite set defined by a formula (in the language for the model) has the power of the whole model. Hence  $|\phi(\mathfrak{A}, a_1, \dots, a_n)| = \alpha$  or  $|\phi(\mathfrak{A}, a_1, \dots, a_n)| < \aleph_0$ .

COROLLARY 2.6. *Let  $T$  be as in Lemma 2.5. Then for every formula  $\phi(x_1, \dots, x_n)$  in  $L(Q)$  there exists a formula  $\psi(x_1, \dots, x_n)$  in  $L$  such that  $T(Q) \models_\alpha \phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$  for every  $\alpha \geq \aleph_0$ .*

*Proof.* By Lemmas 2.5, 1.2, 1.1 and Theorem 2.1.

Corollary 2.6 says that the use of the language  $L(Q)$  is dispensible for talking about models of  $T$ ; namely, everything that can be said in  $L(Q)$  about elements in a model of  $T$  can be said about them in  $L$ .

THEOREM 2.7. *Let  $T$  be a theory in a countable first order language  $L$  such that  $T$  is categorical in an uncountable power and also model-complete (in the usual sense). Then*

(1) *For every formula  $\phi(x_1, \dots, x_n)$  in  $L(Q)$  there exist two formulae  $\psi_i(x_1, \dots, x_n)$ ,  $i = 1, 2$ , in  $L$ ,  $\psi_1$  is existential,  $\psi_2$  is universal and  $T(Q) \models_\alpha \phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$  for every  $\alpha \geq \aleph_0$ .*

(2)  *$T$  is  $\alpha$ -model-complete for every  $\alpha \geq \aleph_0$ .*

(3) *If  $L(Q)$  and  $T$  are recursive then there exists an effective procedure to find  $\psi_i$ ,  $i = 1, 2$ , that were mentioned in (1).*

*Proof.* (1) Let  $\phi(x_1, \dots, x_n)$  be a formula in  $L(Q)$ . By Corollary 2.6 there exists a formula  $\psi(x_1, \dots, x_n)$  in  $L$  such that  $T(Q) \models_\alpha \phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$  for every  $\alpha \geq \aleph_0$ . By Robinson [9] (Theorem 3.3.11), since  $T$  is model-complete, there exist two formulae  $\psi_i(x_1, \dots, x_n)$ ,  $\psi_2(x_1, \dots, x_n)$  in  $L$ ,  $\psi_1$  is existential,  $\psi_2$  is universal and  $T \vdash \psi(x_1, \dots, x_n) \leftrightarrow \psi_i(x_1, \dots, x_n)$ ,  $i = 1, 2$ . Therefore

$$T(Q) \models_\alpha \phi(x_1, \dots, x_n) \leftrightarrow \psi_i(x_1, \dots, x_n),$$

$i = 1, 2$ , for every  $\alpha \geq \aleph_0$ .

(2) By the assumption on  $T$  and by Corollary 2.6  $T$  is  $\alpha$ -model-complete for every  $\alpha \cong \aleph_0$ .

(3) Since  $L(Q)$  is recursive there is an effective procedure to count all existential formulae (in  $L$ ) that have exactly  $x_1, \dots, x_n$  as free variables. Let  $\psi'$  be such a formula. By Theorem 2.4 there is an effective procedure to decide whether  $[\phi \leftrightarrow \psi'] \in T(\aleph_1)$  or not. Since there exists an existential formula  $\psi_1$  such that  $[\phi \leftrightarrow \psi_1] \in T(\aleph_1)$  we shall find it after finite number of steps. In the same way we shall find a universal formula  $\psi_2$  such that  $[\phi \leftrightarrow \psi_2] \in T(\aleph_1)$ .

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Received September 24, 1973.

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# Pacific Journal of Mathematics

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